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Some new results on the prime order Cayley graph of given groups

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Abstract. In this paper, we study the prime order Cayley graph assigned to the group $\mathbb{Z}_n$ for different values of n. We specify some of the graph theoretical properties such as chromatic and perfect matching numbers. Furthermore, we determine the adjacency matrices and eigenvalues of the prime order Cayley graph associated with groups $\mathbb{Z}_n$ and D_{2n}.

1 Introduction

The Cayley Graph was first considered for finite groups by Arthur Cayley in 1878. Let G be a group, and let S be a subset of $G\backslash\{1_G\}$. The Cayley graph associated with (G,S) is denoted by $Cay(G,S)$ and defined as the directed graph with vertex set G and arc set $\{(a,b)|a,b \in G, ba^{-1} \in S\}$. The Cayley graph may depend on the choice of a generating set, and it is connected if and only if S generates G. If $S = S^{-1}$, then the Cayley graph is undirected. In this work, we restrict our attention only to the undirected Cayley graphs.

B. Tolue defined the prime and composite order Cayley graphs in [12], and she discussed about some of their properties. For instance, the structure of them for some certain groups was achieved.

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In this research, we determine the prime order Cayley graph assigned to $\mathbb{Z}_n$ for different values of n and study some of the graph theoretical properties such as chromatic and perfect matching numbers. Moreover, we distinguish the adjacency matrices of the prime order Cayley graphs associated to $\mathbb{Z}_n$ and D_{2n} for a given n. We will verify for which n the prime order Cayley graphs assigned to $\mathbb{Z}_n$ and D_{2n} have total perfect code. First we recall some basic definitions and concepts.

Definition 1 *Let G be a group and S be the set of non-identity prime order elements of G. Consider the Cayley graph $Cay_p(G, S)$ associated to the group G relative to S. We call it prime order Cayley graph.*

The set of vertices and edges for the graph Γ is denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. Throughout the article, the notation $v_1 \sim v_2$ denotes that v_1 is adjacent to v_2. For the definition of the adjacency matrix of Γ, which is denoted by $A(\Gamma)$, and more details in this area, one can refer to [2]. If A is an $n \times n$ matrix over the field F, an eigenvalue of A in F is a scalar $\lambda \in F$ such that the matrix $(A - \lambda I)$ is not invertible. Any X such that $AX = \lambda X$ is called an eigenvector of A associated with eigenvalue λ. The set of all eigenvalues is the spectrum of A, and it is denoted by spec(A), and note that spec$(A) = \{\lambda \in \mathbb{C} \mid \det(\lambda I - A) = 0\}$ (see [1] for more details). The eigenvalues of a graph is the eigenvalues of its adjacency matrix and the spectrum of the graph Γ is denoted by spec$(A(\Gamma))$. Furthermore the spectrum of a disconnected graph is simply the disjoint union of the spectra of its components. The spectrum of a clique K_n is $\lambda_1 = n - 1, \lambda_2 = ... = \lambda_n = -1$ (see [5]).

A block matrix is a matrix that has been partitioned into sub-matrices ("blocks") of the same size. Early in this century Issai Schur compute the determinant of block matrices. He considered a $2n \times 2n$ matrix M and partitioned it into four $n \times n$ blocks A, B, C and D as shown below

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

If C and D commute ($CD = DC$), then $\det(M) = \det(AD - BC)$ (see [8]). When $A = D$ and $B = C$, the following formula holds (even if A and B do not commute)

$$M = \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det(A - B)\det(A + B).$$

Remark 1 *[10] A block diagonal matrix is a block matrix which is square such that the main diagonal blocks are square matrices and all off-diagonal* *blocks are zero matrices. If* A *is a block diagonal matrix with diagonal blocks* $A_1, \ldots, A_n$*, then* $\det(A) = \det(A_1)\det(A_2)\ldots\det(A_n)$.

Let Γ_1 and Γ_2 be graphs with vertex sets $V(\Gamma_1)$ and $V(\Gamma_2)$, respectively. The Cartesian product of Γ_1 and Γ_2, denoted by $\Gamma_1 \times \Gamma_2$, is the graph defined as follows. The vertex set of $\Gamma_1 \times \Gamma_2$ is $V(\Gamma_1) \times V(\Gamma_2)$. The vertices (v, w) and $(v^{'}, w^{'})$ are adjacent if either $v = v^{'}$ and $w, w^{'}$ are adjacent in Γ_2, or $w = w^{'}$ and $v, v^{'}$ are adjacent in Γ_1.

Lemma 1 *Let* Γ_1 *and* Γ_2 *be graphs with* m *and* n *vertices, respectively. If* $\lambda_1, \lambda_2, ..., \lambda_m$ *and* $\mu_1, \mu_2, ..., \mu_n$ *are the eigenvalues of* Γ_1 *and* Γ_2*, receptively, then the eigenvalues of* $\Gamma_1 \times \Gamma_2$ *are given by* $\lambda_i + \mu_j$*,* $i = 1, \ldots, m$ $j = 1, \ldots, n$.

The union $\Gamma = \Gamma_1 \cup \Gamma_2$ of graphs Γ_1 and Γ_2 with disjoint point sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. This operation is sometimes also known explicitly as the graph disjoint union (see [9] for more details).

In section 2, we show that $Cay_p(\mathbb{Z}_{p^s \times q^{s'}})$ is isomorphic to disjoint union of $p^{s-1}q^{s'-1}$ copies of $K_p \times K_q$, where p, q are distinct prime numbers $p < q$ and $s, s^{'}$ are positive integers. Moreover, by the structure of $Cay_p(\mathbb{Z}_{p^s \times q^{s'}})$, we present some new results about its perfect matching and total perfect code. We discuss about the adjacency matrix of $Cay_p(\mathbb{Z}_n, S)$ for different values of n, in section 3. Furthermore, we obtain the adjacency matrices of $Cay_p(\mathbb{Z}_{\prod_{i=1}^n p_i}, S)$ and $Cay_p(\mathbb{Z}_{\prod_{i=1}^n p_i^{\alpha_i}}, S)$, where p_i's are distinct prime numbers $p_1 < p_2 < ... < p_n$ and α_i's are positive integers $1 \le i \le n$. Finally, we determine the adjacency matrices of $Cay_p(D_{2n}, S)$ and $Cay_p(Q_{4n}, S)$ for different values of n. Furthermore, by use of the adjacency matrix, we clarify the structure of $Cay_p(D_{2n}, S)$ and $Cay_p(Q_{4n}, S)$.

2 The prime order Cayley graph associated to the group $\mathbb{Z}_{p^s \times q^{s'}}$

In this part, we will study the prime order Cayley graph of the group $\mathbb{Z}_{p^s \times q^{s'}}$, where p, q are distinct prime numbers and $s, s^{'}$ are positive integers. We compute chromatic and perfect matching numbers of $Cay_p(\mathbb{Z}_n, S)$. Moreover, we determine total perfect code and perfect matching sets of it for different values of n say, $p^\alpha, pq, 2q, 2^\alpha q$ and $3^\alpha q$, where p, q are distinct prime numbers and α is a positive integer.

Theorem 1 *The prime order Cayley graph associated to the group* $\mathbb{Z}_{p^s \times q^{s'}}$ *is isomorphic to disjoint union of* $p^{s-1}q^{s'-1}$ *copies of* $K_p \times K_q$*, where* p, q *are distinct prime numbers* $p < q$ *and* s, s' *are positive integers.*

Proof. We check first the component of $Cay_p(\mathbb{Z}_{p^s \times q^{s'}}, S)$ which contains zero. The form of the connection between the vertices, implies that the vertex zero is joined to all vertices of prime orders. We classify these vertices, to provide better understanding. Let x be a vertex of order q. Then we have

$$q = |x| = \frac{p^s \times q^{s'}}{gcd(p^s \times q^{s'}, x)}, \; x = lp^s \times q^{s'-1}, \; 1 \leq l \leq q - 1.$$

So there exist $q-1$ vertices of order q and all of them are joined to the vertex zero. Moreover, if x_1, x_2 are two vertices of order q, then they are adjacent to one another because

$$|x_1 - x_2| = \frac{p^s \times q^{s'}}{gcd(p^s \times q^{s'}, x_1 - x_2)} = q,$$
$$x_i = l_i p^s \times q^{s'-1}, \; 1 \leq i \leq 2, \; 1 \leq l_1, l_2 \leq q - 1.$$

So the vertex zero and all the $q-1$ vertices of order q construct a clique K_q. Now we know there exist vertices of order p and all of them are joined to zero. We name one of them y_1. Obviously y_1 is joined to the vertex zero. Moreover, y_1 is adjacent to all the vertices $x_i + y_1, 1 \leq i \leq q-1$ and the vertices $x_i + y_1$ are joined to $x_j + y_1$, where $1 \leq i, j \leq q-1$. Therefore y_1 together with all the $q-1$ vertices in the form $x_i + y_1, 1 \leq i \leq q-1$ make a clique K_q. Clearly, all the vertices in the first K_q including the vertex zero and all the vertices in the second K_q including y_1 are joined to each other one by one, that is $0 \sim y_1$ and $x_i \sim x_i + y_1$, where $1 \leq i \leq q-1$. Still there exist some elements of order p which are joined to the vertex zero. We consider an arbitrary vertex of order p, say, y_w and we have

$$p = |y_w| = \frac{p^s \times q^{s'}}{gcd(p^s \times q^{s'}, y_w)} \qquad y_w = lp^{s-1}q^{s'}, \; 1 \leq l \leq p - 1.$$

So the set $\{0, y_w \,|\, 1 \leq w \leq p-1\}$ together with the set $\{x_i, x_i + y_w \,|\, 1 \leq w \leq p-1\}$ create a clique K_p. By the group order and the number of vertices in each component, we obtain the number of component which is $\frac{p^s \times q^{s'}}{pq} = p^{s-1}q^{s'-1}$. Hence we have $p^{s-1}q^{s'-1}$ components that are isomorphic to $K_p \times K_q$. □

It is not hard to conclude that $\mathrm{Cay}_p(\mathbb{Z}_{3^2\times 5}, S)$ is formed by three isomorphic components. We will explain the structure of one of these components. It is clear that $S = \{9, 15, 18, 27, 30, 36\}$, and each set $\{0, 9, 18, 27, 36\}$, $\{3, 12, 21, 30, 39\}$ and $\{6, 15, 24, 33, 42\}$ form a clique K_5. Since zero is joined to the vertices with prime order and the vertices 15 and 30 have prime order 3. So the vertex 0 is adjacent to the vertices 15 and 30. Furthermore, vertex 15 is adjacent to 30. Vertex 9 is adjacent to the vertices $9+15 = 24$ and $9+30 = 39$. Moreover, $24 \sim 39$. Similarly, we can check the connection between the other elements. Hence $\mathrm{Cay}_p(\mathbb{Z}_{3^2\times 5}, S)$ is a graph with three isomorphic components. For which one its components has been drawn in Figure 1.

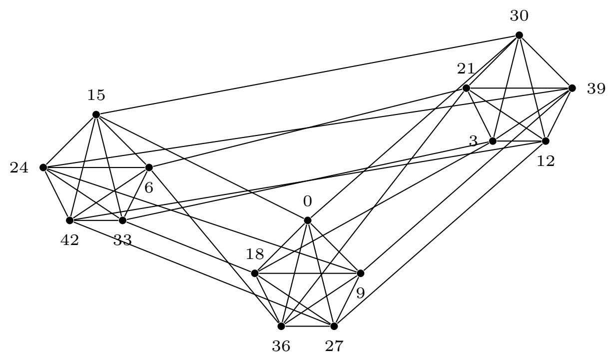

Figure 1:

The $m\times n$ rook graph is the Cartesian product $K_m \times K_n$ of complete graphs. The graph $K_m \times K_n$ has mn vertices and $mn(m+n)/2-mn$ edges (see [4]). By the Theorem 1, we observe that each component of $\mathrm{Cay}_p(\mathbb{Z}_{p^s\times q^{s'}}, S)$ is a rook graph. So each component of $\mathrm{Cay}_p(\mathbb{Z}_{p^s\times q^{s'}}, S)$ has $pq(p+q)/2-pq$ edges and we have $p^{s-1}q^{s'-1}$ components. Therefore this graph has $p^s q^{s'}(p+q)/2-p^s q^{s'}$ edges.

We refer the reader to [3], for the definitions of k-colorable graph Γ and the chromatic number, $\chi(\Gamma)$. Note that $\chi(\mathrm{Cay}_p(\mathbb{Z}_{p^s\times q^{s'}}, S)) = q$, where p, q are distinct prime numbers and $p < q$. Since each component of $\mathrm{Cay}_p(\mathbb{Z}_{p^s\times q^{s'}}, S)$ is isomorphic to $K_p \times K_q$ and by the fact that the chromatic number of a rook graph is $\max(p, q)$, the assertion is clear.

Theorem 2 *The number of triangles of* $\mathrm{Cay}_p(\mathbb{Z}_{p^s\times q^{s'}}, S)$ *is equal to* $p^{s-1}q^{s'-1}(q\binom{p}{3} + p\binom{q}{3})$.

Proof. By the structure of $\mathsf{Cay}_p(\mathbb{Z}_{p^s \times q^{s'}}, S)$, we know there exist $p^{s-1}q^{s'-1}$ components isomorphic to $K_p \times K_q$. It is enough to compute the number of triangles for one of these components. Since each component is isomorphic to $K_p \times K_q$, then the number of K_p and K_q in each component are q and p, respectively. On the other hand, we know that there exist $\binom{p}{3}$ and $\binom{q}{3}$ triangles in K_p and K_q, respectively. Thus each component has $q\binom{p}{3} + p\binom{q}{3}$ triangles. Hence $\mathsf{Cay}_p(\mathbb{Z}_{p^s \times q^{s'}}, S)$ has $p^{s-1}q^{s'-1}(q\binom{p}{3} + p\binom{q}{3})$ triangles. □

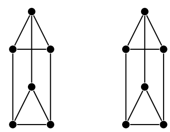

Figure 2: $\mathsf{Cay}_p(\mathbb{Z}_{2^2 \times 3}, S) \cong (K_2 \times K_3) \cup (K_2 \times K_3)$

The number of triangles in each component is $2\binom{3}{3} + 3\binom{2}{3} = 2$.

We denote the matching number of the graph Γ by $\upsilon(\Gamma)$ (for the details see [3]). A perfect matching is a matching containing $\frac{n}{2}$ edges, meaning perfect matching are only possible on graphs with an even number of vertices (see [11]). In the next theorem we specify the perfect matching number and the perfect matching set for $\mathsf{Cay}_p(\mathbb{Z}_n, S)$, when n is determined.

Theorem 3 *(i) The perfect matching number of the graph $\mathsf{Cay}_p(\mathbb{Z}_{p^\alpha}, S)$ is equal to $p^{\alpha-1}$, where p is a prime number and α is a positive integer.*

(ii) The perfect matching number of the graph $\mathsf{Cay}_p(\mathbb{Z}_{2q}, S)$ is equal to q. Moreover, the perfect matching set has the form $\{\{x, x+q\} | x \in V(\Gamma)\}$, where q is a prime number and $q > 2$.

(iii) $\mathsf{Cay}_p(\mathbb{Z}_{pq}, S)$ has no perfect matching. But it has a matching of the form $\{i, q+i | 0 \leq i \leq q-1\}$, where p, q are distinct prime numbers $p < q$ and $p, q \neq 2$.

(iv) The perfect matching number of $\mathsf{Cay}_p(\mathbb{Z}_{2^\alpha q}, S)$ is $2^{\alpha-1}q$, where q is a prime number and α is a positive integer.

Proof. (i) We know $\mathsf{Cay}_p(\mathbb{Z}_{p^\alpha}, S)$ is isomorphic to the disjoint union of $p^{\alpha-1}$ complete components on p vertices. By the definition of the perfect matching number, it is enough to take one edge from each component. Thus the perfect matching number is $p^{\alpha-1}$.

(ii) $Cay_p(\mathbb{Z}_{2q}, S)$ have two cliques on q vertices and there exist some edges between these two complete graphs. Two vertices x, y of each clique, are joined to each other, if their difference is equal to q. Moreover, each vertex that belongs to the first K_q is joined to exactly one vertex that belongs to the second. It is clear that the edges between these two complete graphs form a perfect matching for $Cay_p(\mathbb{Z}_{2q}, S)$.
(iii) $\Gamma = Cay_p(\mathbb{Z}_{pq}, S)$ has p cliques on q vertices and there exist some edges between these two cliques. Since the number of the components is odd, then we have no perfect matching. The edges between these two cliques are $\{x, x+p \mid x \in V(\Gamma)\}$ and $\{x, x+q \mid x \in V(\Gamma)\}$. Now we omit edges that have a vertex in common and we obtain a matching $\{i, q+i \mid 0 \leq i \leq q-1\}$.
(iv) Note that $Cay_p(\mathbb{Z}_{2^\alpha q}, S)$ has $2^{\alpha-1}$ components and each component is isomorphic to $P_2 \times K_q$. Consider the component which contains zero. The vertex zero and all the $q-1$ elements of order q, make a clique K_q. We denoted two of these $q-1$ vertices by x, y. The next K_q is consists of the vertex t of order 2 and $x+t, y+t$. In addition $0, x$ and y are joined to $t, x+t$ and $y+t$, respectively. In general, ith component of $Cay_p(\mathbb{Z}_{2^\alpha q}, S)$ has two cliques of size q, and there exist some edge between them, where $1 \leq i \leq 2^\alpha - 1$. In fact the vertices $i, x+i$ and $y+i$ of the first clique are joined to the vertices $t+i, x+t+i$ and $y+t+i$ of the second clique, respectively. So according to our observation above, the perfect matching number for each component is equal to q. Since we have $2^\alpha - 1$ components that do not overlap, then the perfect matching number is equal to $\underbrace{q+q+...+q}_{2^\alpha-1}$. □

A perfect code in a graph $\Gamma = (V, E)$ is a subset C of V that is an independent set such that every vertex in $V \backslash C$ is adjacent to exactly one vertex in C.
A total perfect code in Γ is a subset C of V such that every vertex of V is adjacent to exactly one vertex in C (see [7]). In the following theorem, we will specify the total perfect code for $Cay_p(\mathbb{Z}_n, S)$, when n is given.

Theorem 4 *(i) If* $n = 2q$, *then one of the total perfect codes for* $Cay_p(\mathbb{Z}_{2q}, S)$ *is the set* $\{0, q\}$, *where* q *is a prime number and* $q > 2$.

(ii) If $n = pq$, *then* $Cay_p(\mathbb{Z}_{pq}, S)$ *has no total perfect code, where* p, q *are prime numbers* $p < q$ *and* $p, q \neq 2$.

(iii) If $n = p^\alpha$, *then* $Cay_p(\mathbb{Z}_{p^\alpha}, S)$ *has no total perfect code. But it has a perfect code which is equal to the set* $\{0, 1, ..., p^{\alpha-1} - 1\}$, *where* p *is a prime number and* α *is a positive integer.*

Proof. By the structure of prime order Cayley graph and total perfect code definition, the proof is clear. □

In the next theorem, we discuss $\mathsf{Cay_p}(\mathbb{Z}_n, S)$ planarity.

Theorem 5 *The graph* $\mathsf{Cay_p}(\mathbb{Z}_n, S)$ *is planar if and only if* $n = 1, 2, 3, 2^\alpha \times 3^\beta$, *where* α, β *are positive integers and* $\alpha, \beta \geq 1$.

Proof. If $n = 1, 2, 3, 4$, then by the Figure 3, it is clear that $\mathsf{Cay_p}(\mathbb{Z}_n, S)$ is planar. If $n = 2^\alpha \times 3^\beta$, we know that $\mathsf{Cay_p}(\mathbb{Z}_{2^\alpha \times 3^\beta}, S)$ is isomorphic to disjoint

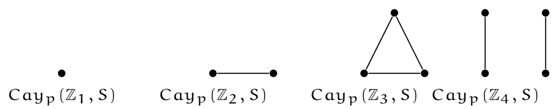

Figure 3:

union of $2^{\alpha-1} \times 3^{\beta-1}$ copies of $K_2 \times K_3$. So it is enough to discuss about one of its components. Clearly $K_2 \times K_3$ has neither K_5 nor $K_{3,3}$ subdivision. Therefore it is planar. Now, suppose that $\mathsf{Cay_p}(\mathbb{Z}_n, S)$ be a planar graph. If 5 devides n, then we obtain the number of elements of order 5 in $\mathbb{Z}_n$,

$$x \in \mathbb{Z}_n,\ 5 = |x| = \frac{n}{\gcd(x, n)} = \frac{5w}{\gcd(x, 5w)}.$$

Therefore $\gcd(x, 5w) = wl$, $1 \leq l \leq 4$. These 4 elements are of order 5, so they belong to S and together with 0, form a clique K_5. Thus in this case, $\mathsf{Cay_p}(\mathbb{Z}_n, S)$ can not be a planar graph. Hence 5 does not divide n. Moreover, if $m > 5$ and m divides n, again we can find a clique K_m, which contains a K_5 subdivision. Therefore the only prime numbers that can be divisors of n are 2 and 3. □

3 The adjacency matrix of $\mathsf{Cay_p}(\mathbb{Z}_n, S)$

In this section, we compute the adjacency matrix of the prime order Cayley graph assigned to $\mathbb{Z}_n$ for different values of n. Moreover, we will calculate its determinant and eigenvalues. Let us start with this necessary definition.

Definition 2 *A circulant matrix is fully specified by one vector* c, *which appears as the first row (or column) of* C. *The remaining rows (and columns, respectively) of* C *are each cyclic permutations of the vector* c *with offset equal* *to the row (or column) index, if lines are indexed from* 1 *to* n *(see [6]). The corresponding eigenvalues are given by* $\lambda_j = c_1 + c_2\omega_j + c_3{\omega_j}^2 + ... + c_n{\omega_j}^{n-1}$, $\omega_j = e^{\frac{2\pi j}{n}}$, $j = 0, 1, ..., n-1$. *Therefore the determinant of a circulant matrix can be computed as*

$$\det(C) = \prod_{j=0}^{n-1}(c_1 + c_2\omega_j + c_3{\omega_j}^2 + ... + c_n{\omega_j}^{n-1}).$$

The following proposition will pave the way for proof of the next theorems in this section.

Proposition 1 *The adjacency matrix of* $Cay_p(\mathbb{Z}_{pq}, S)$ *is circulant, where* p, q *are distinct prime numbers.*

Proof. Obviously, $S = \{p, 2p, ..., (q-1)p, q, ..., (p-1)q\}$. By the structure of the prime order Cayley graph, we know that the vertex zero is joined to all elements belong to S. Hence in the first row, all entries in the $(1, kp+1), 1 \leq k \leq q-1$ and $(1, k'q+1), 1 \leq k' \leq p-1$ positions, are equal to one. The next vertex, $i.e.$ vertex 1 differs with zero one unit. Therefore its corresponding row can be determined by shifting the entries in the row corresponding to zero, one unit to the right. Since the vertex i differs with the vertex zero, i unit, then the row corresponding to the vertex i can be specified by shifting entries in the row corresponding to zero, i unit to the right. □

In the next theorem the notation $\frac{\lceil M \rceil}{\lfloor m \rfloor}$ stands for the multiplication of m prime numbers belong to M, where M is the set of all the prime numbers, which divide $\prod_{i=1}^{n} p_i$ and p_i's are distinct prime numbers and $p_1 < p_2 < ... < p_n$ $(1 \leq m \leq n)$.

Theorem 6 *(i) In the* j*th row of the adjacency matrix of* $Cay_p(\mathbb{Z}_{\prod_{i=1}^{n} p_i}, S)$, *the columns* $\frac{\lceil M \rceil}{\lfloor n-1 \rfloor} l + j$, $1 \leq j \leq \prod_{i=1}^{n} p_i$, $1 \leq l \leq p_i - 1$ *are equal to one, where* p_i*'s are distinct prime numbers and* $p_1 < p_2 < ... < p_n$.

(ii) In the j*th row of the adjacency matrix of* $Cay_p(\mathbb{Z}_{\prod_{i=1}^{n} p_i^{\alpha_i}}, S)$, *the columns* $\frac{\lceil M \rceil}{\lfloor n-1 \rfloor} p_k^{\alpha_k - 1} l + j$, $1 \leq l \leq p_k - 1$ *are equal to one, where* p_i*'s are prime numbers* $p_1 < p_2 < ... < p_n$ *and* α_i*'s are positive integers* $1 \leq i \leq n$.

Proof. (i) First note that

$$S = \{x \in \mathbb{Z}_{\prod_{i=1}^{n} p_i} \mid |x| = p_i = \frac{\prod_{i=1}^{n} p_i}{gcd(x, \prod_{i=1}^{n} p_i)}\}$$

$$= \{x \in \mathbb{Z}_{\prod_{i=1}^n p_i} \mid x = \frac{\lceil M \rceil}{\lfloor n-1 \rfloor} l, \; 1 \le l \le p_i - 1\},$$

where p_i is a prime number that does not appear in the multiplication $\frac{\lceil M \rceil}{\lfloor n-1 \rfloor}$. Clearly, the vertex zero is joined to all vertices in S. Thus the $p_i - 1$ columns $\frac{\lceil M \rceil}{\lfloor n-1 \rfloor} l$, $1 \le l \le p_i - 1$ in the first row are equal to one. Therefore, the number of ones in the first row will be $\sum_{i=1}^n (p_i - 1)$. Since, the adjacency matrix of $Cay_p(\mathbb{Z}_{\prod_{i=1}^n p_i}, S)$ is circulant, we can write the next rows from the first one. For instance, the second row, corresponding to vertex one is obtained by shifting all entries of the first row, one unit to the right. Hence the columns $\frac{\lceil M \rceil}{\lfloor n-1 \rfloor} l + 1$, $1 \le l \le p_i - 1$ are equal to one. Continuing this way, the columns $\frac{\lceil M \rceil}{\lfloor n-1 \rfloor} l + j$, $1 \le j \le \prod_{i=1}^n p_i$, $1 \le l \le p_i - 1$ in the jth row are equal to one.
(ii) We know that $M = \{p_i^{\alpha_i} \mid 1 \le i \le n\}$. So the set S is as

$$S = \{x \in \mathbb{Z}_{\prod_{i=1}^n p_i} \mid |x| = p_i = \frac{\prod_{i=1}^n p_i^{\alpha_i}}{gcd(x, \prod_{i=1}^n p_i^{\alpha_i})}, \; 1 \le l \le p_k\}$$
$$= \{x \in \mathbb{Z}_{\prod_{i=1}^n p_i^{\alpha_i}} \mid x = \frac{\lceil M \rceil}{\lfloor n-1 \rfloor} p_k^{\alpha_k - 1} l, \; 1 \le l \le p_k - 1\},$$

where p_k is a prime number that does not appear in $\frac{\lceil M \rceil}{\lfloor n-1 \rfloor}$. It is clear that vertex zero is joined to all vertices in S. Thus the columns $\frac{\lceil M \rceil}{\lfloor n-1 \rfloor} p_k^{\alpha_k - 1} l + 1$, $1 \le l \le p_k - 1$, in the first row are equal to one. Since the adjacency matrix of $Cay_p(\mathbb{Z}_{\prod_{i=1}^n p_i^{\alpha_i}}, S)$ is circulant, then the jth row corresponding to vertex $j-1$ is obtained by shifting the entries in the first row j unit to the right. So the columns $\frac{\lceil M \rceil}{\lfloor n-1 \rfloor} p_k^{\alpha_k - 1} l + j$, $1 \le l \le p_k - 1$ in the jth row are equal to one. □

Now, consider $Cay_p(\mathbb{Z}_{30}, S)$. In order to obtain its adjacency matrix, it is enough to find out vertices of prime order which are adjacent to zero. By this process, we can present the first row of the adjacency matrix. By use of the proof of Theorem 6 the position of such vertices are $(1, \frac{\lceil M \rceil}{\lfloor 2 \rfloor} l + 1)$, where $M = \{2, 3, 5\}$ and $1 \le l \le p - 1$ for all $p \in M$. Therefore, the columns $\frac{\lceil M \rceil}{\lfloor 2 \rfloor} l + 1$ are 1 and the rest columns are 0, in the first row. Consequently,

Columns	1	..	7	..	11	..	19	..	21	..	25	..
First row	0	0...0	1	0...0	1	0...0	1	0...0	1	0...0	1	0...0

shows the first row.

Theorem 7 *The spectrum of the graph* $Cay_p(\mathbb{Z}_{p^s \times q^{s'}}, S)$ *is the set* $\{p + q - 2, q - 2, p - 2, -2\}$.

Proof. We remind that the graph $\mathrm{Cay}_p(\mathbb{Z}_{p^s \times q^{s'}}, S)$ is isomorphic to disjoint union of $p^{s-1}q^{s'-1}$ copies of $K_p \times K_q$. Moreover, the spectrum of a disconnected graph is the disjoint union of the spectra of its components. So we need to find the spectrum of one of its components. Each component has the form $K_p \times K_q$ and by Proposition 1 we have $\mathrm{spec}(A(K_p \times K_q)) = \{\lambda + \mu \,|\, \lambda \in \mathrm{spec}(K_p), \mu \in \mathrm{spec}(K_q)\}$. On the other hand, the spectrum of the graph K_p and K_q are $\lambda_1 = p-1, \lambda_2 = \dots = \lambda_p = -1$ and $\mu_1 = q-1, \mu_2 = \dots = \mu_q = -1$, respectively. Hence the spectrum of $A(K_p \times K_q)$ is $\lambda_1' = p+q-2$, $\lambda_2' = q-2$, $\lambda_3' = p-2$, $\lambda_4' = -2$ with multiplicity 1, $p-1$, $q-1$ and $(p-1)(q-1)$, respectively. □

Corollary 1 *The determinant of the adjacency matrix of* $\mathrm{Cay}_p(\mathbb{Z}_{p^s \times q^{s'}}, S)$ *is*

$$((p+q-2)(q-2)^{p-1}(p-2)^{q-1}(-2)^{(p-1)(q-1)})^{p^{s-1}q^{s'-1}}.$$

Proof. By the structure of $\mathrm{Cay}_p(\mathbb{Z}_{p^s \times q^{s'}}, S)$, we label the vertices, such that the adjacency matrix has the following form

$$A(\mathrm{Cay}_p(\mathbb{Z}_{p^s \times q^{s'}}, S)) = \begin{pmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_{p^{s-1}q^{s'-1}} \end{pmatrix}$$

where $B_i = A(K_p \times K_q)$, $1 \leq i \leq p^{s-1}q^{s'-1}$. By the Theorem 7 we obtain $\det(B_i) = \det(A(K_p \times K_q)) = (p+q-2)(q-2)^{p-1}(p-2)^{q-1}(-2)^{(p-1)(q-1)}$. Moreover, by Remark 1,

$$\det(A(\mathrm{Cay}_p(\mathbb{Z}_{p^s \times q^{s'}}, S))) = \prod_{i=1}^{p^{s-1}q^{s'-1}} \det(B_i)$$

$$= ((p+q-2)(q-2)^{p-1}(p-2)^{q-1}(-2)^{(p-1)(q-1)})^{p^{s-1}q^{s'-1}}.$$

□

4 The prime order Cayley graph of groups D_{2n} and Q_{4n}

In this section, we compute the adjacency matrices of the prime Cayley graphs $\mathrm{Cay}_p(D_{2n}, S)$ and $\mathrm{Cay}_p(Q_{4n}, S)$ for different values of n. Furthermore, by use of the adjacency matrix, we clarify the structure of $Cay_p(D_{2n}, S)$ and $Cay_p(Q_{4n}, S)$. Recall that $D_{2n} =< x, a \,|\, a^n = x^2 = e, xax^{-1} = a^{-1} >$ is the Dihedral group of order $2n$. In general, the adjacency matrix of $Cay_p(D_{2n}, S)$, when n is given, has the form

$$A(Cay_p(D_{2n}, S)) = \begin{pmatrix} B & J_n \\ J_n & B \end{pmatrix},$$

where J_n is an $n \times n$ matrix of all ones and the rows and the columns of B are indexed by the elements $\{e, a, a^2, ..., a^{n-1}\}$ of the group D_{2n}. According to the structure of $A(D_{2n})$, it is enough to characterize the matrix B. We know that B is a circulant matrix, so according to the definition of the circulant matrix, we need to determine the first row of B.

Proposition 2 *The structure of the block* B *of the adjacency matrix of* $Cay_p(D_{2n}, S)$*, when* n *takes different values is as follows:*

(i) If $n = 2^\alpha$*, then in the first row of the block matrix* B*, the entry in the* $(1, 2^{\alpha-1} + 1)$ *position is one and all the other entries are equal to zero, where* α *is a positive integer and* $\alpha > 1$*.*

(ii) If $n = p^\alpha$*, then in the first row of the block matrix* B*, all entries in the* $(1, kp^{\alpha-1} + 1)$ *positions,* $1 \leq k \leq p - 1$ *are equal to one and all the other entries are equal to zero, where* p *is a prime number* $p > 2$ *and* α *is a positive integer* $\alpha > 1$*.*

(iii) If $n = 2q$*, then in the first row of the block matrix* B*, all entries in the* $(1, 1 + q)$ *and* $(1, 2l + 1)$ *positions,* $1 \leq l \leq q - 1$ *are equal to one and all the other entries are equal to zero and we have* q *entries equal to one, where* q *is a prime number* $q \neq 2$*.*

(iv) If $n = 2q^\alpha$*, then in the first row of the block matrix* B*, all entries in the* $(1, 1 + q^\alpha)$ *and* $(1, 2q^{\alpha-1}l + 1)$ *positions,* $1 \leq l \leq q - 1$ *are equal to one and all the other entries are equal to zero and we have* q *entries equal to one, where* q *is a prime number* $q \neq 2$ *and* α *is a positive integer* $\alpha \neq 1$*.*

(v) If $n = \prod_{s=1}^{t} p_s$*, then in the first row of the block matrix* B*, all entries in the* $(1, l \prod_{s'=1, s \neq s'}^{t} p_{s'} + 1)$ *positions,* $1 \leq l \leq p_s - 1$ *are equal to one and all the other entries are equal to zero, where* p_s*'s are prime numbers and* $p_s \neq 2$*,* $1 \leq s \leq t$*.*

(vi) *If* $n = \prod_{s=1}^{t} p_s^{\alpha_s}$, *then in the first row of the block matrix* B, *all entries in the* $(1, lp_1^{\alpha_1}...p_{s-1}^{\alpha_{s-1}}p_s^{\alpha_s - 1}p_s^{\alpha_{s+1}}...p_t^{\alpha_t} + 1)$ *positions,* $1 \leq l \leq p_s - 1$ *are equal to one and all the other entries are equal to zero, where* p_s*'s are prime numbers (*$p_s \neq 2, 1 \leq s \leq t$*) and* α_s*'s are positive integers* $\alpha_s > 1$.

(vii) *If* $n = 2\prod_{s=1}^{t} p_s^{\alpha_s}$, *then in the first row of the block matrix* B, *all entries in the* $(1, \prod_{s=1}^{t} p_s^{\alpha_s} + 1)$ *and* $(1, l2p_1^{\alpha_1}...p_{s-1}^{\alpha_{s-1}}p_s^{\alpha_s - 1}p_s^{\alpha_{s+1}}...p_t^{\alpha_t} + 1)$ *positions,* $1 \leq l \leq p_s - 1$ *are equal to one and all the other entries are equal to zero, where* p_s*'s are distinct prime numbers* $p_s \neq 2, 1 \leq s \leq t$ *and* α_s*'s are positive integers* $\alpha_s > 1$.

Proof. (i) We determine the first row of B. For this, we must specify that e is joined to which powers of a. It is enough to find powers of a, namely i, such that a^i has prime order. We have $2 = |a^i| = \frac{2^\alpha}{gcd(2^\alpha, i)}$ and therefore $gcd(2^\alpha, i) = 2^{\alpha-1}$ which implies that $i = k2^{\alpha-1}$, $1 \leq k < 2$. Hence e is joined to $a^{2^{\alpha-1}}$, and the $(1, 2^{\alpha-1} + 1)$-entry of B is 1. The proof of (ii), (iii), (iv), (v) and (vi) are similar to the proof of part (i). So the assertion is clear. $\square$

Now by the above proposition, we can easily determine $A(Cay_p(D_{2n}, S))$ and compute its determinant and eigenvalues, where n is the multiplication of distinct prime numbers.

Theorem 8 *Let* $n = \prod_{s=1}^{t} p_s^{\alpha_s}$. *Then we have the following results for* $A(Cay_p(D_{2n}, S))$, *where* p_s*'s are distinct prime numbers (*$p_s \neq 2, 1 \leq s \leq t$*).*

(i) *In the first row of* $B + J_n$, *we have* 2 *where* e *is joined to* a^i *and all the other entries are equal to* 1, *where* $i = lp_1^{\alpha_1}...p_{s-1}^{\alpha_{s-1}}p_s^{\alpha_s - 1}p_s^{\alpha_{s+1}}...p_t^{\alpha_t}$, $1 \leq l \leq p_s - 1$.

(ii) *If* $[x_1, x_2, ..., x_n]$ *is the first row of* $A(Cay_p(D_{2n}, S))$, *then*

$$\lambda_j = x_1 + x_2\omega_j + x_3{\omega_j}^2 + ... + x_n{\omega_j}^{n-1},$$

where $\omega_j = e^{\frac{2\pi j}{n}}$, $j = 0, 1, ..., n - 1$. *Moreover,*

$$\begin{aligned}\det(A(Cay_p(D_{2n}, S))) &= \prod_{j=0}^{n-1}(x_1 + x_2\omega_j + x_3{\omega_j}^2 + ... + x_n{\omega_j}^{n-1}) \\ &= \prod_{j=0}^{n-1}(\sum_{a^k \not\sim e} \omega_j^k + \sum_{a^k \sim e} 2\omega_j^k).\end{aligned}$$

Proof. (i) By the argument before Proposition 2, $A(Cay_p(D_{2n}, S))$ has the form

$$A(Cay_p(D_{2n}, S)) = \begin{pmatrix} B & J_n \\ J_n & B \end{pmatrix}.$$

By the definitions of the prime order Cayley graph and D_{2n}, clearly a is joined to p_s elements of D_{2n} and the identity element e is joined to a^i, where $i = lp_1^{\alpha_1}...p_{s-1}^{\alpha_{s-1}} p_s^{\alpha_s - 1} p_s^{\alpha_{s+1}}...p_t^{\alpha_t}$, $1 \leq l \leq p_s - 1$.
(ii) By the Definition 2 and the discussion after it, the assertion is clear. □

In the following remark, we will classify $Cay_p(D_{2n}, S)$ vertices. This vertex classification is useful to obtain some facts about the structure of the graph such as the chromatic number.

Remark 2 *Consider* $Cay_p(D_{2n}, S)$.

(i) If $n = 2^\alpha$ $(\alpha > 1)$*, then the graph vertices is partitioned into the sets* $A_1 = \{a^i, a^{2^{\alpha-1}+i}\}$ *and* $A_2 = \{a^i b, a^{2^{\alpha-1}+i} b\}$*, where* $0 \leq i < \alpha$*. In each of these two sets, we have* $a^i \sim a^{2^{\alpha-1}+i}$ *and* $a^i b \sim a^{2^{\alpha-1}+i} b$*, for every* $0 \leq i < \alpha$*. Furthermore, each vertex of* A_1 *is adjacent to each vertex* A_2*.*

(ii) If $n = p^\alpha, p > 2, \alpha \geq 1$*, then the vertices is classified to the sets* $A_i = \{a^i, a^{i+p}, a^{i+2p}, ..., a^{i+(p-1)p}\}$ *and* $B_i = \{a^i b, a^{i+p} b, a^{i+2p} b, \ldots, a^{i+(p-1)p} b\}$*, where* $0 \leq i < p$*. All the vertices in each* A_i *are adjacent for* $0 \leq i < p$*. But there's no connection between any two vertices of* A_i *and* A_j*, where* $i \neq j$ $(0 \leq i, j < p)$*. Similarly, all the vertices in each* B_i *are adjacent for* $0 \leq i < p$*. Moreover, there's no connection between* B_i *and* B_j*, for* $i \neq j$*. Note that every vertex that belongs to* A_i *is adjacent to every vertex in* B_i*, for* $0 \leq i < p$*.*

(iii) If $n = 2q, q \neq 2$*, then the graph vertices is partitioned to* $A_1 = \{a^{2k} \mid 0 \leq k \leq q-1\}$*,* $A_2 = \{a^{2k'+1} \mid 0 \leq k' \leq q-1\}$ $A_3 = \{a^{2l} \mid 0 \leq l \leq q-1\}$*,* $A_4 = \{a^{2l'+1} \mid 0 \leq l' \leq q-1\}$*. The vertices belonging to* A_1 *form a clique* K_q *and similarly* A_2*. Moreover, there is some connection between* A_1 *and* A_2*. The vertices* a^i *and* a^{q+i} *are adjacent* $(0 \leq i \leq q-1)$*. All the vertices of* A_3 *form a clique* K_q *and similarly for* A_4*. The connection between these two sets is between the vertices* $a^i b$ *and* $a^{q+i} b$ $(0 \leq i \leq q-1)$*. Note that each vertex in one of the sets* A_1 *and* A_2 *is adjacent to each vertex in sets* A_3 *and* A_4*.*

(iv) If $n = 2q^\alpha, q > 2, \alpha > 1$*, then the graph vertices is partitioned to*

$$A_i = \{a^i, a^{i+2q}, a^{i+2(2q)}, ..., a^{i+(q-1)2q}\},$$

$$B_i = \{a^i b, a^{i+2q} b, a^{i+2(2q)} b, ..., a^{i+(q-1)2q} b\},$$

$0 \leq i \leq 2q - 1$. *For each i, all the vertices belonging to* A_i *are connected and, for* B_i*, we have the same. Furthermore, there's some connection between some of* A_i*'s. The vertices* a^t *and* a^{t+q^α} *are adjacent for* $0 \leq t \leq q^\alpha - 1$. *Similarly, there are some edges between some of the sets* B_i. *The vertices* $a^{t'} b$ *and* $a^{t'+q^\alpha} b$ *are adjacent for each* $0 \leq t' \leq q^\alpha - 1$.

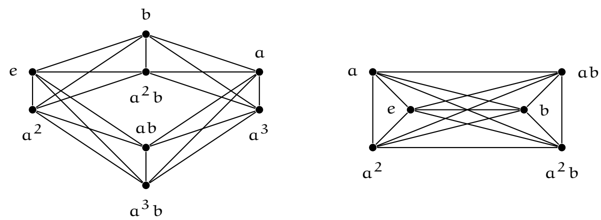

Figure 4: $Cay_p(D_8, S)$ and $Cay_p(D_6, S)$

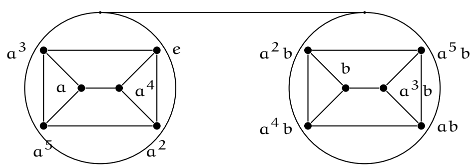

Figure 5: $Cay_p(D_{12}, S)$

The line between two circles represents the edges connect all the vertices in each component.

Note that in Figure 5, the line between two circles means that all the vertices in the first circle are adjacent to all the vertices in the second. We create this display for simplicity. In the next theorem, we compute the chromatic number of $Cay_p(D_{2n}, S)$ by using the Remark 2 and its notations.

Theorem 9 *(i) The chromatic number of graph* $Cay_p(D_{2n}, S)$ *is* 4*, where* $n = 2^\alpha, \alpha > 1$.

(ii) The chromatic number of $Cay_p(D_{2n}, S)$ *is* $2p$*, where* $n = p^\alpha, p > 1$, $\alpha > 1$.

(iii) *The chromatic number of* $\mathsf{Cay}_p(D_{2n}, S)$ *is* $2q$, *where* $n = 2q, q \neq 2$.

(iv) *The chromatic number of* $\mathsf{Cay}_p(D_{2n}, S)$ *is* $2q$, *where* $n = 2q^{\alpha}, q \neq 2$, $\alpha \neq 1$.

Proof. (i) By Remark 2 and using its notations, we color the vertices of A_1 first. For each i, we color a^i and $a^{2^{\alpha-1}+i}$ with two distinct colors, say l_1 and l_2, respectively. Since $a^{2^{\alpha-1}+i} \not\sim a^j$, for $i \neq j$ $(0 \leq i, j < \alpha)$, the vertices a^j and $a^{2^{\alpha-1}+j}$ which $i \neq j$ can be colored with l_1 and l_2, respectively. Similarly, we can color the vertices of A_2 with colors l_3 and l_4. Hence $\chi(\mathsf{Cay}_p(D_{2n}, S)) = 4$, where $n = 2^{\alpha}$ $(\alpha > 1)$.
(ii) For coloring the vertices of an arbitraryA_i $(0 \leq i < p)$, we need p colors. We can color the sets A_j which $i \neq j$ with these p colors. We have the same for B_i's. Since each vertex of A_i's is adjacent to each vertex of B_i's, we need $2p$ colors for coloring the graph.
(iii) We color the vertices of A_1 with q colors namely $l_1, ..., l_q$. For each k $(0 \leq k \leq q-1)$, $a^{2k} \sim a^{2k+q}$. So if a^{2k} has been colored with l_t $(1 \leq t \leq q)$, then we can color $a^{2k+1}$of the set A_2 with l_t. All the vertices in A_1 and A_2 are adjacent to each vertex in A_3 and A_4. So we need q new colors for coloring A_3 and A_4. We color A_3 and A_4 in the same way. Therefore, $\chi(\mathsf{Cay}_p(D_{2n}, S)) = 2q$, where $n = 2q$ $(q \neq 2)$.
(iv) The proof of this part is similar to the proof of part (iii). □

Now we determine the adjacency matrix of the prime order Cayley graph associated with Q_{4n}, for different values of n. The generalized quaternion group of order $4n$ is defined as $Q_{4n} =< a, b \,|\, a^{2n} = 1,\ a^n = b^2,\ bab^{-1} = a^{-1} >$. In general, the adjacency matrix of $\mathsf{Cay}_p(Q_{4n}, S)$, when n is given, has the following form

$$A(\mathsf{Cay}_p(Q_{4n}, S)) = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix},$$

where the columns of B are indexed by the elements $\{e, a, a^2, ..., a^{2n-1}\}$ of the group Q_{4n}. So it is enough to determine matrix B. Since B is a circulant matrix, we only need to specify the first row of B.

Proposition 3 *The structure of the block* B *of the adjacency matrix of* $\mathsf{Cay}_p(Q_{4n}, S)$ *is as follows:*

(i) *In block* B *of* $A(\mathsf{Cay}_p(Q_{4n}, S))$, *the entry in the* $(1, 2^{\alpha} + 1)$ *position is equal to one and all the other entries are equal to zero, where* $n = 2^{\alpha}$, *where* $\alpha > 1$.

(ii) *In block* B *of* $A(Cay_p(Q_{4n}, S))$, *the entries in the* $(1, p^\alpha + 1)$ *and* $(1, 2lp^{\alpha-1}+1)$ *positions are equal to one and all the other entries are equal to zero, where* $n = p^\alpha$, *where* $\alpha > 1, p > 2, 1 \leq l \leq p-1$.

(iii) *In block* B *of* $A(Cay_p(Q_{4n}, S))$, *the entries in the* $(1, 4l+1)$ *and* $(1, 2q+1)$ *positions are equal to one and all the other entries are equal to zero, where* $n = 2q$, *where* $q \neq 2$, $1 \leq l \leq q-1$.

(iv) *In block* B *of* $A(Cay_p(Q_{4n}, S))$, *the entries in the* $(1, 2q^\alpha+1)$ *and* $(1, 2q^{\alpha-1} l + 1)$ *positions are equal to one and all the other entries are equal to zero, where* $n = 2q^\alpha$, $\alpha > 1, q \neq 2, 1 \leq l \leq q-1$.

(v) *In block* B *of* $A(Cay_p(Q_{4n}, S))$, *the entries in the* $(1, n+1)$ *and* $(1, 2\prod_{s=1}^{t} p_s l + 1)$ *positions are equal to one and all the other entries are equal to zero, where* $n = \prod_{s=1}^{t} p_s$, *where* $p_s \neq 2, 1 \leq l \leq p_{s'} - 1)$.

Proof. For determining the first row of B, we must specify the powers of a which e is joined to them. For case (i), we have $2 = \mid a^i \mid = \frac{2^{\alpha+1}}{gcd(2^{\alpha+1}, i)}$, therefore $gcd(2^{\alpha+1}, i) = 2^\alpha$ and $i = 2^\alpha k$ $(1 \leq k < 2)$. So e is joined to a^{2^α} and the $(1, a^{2^\alpha} + 1)$-entry of B is 1. For the other cases, the proof is similar to case (i). □

In the next theorem, we present the appearance of the graph $Cay_p(Q_{4n}, S)$ for different n's.

Theorem 10 *(i) The graph* $Cay_p(Q_{4n}, S)$ *is the disjoint union of* $2^{\alpha+1}$ *copies of the complete graphs* K_2, *where* $n = 2^\alpha$ $(\alpha > 1)$.

(ii) *The graph* $Cay_p(Q_{4n}, S)$ *is the disjoint union of* $2p^{\alpha-1}$ *copies of the Cartesian product of* K_2 *and* K_p, *where* $n = p^\alpha$ $(\alpha > 1,\ p > 2)$.

(iii) *The graph* $Cay_p(Q_{4n}, S)$ *is the disjoint union of* $\frac{4n}{2q}$ *copies of the Cartesian product of* K_2 *and* K_q, *where* $n = 2q^\alpha$ $(\alpha \geq 1\ q \neq 2)$.

Proof. (i) We know $S = \{a^{2^\alpha}\}$ and for each i $(0 \leq i \leq 2^\alpha - 1)$, $a^i \sim a^{2^\alpha + i}$ and $a^i b \sim a^{2^\alpha + i} b$. Since there is not any other connection between the vertices, we have $2^{\alpha+1}$ complete graphs K_2.
(ii) Note that $S = \{a^{p^\alpha}, a^{2lp^{\alpha-1}}, 1 \leq l \leq p-1\}$ and for each $i\,(0 \leq i \leq p^{\alpha-1} - 1)$, $a^i \sim a^{p^\alpha + i}$, $a^i \sim a^{2lp^{\alpha-1}+i}$, $a^i b \sim a^{p^\alpha + i} b$ and $a^i b \sim a^{2lp^{\alpha-1}+i} b$, $(1 \leq l \leq p-1)$. The vertices a^i and $a^{2lp^{\alpha-1}+i}, 1 \leq l \leq p-1$ form a complete graph K_p. Furthermore, each K_p on the vertices a^i and $a^{2lp^{\alpha-1}+i}$ is connected to another K_p that is formed by the vertices $a^{i+p^{\alpha-1}}$ and $a^{i+(2l'+1)p^{\alpha-1}}$ $(0 \leq$

$i \leq p^{\alpha-1} - 1$, $1 \leq l, l' \leq p-1$). More precisely for $1 \leq l \leq [\frac{p}{2}]$, $a^i \sim a^{p^\alpha+i}$ and $a^{2lp^{\alpha-1}+i} \sim a^{2lp^{\alpha-1}+i+p^\alpha}$ and for $1+[\frac{p}{2}] \leq l \leq p-1$, $a^{2lp^{\alpha-1}+i} \sim a^{2lp^{\alpha-1}+i-p^\alpha}$. These vertices form the Cartesian product of two graphs K_2 and K_p. We have the same argument for the vertices $a^i b$ and $a^{2lp^{\alpha-1}+i}b, 1 \leq l \leq p-1$. All these vertices form $p^{\alpha-1}$ copies of $K_2 \times K_p$. Hence we have $2p^{\alpha-1}$ copies of $K_2 \times K_p$.
(iii) It is similar to proof of the case (ii). □

By Theorem 10, the structure of $Cay_p(Q_{4n}, S)$ and the argument above the Theorem 2, we have the following result.

Corollary 2 *(i)* $\chi(Cay_p(Q_{4n}, S)) = 2$, *where* $n = 2^\alpha$ $(\alpha > 1)$.

(ii) $\chi(Cay_p(Q_{4n}, S)) = p^{\alpha-1}$, *where* $n = p^\alpha$ $(p > 2, \alpha > 1)$.

(iii) $\chi(Cay_p(Q_{4n}, S)) = q$, *where* $n = 2q^\alpha$ $(q \neq 2, \alpha \geq 1)$.

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