Acta Univ. Sapientiae, Mathematica, **15**, 2 (2023) 244–258

DOI: 10.2478/ausm-2023-0013

Norm and almost everywhere convergence of matrix transform means of Walsh-Fourier series

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Abstract. We show the uniformly boundedness of the L_1 norm of general matrix transform kernel functions with respect to the Walsh-Paley system. Special such matrix means are the well-known Cesàro, Riesz, Bohner-Riesz means. Under some conditions, we verify that the kernels $K_n^T = \sum_{k=1}^{n} t_{k,n} D_k$, (where D_k is the kth Dirichlet kernel) satisfy

$$\left\|K_n^T\right\|_1 \le c.$$

As a result of this we prove that for any $1 \le p < \infty$ and $f \in L_p$ the L_p-norm convergence $\sum_{k=1}^{n} t_{k,n} S_k(f) \to f$ holds. Besides, for each integrable function f we have that these means converge to f almost everywhere.

1 Definitions and notations

We follow the standard notions of dyadic analysis introduced by F. Schipp, W. R. Wade, P. Simon, and J. Pál [18] and others.

2010 Mathematics Subject Classification: 42C10
Key words and phrases: character system, Fourierseries, Walsh-Paley system, kernel function, matrix transform, almost everywhere convergence

Let $\mathbb{P}$ be the set of positive natural numbers and $\mathbb{N} := \mathbb{P} \cup \{0\}$. Let denote by $\mathbb{Z}_2$ the discrete cyclic group of order 2, the group operation is the modulo 2 addition. Let be every subset open. The normalized Haar measure μ on $\mathbb{Z}_2$ is given in the way that $\mu(\{0\}) = \mu(\{1\}) = 1/2$. $G := \mathop{\times}\limits_{k=0}^{\infty} \mathbb{Z}_2$, G is called the Walsh group. The elements of Walsh group G are sequences of numbers 0 and 1, that is $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0, 1\}$ $(k \in \mathbb{N})$.

The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure μ is the product measure and the topology is the product topology. Dyadic intervals are defined in the usual way

$$I_0(x) := G, \; I_n(x) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\}$$

for $x \in G, n \in \mathbb{P}$. They form a base for the neighbourhoods of G. Let $0 := (0 : i \in \mathbb{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \mathbb{N}$.

Let $L_p(G)$ denote the usual Lebesgue spaces on G (with the corresponding norm $\|.\|_p$), where $1 \le p < \infty$.

We introduce some concepts of Walsh-Fourier analysis. The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \; (x \in G, k \in \mathbb{N}).$$

The Walsh-Paley functions are the product functions of the Rademacher functions. Namely, each natural number n can be uniquely expressed in the number system based 2, in the form

$$n = \sum_{k=0}^{\infty} n_k 2^k, \; n_k \in \{0, 1\} \; (k \in \mathbb{N}),$$

where only a finite number of n_k's different from zero. Let the order of $n > 0$ be denoted by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$. Walsh-Paley functions are $w_0 := 1$ and for $n \ge 1$

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = (-1)^{\sum_{k=0}^{|n|} n_k x_k}.$$

Let $\mathcal{P}_n$ be the collection of Walsh polynomials of order less than n, that is, functions of the form

$$P(x) = \sum_{k=0}^{n-1} a_k w_k(x),$$

where $n \in \mathbb{P}$ and $\{a_k\}$ is a sequence of complex numbers.

It is known [11, 18] that the system $(w_n, n \in \mathbb{N})$ is a character system of $(G, +)$. The nth Fourier-coefficient, the nth partial sum of the Fourier series and the nth Dirichlet kernel is defined by

$$\hat{f}(n) := \int_G f w_n \, d\mu, \; S_n(f) := \sum_{k=0}^{n-1} \hat{f}(k) w_k, \; \sigma_n(f) := \frac{1}{n} \sum_{k=1}^{n} S_k(f),$$

$$D_n := \sum_{k=0}^{n-1} w_k, \; D_0 := 0.$$

Fejér kernels are defined as the arithmetical means of Dirichlet kernels, that is,

$$K_n := \frac{1}{n} \sum_{k=1}^{n} D_k.$$

Let $T := (t_{i,j})_{i,j=1}^{\infty}$ be a doubly infinite matrix of numbers. It is always supposed that matrix T is triangular. Let us define the nth matrix transform mean (or linear mean) determined by the matrix T

$$\sigma_n^T(f) := \sum_{k=1}^{n} t_{k,n} S_k(f),$$

where $\{t_{k,n} : 1 \le k \le n, \; k \in \mathbb{P}\}$ be a finite sequence of non-negative numbers for each $n \in \mathbb{P}$.

It is a common generalisation of the partial sum of Fourier series (see Remark 2) and several well-known (e.g. Fejér, Cesàro, Nörlund, weighted, Riesz) means.

The approximation properties of these means with respect to Walsh system was studied by several mathematicians. For example, including but not limited to (in alphabetic order) Baramidze [1], Blahota [5], Gát [8], Goginava [9], Fridli [7], Jastrebova [12], Oniani [10], Manchanda [7], Marcinkiewicz [13], Memić [14], Móricz [15], Nagy [16], Paley [17], Pál [18], Persson [1], Rhoades [15], Schipp [18], Siddiqi [7], Simon [18], Singh [1], Skvortsov [19], Tephnadze [1], Toledo [5], Wade [18], Weisz [23], Yano [21], Walsh [22], Zygmund [13] and several others. (Mentioned only one paper or book per authors.)

The nth matrix transform kernel (T kernel) is defined by

$$K_n^T := \sum_{k=1}^{n} t_{k,n} D_k.$$

It is easily seen that

$$\sigma_n^T(f;x) = \int_G f(u)K_n^T(u+x)d\mu(u).$$

This equality (and its analogous versions for special means) shows us the necessity of observing kernel functions.

We introduce the notation $\Delta t_{k,n} := t_{k,n} - t_{k+1,n}$, where $k \in \{1,\dots,n\}$ and $t_{n+1,n} := 0$.

For some other results on matrix transform means see e.g. [2], [3] and [4].

In the sequel, we summarise the main results of this paper. Let $\{t_{k,n} : 1 \le k \le n\}$ be a finite and monotone (not necessarily in the same sense for different n's) sequence of non-negative numbers such that $\sum_{k=1}^{n} t_{k,n} = 1$. Then we prove the almost everywhere convergence $\sigma_n^T(f) \to f$ for each integrable f and also the norm convergence $\sigma_n^T(f) \to f$ for any $f \in L_p(G)$ for each $1 \le p < \infty$ and also for continuous functions (that is, for functions in $C(G)$).

2 Auxiliary results

To prove Theorem 1 we need the following Lemmas.

Lemma 1 (Paley's Lemma [18], p. 7.) *For* $n \in \mathbb{N}$

$$D_{2^n}(x) = \begin{cases} 2^n, & \textit{if } x \in I_n, \\ 0, & \textit{if } x \notin I_n. \end{cases}$$

Lemma 2 ([18], p. 34.) *For* $k, n \in \mathbb{N},\ k < 2^n$

$$D_{2^n+k} = D_{2^n} + r_n D_k.$$

Lemma 3 (Yano's Lemma [21]) *For* $n \in \mathbb{N}$

$$\|K_n\|_1 \le 2.$$

In 2018 Toledo [20] improved this result, but for our proof the knowledge of the exact supremum of $\|K_n\|_1$ is not necessary, just its boundedness.

In the next lemma, we give a decomposition of the kernels K_n^T.

Lemma 4 *Let* $n \in \mathbb{P}$. *Then we have*

$$K_n^T = \sum_{j=0}^{|n|-1} \sum_{k=0}^{2^j-1} t_{2^j+k,n} D_{2^j}$$

$$\begin{aligned}
&+\sum_{j=0}^{|n|-1} r_j \sum_{k=1}^{2^j-2} \Delta t_{2^j+k,n} k K_k + \sum_{j=0}^{|n|-1} r_j t_{2^{j+1}-1,n}(2^j-1)K_{2^j-1} \\
&+\sum_{k=0}^{n-2^{|n|}} t_{2^{|n|}+k,n} D_{2^{|n|}} + r_{|n|} \sum_{k=1}^{n-2^{|n|}} \Delta t_{2^{|n|}+k,n} k K_k \\
=: &\sum_{j=1}^{5} K_{j,n}.
\end{aligned}$$

Proof. We write

$$K_n^T = \sum_{j=0}^{|n|-1} \sum_{l=2^j}^{2^{j+1}-1} t_{l,n} D_l + \sum_{l=2^{|n|}}^{n} t_{l,n} D_l =: K_n^A + K_n^B.$$

Now, we apply Lemma 2 for the expressions K_n^A and K_n^B. We get

$$\begin{aligned}
K_n^A &= \sum_{j=0}^{|n|-1} \sum_{k=0}^{2^j-1} t_{2^j+k,n} D_{2^j+k} \\
&= \sum_{j=0}^{|n|-1} \sum_{k=0}^{2^j-1} t_{2^j+k,n} D_{2^j} + \sum_{j=0}^{|n|-1} r_j \sum_{k=1}^{2^j-1} t_{2^j+k,n} D_k.
\end{aligned}$$

Using Abel-transform

$$\sum_{k=1}^{2^j-1} t_{2^j+k,n} D_k = \sum_{k=1}^{2^j-2} \Delta t_{2^j+k,n} k K_k + t_{2^{j+1}-1,n}(2^j-1)K_{2^j-1}.$$

Similarly,

$$\begin{aligned}
K_n^B &= \sum_{k=0}^{n-2^{|n|}} t_{2^{|n|}+k,n} D_{2^{|n|}+k} \\
&= \sum_{k=0}^{n-2^{|n|}} t_{2^{|n|}+k,n} D_{2^{|n|}} + r_{|n|} \sum_{k=1}^{n-2^{|n|}} t_{2^{|n|}+k,n} D_k.
\end{aligned}$$

Using $t_{n+1,n} = 0$ and Abel-transform again we obtain

$$\sum_{k=1}^{n-2^{|n|}} t_{2^{|n|}+k,n} D_k = \sum_{k=1}^{n-2^{|n|}-1} \Delta t_{2^{|n|}+k,n} k K_k + t_{n,n}(n-2^{|n|})K_{n-2^{|n|}}$$

$$= \sum_{k=1}^{n-2^{|n|}} \Delta t_{2^{|n|}+k,n} k K_k.$$

It completes the proof of Lemma 4. □

3 Boundedness of the L_1 norm of matrix transform kernel

Theorem 1 *For every $n \in \mathbb{P}$, $\{t_{k,n} : 1 \leq k \leq n\}$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=1}^{n} t_{k,n} = O(1) \tag{1}$$

is satisfied.

a) *If the finite sequence $\{t_{k,n} : 1 \leq k \leq n\}$ is non-decreasing as a function of k for all fixed n and the condition*

$$t_{n,n} = O\left(\frac{1}{n}\right) \tag{2}$$

is satisfied, or

b) *if the finite sequence $\{t_{k,n} : 1 \leq k \leq n\}$ is non-increasing as a function of k for all fixed n,*

then the L_1-norm of the T kernel is bounded uniformly. Namely,

$$\left\| K_n^T \right\|_1 \leq c$$

holds.

Proof. During our proofs c denotes a positive constant, which may vary at different appearances.

We use Lemma 4

$$\begin{aligned} \left\| K_n^T \right\|_1 &= \int_G \left| K_n^T(x) \right| d\mu(x) \\ &\leq \sum_{j=1}^{5} \int_G |K_{j,n}(x)| d\mu(x) \end{aligned}$$

$$=: \sum_{j=1}^{5} I_{j,n}.$$

Using Lemma 1 for the expressions $I_{1,n}$ and $I_{4,n}$, we obtain

$$\begin{aligned} I_{1,n} &\le \sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-1} t_{2^j+k,n}\int_G |D_{2^j}(x)|d\mu(x) \\ &= \sum_{j=0}^{|n|-1}\sum_{k=0}^{2^j-1} t_{2^j+k,n} \\ &= \sum_{k=1}^{2^{|n|}-1} t_{k,n} \le c \end{aligned}$$

and also using Condition (1)

$$\begin{aligned} I_{4,n} &\le \sum_{k=2^{|n|}}^{n} t_{k,n}\int_G |D_{2^{|n|}}(x)|d\mu(x) \\ &= \sum_{k=2^{|n|}}^{n} t_{k,n} \le c. \end{aligned}$$

Applying Lemma 3 we get

$$\begin{aligned} I_{2,n} &\le \sum_{j=0}^{|n|-1}\sum_{k=1}^{2^j-2} |\Delta t_{2^j+k,n}|\, k\int_G |r_j(x)K_k(x)|d\mu(x) \\ &= \sum_{j=0}^{|n|-1}\sum_{k=1}^{2^j-2} |\Delta t_{2^j+k,n}|\, k\|K_k\|_1 \\ &\le 2\sum_{j=0}^{|n|-1}\sum_{k=1}^{2^j-2} |\Delta t_{2^j+k,n}|\, k. \end{aligned}$$

We write in case a)

$$\sum_{k=1}^{2^j-2} |\Delta t_{2^j+k,n}| k = \sum_{k=1}^{2^j-2} (t_{2^j+k+1,n} - t_{2^j+k,n})k$$

$$
\begin{aligned}
&= (2^j - 2)t_{2^{j+1}-1,n} - \sum_{k=1}^{2^j-2} t_{2^j+k,n} \\
&\leq 2^j t_{2^{j+1}-1,n}
\end{aligned}
$$

and

$$
\begin{aligned}
I_{2,n} &\leq 2\sum_{j=0}^{|n|-1} 2^j t_{2^{j+1}-1,n} \\
&\leq 2n t_{n,n} \leq c.
\end{aligned}
$$

We have in case b)

$$
\begin{aligned}
\sum_{k=1}^{2^j-2} |\Delta t_{2^j+k,n}|\, k &= \sum_{k=1}^{2^j-2} t_{2^j+k,n} - (2^j-2)t_{2^{j+1}-1,n} \\
&\leq \sum_{k=1}^{2^j-2} t_{2^j+k,n}
\end{aligned}
$$

and

$$
\begin{aligned}
I_{2,n} &\leq 2\sum_{j=0}^{|n|-1} \sum_{k=1}^{2^j-2} t_{2^j+k,n} \\
&\leq 2\sum_{k=1}^{2^{|n|}-2} t_{k,n} \leq c.
\end{aligned}
$$

We estimate the expression $I_{3,n}$. Lemma 3 yields

$$
\begin{aligned}
I_{3,n} &\leq \sum_{j=0}^{|n|-1} 2^j t_{2^{j+1}-1,n} \int_G |r_j(x) K_{2^j-1}(x)| d\mu(x) \\
&= \sum_{j=0}^{|n|-1} 2^j t_{2^{j+1}-1,n} \|K_{2^j-1}\|_1 \\
&\leq 2\sum_{j=0}^{|n|-1} 2^j t_{2^{j+1}-1,n}.
\end{aligned}
$$

So, in case a) we have got the same inequality, as in $I_{2,n}$ case a)

$$I_{3,n} \leq 2\sum_{j=0}^{|n|-1} 2^j t_{2^{j+1}-1,n} \leq c.$$

We have in case b)

$$\begin{aligned} I_{3,n} &\leq 2\sum_{j=0}^{|n|-1} 2^j t_{2^{j+1}-1,n} \\ &\leq 2\sum_{k=1}^{2^{|n|}-1} t_{k,n} \leq c. \end{aligned}$$

Now, we estimate the expression $I_{5,n}$. Lemma 3 yields again

$$\begin{aligned} I_{5,n} &\leq \sum_{k=1}^{n-2^{|n|}} \left|\Delta t_{2^{|n|}+k,n}\right| k \int_G |r_{|n|}(x) K_k(x)| d\mu(x) \\ &= \sum_{k=1}^{n-2^{|n|}} \left|\Delta t_{2^{|n|}+k,n}\right| k \|K_k\|_1 \\ &\leq 2\sum_{k=1}^{n-2^{|n|}} \left|\Delta t_{2^{|n|}+k,n}\right| k. \end{aligned}$$

We get in case a)

$$\begin{aligned} \sum_{k=1}^{n-2^{|n|}} \left|\Delta t_{2^{|n|}+k,n}\right| k &= \sum_{k=1}^{n-2^{|n|}-1} (t_{2^{|n|}+k+1,n} - t_{2^{|n|}+k,n})k + t_{n,n}(n-2^{|n|}) \\ &= t_{n,n}(n-2^{|n|}-1) - \sum_{k=1}^{n-2^{|n|}-1} t_{2^{|n|}+k,n} + t_{n,n}(n-2^{|n|}) \\ &\leq 2n t_{n,n} \end{aligned}$$

and using Condition (2)

$$I_{5,n} \leq 4n t_{n,n} \leq c.$$

We have in case b)

$$\sum_{k=1}^{n-2^{|n|}} \left|\Delta t_{2^{|n|}+k,n}\right| k = \sum_{k=1}^{n-2^{|n|}} t_{2^{|n|}+k,n}$$

$$= \sum_{k=2^{|n|}+1}^{n} t_{k,n}$$

and using Condition (1)

$$I_{5,n} \leq 2 \sum_{k=2^{|n|}}^{n} t_{k,n} \leq c.$$

This completes the proof of our Theorem 1. □

Corollary 1 *Suppose that* $f \in L_p(G)$ *for some* $1 \leq p < \infty$ *and for every* $n \in \mathbb{P}$, $\{t_{k,n} : 1 \leq k \leq n\}$ *be a finite sequence of non-negative numbers such that*

$$\sum_{k=1}^{n} t_{k,n} = 1$$

is satisfied.

a) *If the finite sequence* $\{t_{k,n} : 1 \leq k \leq n\}$ *is non-decreasing as a function of* k *for all fixed* n *and the condition*

$$t_{n,n} = O\left(\frac{1}{n}\right)$$

is satisfied, or

b) *if the finite sequence* $\{t_{k,n} : 1 \leq k \leq n\}$ *is non-increasing as a function of* k *for all fixed* n *and we also have*

$$\lim_{n\to\infty} t_{k,n} = 0 \tag{3}$$

for any fixed k,

then we have the L_p*-norm convergence* $\sigma_n^T(f) \to f$.

Proof. We remark, that in the non-decreasing situation (case a) Condition (3) trivially holds. The proof is a straightforward consequence of Theorem 1 and the usual density argument. That is, the set of Walsh polynomials is dense in $L_p(G)$ for each $1 \leq p < \infty$. Besides, for any Walsh polynomial P we have $S_n(P) = P$ for sufficiently large n. Say for $n \geq k_0$. Then, using Condition (3) we have $\sigma_n^T(P) = \sum_{k=1}^{k_0-1} t_{k,n} S_k(P) + \sum_{k=k_0}^{n} t_{k,n} P \to P$ in norm and also everywhere. □

4 Almost everywhere convergence

Norm convergence results for Walsh-Paley and more general systems on matrix transform means are known, see e.g. [2], [3] and [4]. These results even give the rate of approximation.

Now let us speak about almost everywhere convergence.

Theorem 2 *Let* $f \in L_1(G)$*. Let the members of the finite sequences* $\{t_{k,n} : 0 \le k \le n\}$ *be non-negative numbers. Let* $\{t_{k,n} : 1 \le k \le n\}$ *be a finite and monotone (not necessarily in the same sense for different* n*'s) sequence of non-negative numbers such that*

$$\lim_{n\to\infty} t_{k,n} = 0 \tag{4}$$

for any fixed k*. Besides,*

$$\sum_{k=1}^{n} t_{k,n} = 1 \tag{5}$$

and

$$t_{n,n} = O\left(\frac{1}{n}\right). \tag{6}$$

If $n \to \infty$*, then*

$$\sigma_n^T(f) \to f$$

almost everywhere.

Proof. Using Abel transformation

$$\sum_{k=1}^{n} t_{k,n} S_k(f) = \sum_{k=1}^{n-1} \Delta t_{k,n} k \sigma_k(f) + t_{n,n} n \sigma_n(f).$$

Since

$$\sum_{k=1}^{n-1} \Delta t_{k,n} k \quad = \quad \sum_{k=1}^{n} t_{k,n} - n t_{n,n},$$

from monotonicity, using (5) and (6) we get

$$\sum_{k=1}^{n-1} |\Delta t_{k,n} k| \quad \le \quad \sum_{k=1}^{n} t_{k,n} + n t_{n,n}$$

$$\leq \quad 1 + c \leq c.$$

So, using (6) again

$$\begin{aligned} \left| \sum_{k=1}^{n} t_{k,n} S_k(f) \right| &\leq \sum_{k=1}^{n-1} |\Delta t_{k,n} k| \sup_{k \in \{1,\dots,n-1\}} |\sigma_k(f)| + t_{n,n} n |\sigma_n(f)| \\ &\leq c \sup_{k \in \{1,\dots,n-1\}} |\sigma_k(f)| + c|\sigma_n(f)| \\ &\leq c \sup_{k \in \{1,\dots,n\}} |\sigma_k(f)| \\ &\leq c \sup_{k \in \mathbb{P}} |\sigma_k(f)| \\ &= c\sigma^*(f), \end{aligned}$$

where

$$\sigma^*(f) := \sup_{k \in \mathbb{P}} |\sigma_k(f)|.$$

This inequality implies

$$\sup_{n \in \mathbb{P}} \left| \sum_{k=1}^{n} t_{k,n} S_k(f) \right| \quad \leq \quad c\sigma^*(f). \tag{7}$$

It is known, that operator $f \to \sigma^*(f)$ is type of weak $(1,1)$, therefore (7) inequality yields, that operator

$$f \to \sup_{n \in \mathbb{P}} \left| \sum_{k=1}^{n} t_{k,n} S_k(f) \right|$$

is also type of weak $(1,1)$.

From this we obtain with the standard technique (using Condition (4))

$$\lim_{n \to \infty} \sum_{k=1}^{n} t_{k,n} S_k(f) = f$$

convergence almost everywhere. □

5 Remarks

Remark 1 *We mention, that assuming* (1) *is natural, because many well-known means satisfy* (5)*, namely*

$$\sum_{k=1}^{n} t_{k,n} = 1$$

equality, and it is a part of regularity conditions [24, page 74.].

Remark 2 *In case a) in Theorem 1, if we omit Condition* (2)*, then* $\|K_n^T\| \le c$ *is not true in every cases. For example,*

$$t_{k,n} := \begin{cases} 0, & \textit{if } 1 \le k < n, \\ 1, & \textit{if } k = n. \end{cases}$$

Then $\sigma_n^T(f) = S_n(f)$*, so* $\|K_n^T\|_1 = \|D_n\|_1 = L_n$ *(the* n*-th Lebesgue constant), and Fine ([6], p. 387-388) proved, that the average order of* L_n *is* $\frac{1}{4}\log_2 n$*, which tends to infinity, as* $n \to \infty$*, hence in this case* $\limsup_{n\to\infty} \|K_n^T\|_1 = \infty$*.*

Remark 3 *The proof of Theorem 2 is very general. We used only the following properties of the system:*

- *week* $(1,1)$ *type of the operator* σ^**,*
- *density of polynomials of the observed system in* L_1*.*

Remark 4 *For the proof of Theorem 2 we supposed equality* (5)*. It would have been enough to use* (1) *for the proof of property week type* $(1,1)$*, but* (4) *needs* (5) *of course.*

Remark 5 *In Corollary 1 (in case b)) and Theorem 2, if we omit Condition* (3) *and* (4)*, then convergence is not true in every cases. For example, if*

$$t_{k,n} := \begin{cases} 1, & \textit{if } k = 1, \\ 0 & \textit{otherwise}, \end{cases}$$

then $\sigma_n^T(f) = S_1(f)$ *and in general* $S_1(f) \ne f$*.*

Acknowledgment

The authors would like to thank the anonymous reviewer for his/her help and work.

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Received: June 22, 2022