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On the spectral radius of D_α-matrix of a connected graph

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Abstract. In this paper, we further study the convex combinations $D_\alpha(G)$ of $Tr(G)$ and $D(G)$, defined as $D_\alpha(G) = \alpha Tr(G)+(1-\alpha)D(G)$, $0 \leq \alpha \leq 1$, where $D(G)$ and $Tr(G)$ denote the distance matrix and diagonal matrix of the vertex transmissions of a simple connected graph G, respectively. We obtain some upper and lower bounds for the spectral radius of the generalized distance matrix, in terms of various graph parameters and characterize the extremal graphs. We also obtain a lower bound for the generalized distance spectral radius of a graph with given edge connectivity, in terms of the order n, the edge connectivity r and the parameter α. Further, we obtain a lower bound for the generalized distance spectral radius of a tree, in terms of the order n, the diameter d and the parameter α. We characterize the extremal graphs for some values of diameter d.

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1 Introduction

In this paper, we consider only connected, undirected, simple and finite graphs. A graph is denoted by $G = (V(G), E(G))$, where $V(G) = \{v_1, v_2, \dots, v_n\}$ is its vertex set and $E(G)$ is its edge set. The *order* of G is the number $n = |V(G)|$ and its *size* is the number $m = |E(G)|$. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, refers to the *neighborhood* of v. The *degree* of v, denoted by $d_G(v)$ (we simply write d_v if it is clear from the context) means the cardinality of $N(v)$. A graph is called *regular* if each of its vertex has the same degree. The *distance* between two vertices $u, v \in V(G)$, denoted by d_{uv}, is defined as the length of a shortest path between u and v in G. The *diameter* of G is the maximum distance between any two vertices of G. The *distance matrix* of G, denoted by $D(G)$ is defined as $D(G) = (d_{uv})_{u,v \in V(G)}$. We direct the interested reader to consult the survey [6] for some spectral properties of the distance matrix of graphs. The *transmission* $Tr_G(v)$ of a vertex v is defined as the sum of the distances from v to all other vertices in G, that is, $Tr_G(v) = \sum\limits_{u \in V(G)} d_{uv}$. A graph G is said to be k-*transmission regular* if $Tr_G(v) = k$, for each $v \in V(G)$. The *transmission* of a graph G, denoted by $W(G)$, is the sum of distances between all unordered pairs of vertices in G. Clearly, $W(G) = \frac{1}{2} \sum\limits_{v \in V(G)} Tr_G(v)$. For any vertex $v_i \in V(G)$, the transmission $Tr_G(v_i)$ is called the *transmission degree*, shortly denoted by Tr_i and the sequence $\{Tr_1, Tr_2, \dots, Tr_n\}$ is called the *transmission degree sequence* of the graph G. The *second transmission degree* of v_i, denoted by T_i is given by $T_i = \sum\limits_{j=1}^{n} d_{ij} Tr_j$.

Let $Tr(G) = diag(Tr_1, Tr_2, \dots, Tr_n)$ be the diagonal matrix of vertex transmissions of G. M. Aouchiche and P. Hansen [7, 8, 9] introduced the Laplacian and the signless Laplacian for the distance matrix of a connected graph. The matrix $D^L(G) = Tr(G) - D(G)$ is called the *distance Laplacian matrix* of G, while the matrix $D^Q(G) = Tr(G) + D(G)$ is called the *distance signless Laplacian matrix of* G. The spectral properties of $D(G), D^L(G)$ and $D^Q(G)$ have attracted much more attention of the researchers and a large number of papers have been published regarding their spectral properties, like spectral radius, second largest eigenvalue, smallest eigenvalue, etc. For some recent works we refer to [1, 6, 7, 8, 9, 15, 16, 18] and the references therein.

In [11], Cui et al. introduced the *generalized distance matrix* $D_\alpha(G)$ defined as $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$, for $0 \leq \alpha \leq 1$. Since $D_0(G) = D(G)$, $2D_{\frac{1}{2}}(G) = D^Q(G)$, $D_1(G) = Tr(G)$ and $D_\alpha(G) - D_\beta(G) = (\alpha -$

$\beta)D^L(G)$, any result regarding the spectral properties of generalized distance matrix, has its counterpart for each of these particular graph matrices, and these counterparts follow immediately from a single proof. In fact, this matrix reduces to merging the distance spectral and distance signless Laplacian spectral theories. Since the matrix $D_\alpha(G)$ is real symmetric, all its eigenvalues are real. Therefore, we can arrange them as $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$. The largest eigenvalue ∂_1 of the matrix $D_\alpha(G)$ is called the *generalized distance spectral radius* of G (From now onwards, we will denote $\partial_1(G)$ by $\partial(G)$). As $D_\alpha(G)$ is non-negative and irreducible, by the Perron-Frobenius theorem, $\partial(G)$ is a simple (with multiplicity one) eigenvalue and there is a unique positive unit eigenvector X corresponding to $\partial(G)$, which is called the *generalized distance Perron vector* of G.

A column vector $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ can be considered as a function defined on $V(G)$ which maps vertex v_i to x_i, i.e., $X(v_i) = x_i$ for $i = 1, 2, \ldots, n$. Then,

$$X^T D_\alpha(G) X = \alpha \sum_{i=1}^{n} Tr(v_i) x_i^2 + 2(1-\alpha) \sum_{1 \leq i < j \leq n} d(v_i, v_j) x_i x_j,$$

and λ is an eigenvalue of $D_\alpha(G)$ corresponding to the eigenvector X if and only if $X \neq \mathbf{0}$ and,

$$\lambda x_{v_i} = \alpha Tr(v_i) x_i + (1-\alpha) \sum_{j=1}^{n} d(v_i, v_j) x_j.$$

These equations are called the (λ, x)-*eigenequations* of G. For some spectral properties of the generalized distance matrix of graphs, we direct the interested reader to consult the papers [2, 3, 11, 12, 19, 20, 14].

The remainder of the paper is organized as follows. In Section 2, we obtain some upper and lower bounds for the spectral radius of the matrix $D_\alpha(G)$, involving different graph parameters, and characterize the extremal graphs. In Section 3, we obtain a lower bound for the generalized distance spectral radius of a tree, in terms of the order n, the diameter d and the parameter α. We also characterize the extremal graphs for some values of diameter d. Finally, in Section 4, we obtain a lower bound for the generalized distance spectral radius of a graph with given edge connectivity, in terms of the order n, the edge connectivity r and the parameter α.

2 Bounds on the generalized distance spectral radius of graphs

In this section, we obtain upper and lower bounds for the generalized distance spectral radius of a connected graph G, in terms of various graph parameters associated with the structure of the graph. We characterize the extremal graphs attaining these bounds.

We start by mentioning two previously known results that will be needed in the sequel. The following lemmas can be found in [11].

Lemma 1 (See [11]) *Let G be a connected graph of order n. Then,*

$$\partial(G) \geq \frac{2W(G)}{n},$$

with equality if and only if G is a transmission regular graph.

Lemma 2 (See [11]) *Let G be a connected graph of order n and let $\frac{1}{2} \leq \alpha \leq 1$. If G' is a connected graph obtained from G by deleting an edge, then for any $1 \leq i \leq n$,*

$$\partial_i(G') \geq \partial_i(G).$$

The following gives a lower bound for the generalized distance spectral radius $\partial(G)$, in terms of the order n and the size m of the graph G.

Theorem 1 *Let G be a connected graph of order $n \geq 2$ and size m. Then*

$$\partial(G) \geq 2(n-1) - \frac{2m}{n}, \tag{1}$$

with equality if and only if $G \cong K_n$ or G is a transmission regular graph with diameter two.

Proof. We know that the transmission of each vertex $v \in V(G)$ is

$$Tr(v) \geq d(v) + 2(n-1-d(v)) = 2n - d(v) - 2,$$

where $d(v)$ denotes the degree of v in G, with equality if and only if the maximal distance from v to other vertices in G is at most two. With this we have

$$W(G) = \frac{1}{2}\sum_{v \in V(G)} Tr_G(v) \geq \frac{1}{2}\sum_{v \in V(G)} (2n - d(v) - 2) = n(n-1) - m,$$

with equality if and only if G is of diameter at most two. Using Lemma 1, and the above observation, the result follows. Suppose equality holds in (1), then equality holds in Lemma 1 and $W(G) = m + 2\left(\frac{n(n-1)}{2} - m\right) = n(n-1) - m$. Which is possible, if G is transmission regular and the diameter of G is at most two, that is, $G \cong K_n$ or G is a transmission regular graph of diameter two.

Conversely, if $G \cong K_n$ or G is a transmission regular graph of diameter two, then it is easy to see that (1) is an equality. □

The following gives a lower bound for the generalized distance spectral radius $\partial(G)$ of triangle-free and quadrangle-free graphs.

Corollary 1 *Let G be a triangle-free and quadrangle-free connected graph of order $n \geq 2$ and size m. Then*

$$\partial(G) \geq 3(n-1) - \frac{1}{n}\sum_{i=1}^{n} d^2(v_i) - \frac{2m}{n}, \tag{2}$$

with equality if and only if G is a transmission regular graph and the diameter of G is at most three.

Proof. For a connected graph G, which is triangle-free and quadrangle-free it is shown in [21] that

$$W(G) \geq \frac{3n(n-1)}{2} - \frac{1}{2}\sum_{i=1}^{n} d^2(v_i) - m,$$

with equality holding if and only if the diameter is at most three. Now, the result follows from Lemma 1. □

The following gives an upper bound for the generalized distance spectral radius $\partial(G)$, in terms of the transmission degrees Tr_i, the second transmission degrees T_i and the parameter α.

Theorem 2 *Let G be a connected graph of order $n \geq 2$ and let $\alpha \in [0, 1)$. Let $\{Tr_1, Tr_2, \ldots, Tr_n\}$ be the transmission degree sequence and $\{T_1, T_2, \ldots, T_n\}$ be the second transmission degree sequence of the graph G. Then*

$$\partial(G) \leq \frac{1}{2}\max_{1\leq i\neq j\leq n}\left\{\alpha(Tr_i + Tr_j) + \sqrt{\alpha^2(Tr_i - Tr_j)^2 + 4(1-\alpha)^2\left(\frac{T_i}{Tr_i}\right)\left(\frac{T_j}{Tr_j}\right)}\right\}. \tag{3}$$

Moreover, if $\frac{1}{2} \leq \alpha < 1$, the equality holds if and only if G is a transmission regular graph.

Proof. Let $Tr = Tr(G)$ be the diagonal matrix of vertex transmissions of the connected graph G, then the matrix Tr^{-1} exists. Since the matrices $D_\alpha(G)$ and $Tr^{-1}D_\alpha(G)Tr$ are similar and similar matrices have same spectrum, it follows that $\partial(G)$ is the spectral radius of the matrix $Tr^{-1}D_\alpha(G)Tr$. Let $X = (x_1, x_2, \dots, x_n)^T$ be an eigenvector of $Tr^{-1}D_\alpha(G)Tr$ corresponding to $\partial(G)$. Suppose $x_s = \max\{x_i | \ i = 1, 2, \dots, n\}$ and $x_t = \max\{x_i | \ x_i \neq x_s, \ i = 1, 2, \dots, n\}$. Now, the (i, j)-th entry of $Tr^{-1}D_\alpha(G)Tr$ is αTr_i if $i = j$ and $\frac{Tr_j}{Tr_i}(1-\alpha)d_{ij}$ if $i \neq j$. We have

$$Tr^{-1}D_\alpha(G)TrX = \partial(G)X. \tag{4}$$

From the s-th equation of (4), we have

$$\begin{aligned}(\partial(G) - \alpha Tr_s)x_s &= \sum_{i=1}^{n} \frac{Tr_i}{Tr_s}(1-\alpha)d_{si}x_i \\ &\leq \frac{(1-\alpha)x_t}{Tr_s}\sum_{i=1}^{n} d_{si}Tr_i = \frac{(1-\alpha)T_s}{Tr_s}x_t. \end{aligned} \tag{5}$$

Similarly, from the t-th equation of (4), we have

$$\begin{aligned}(\partial(G) - \alpha Tr_t)x_t &= \sum_{i=1}^{n} \frac{Tr_i}{Tr_t}(1-\alpha)d_{ti}x_i \\ &\leq \frac{(1-\alpha)x_s}{Tr_t}\sum_{i=1}^{n} d_{ti}Tr_i = \frac{(1-\alpha)T_t}{Tr_t}x_s. \end{aligned} \tag{6}$$

Combining (5) and (6) we get,

$$(\partial(G) - \alpha Tr_s)(\partial(G) - \alpha Tr_t)x_s x_t \leq \frac{(1-\alpha)T_s}{Tr_s}\frac{(1-\alpha)T_t}{Tr_t}x_t x_s,$$

which implies that

$$\partial^2(G) - \alpha(Tr_s + Tr_t)\partial(G) + \alpha^2 Tr_s Tr_t - \left(\frac{(1-\alpha)T_s}{Tr_s}\right)\left(\frac{(1-\alpha)T_t}{Tr_t}\right) \leq 0.$$

Thus, we have

$$\partial(G) \leq \frac{1}{2}\Big(\alpha(Tr_s + Tr_t) + \sqrt{\alpha^2(Tr_s - Tr_t)^2 + 4(1-\alpha)^2\left(\frac{T_s}{Tr_s}\right)\left(\frac{T_t}{Tr_t}\right)}\Big).$$

From this the result follows. Assume that G is a k-transmission regular graph. Then $Tr_i = k, T_i = k^2$ for all $i = 1, 2, \ldots, n$, and $\partial(G) = k$. It is now easy to see that equality in (3) holds.

Conversely, suppose that equality holds in (3), then all the inequalities in the above argument must hold as equalities. In particular, from (5) and (6), we have $x_1 = x_2 = \cdots = x_n$. Hence, $\partial(G) = \alpha Tr_1 + \frac{(1-\alpha)T_1}{Tr_1} = \alpha Tr_2 + \frac{(1-\alpha)T_2}{Tr_2} = \cdots = \alpha Tr_n + \frac{(1-\alpha)T_n}{Tr_n}$. Let $Tr_{\max}$ and $Tr_{\min}$ denote the maximum and minimum vertex transmission, respectively. Without loss of generality, assume that $Tr_i = Tr_{\max}$ and $Tr_j = Tr_{\min}$. Therefore, $\alpha Tr_{\max} + \frac{(1-\alpha)T_i}{Tr_{\max}} = \alpha Tr_{\min} + \frac{(1-\alpha)T_j}{Tr_{\min}}$. Since $T_i \geq Tr_{\max}Tr_{\min}$ and $T_j \leq Tr_{\max}Tr_{\min}$, we have

$$\begin{aligned}\alpha Tr_{\max} + (1-\alpha)Tr_{\min} \leq \alpha Tr_{\max} + \frac{(1-\alpha)T_i}{Tr_{\max}} &= \alpha Tr_{\min} + \frac{(1-\alpha)T_j}{Tr_{\min}} \\ &\leq (1-\alpha)Tr_{\max} + \alpha Tr_{\min},\end{aligned}$$

which implies that $Tr_{\max} = Tr_{\min}$ for $\frac{1}{2} \leq \alpha < 1$. Hence, G is a transmission regular graph. This completes the proof. □

The following gives a lower bound for the generalized distance spectral radius $\partial(G)$, in terms of the transmission degrees Tr_i, the second transmission degrees T_i and the parameter α.

Theorem 3 *Let G be a connected graph of order $n \geq 2$ and let $\alpha \in [0, 1)$. Let $\{Tr_1, Tr_2, \ldots, Tr_n\}$ be the transmission degree sequence and $\{T_1, T_2, \ldots, T_n\}$ be the second transmission degree sequence of the graph G. Then*

$$\partial(G) \geq \frac{1}{2} \min_{1 \leq i \neq j \leq n} \left\{ \alpha(Tr_i + Tr_j) + \sqrt{\alpha^2(Tr_i - Tr_j)^2 + 4(1-\alpha)^2 \left(\frac{T_i}{Tr_i}\right)\left(\frac{T_j}{Tr_j}\right)} \right\}.$$

Moreover, if $\frac{1}{2} \leq \alpha < 1$, the equality holds if and only if G is a transmission regular graph.

Proof. Let $X = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector of $Tr^{-1}D_\alpha(G)Tr$ corresponding to $\partial(G)$. Suppose $x_s = \min\{x_i | \ i = 1, 2, \ldots, n\}$ and $x_t = \min\{x_i | \ x_i \neq x_s, \ i = 1, 2, \ldots, n\}$. The rest of the proof is similar to that of Theorem 2 and is therefore omitted. □

The following lemma can be found in [17].

Lemma 3 *If A is an $n \times n$ non-negative matrix with the spectral radius $\lambda(A)$ and row sums $r_1, r_2, \ldots, r_n$, then $\min_{1 \leq i \leq n} r_i \leq \lambda(A) \leq \max_{1 \leq i \leq n} r_i$. Moreover, if A is irreducible, then one of the equalities holds if and only if the row sums of A are all equal.*

The following gives an upper bound for the generalized distance spectral radius $\partial(G)$, in terms of the maximum transmission degree $Tr_{\max}$, the second maximum transmission degree $T_{\max}$ and the parameter α.

Theorem 4 *Let G be a connected graph of order $n \geq 2$ and let $\alpha \in [0,1)$. Let $Tr_{\max}$ and $T_{\max}$ be respectively the maximum transmission degree and the second maximum transmission degree of the graph G. Then*

$$\partial(G) \leq \frac{\alpha Tr_{\max} + \sqrt{\alpha^2 Tr_{\max}^2 + 4(1-\alpha)T_{\max}}}{2},$$

Moreover, the equality holds if and only if G is a transmission regular graph.

Proof. For a graph matrix M, let $r_{v_i}(M)$ be the sum of the entries in the row corresponding to the vertex v_i, for $1 \leq i \leq n$. We have $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$, by a simple calculation, it can be seen that $r_{v_i}(D_\alpha(G)) = Tr_i$ and $r_{v_i}(D(G)Tr) = r_{v_i}(D^2(G)) = \sum_{j=1}^{n} d_{ij} Tr_j = T_i$. Then

$$\begin{aligned}
r_{v_i}(D_\alpha^2(G)) &= r_{v_i}\Big(\alpha Tr(G) + (1-\alpha)D(G)\Big)^2 \\
&= r_{v_i}\Big(\alpha^2 Tr^2 + \alpha(1-\alpha)TrD(G) + \alpha(1-\alpha)D(G)Tr + (1-\alpha)^2 D^2(G)\Big) \\
&= r_{v_i}\Big(\alpha Tr(\alpha Tr + (1-\alpha)D(G))\Big) + r_{v_i}\Big(\alpha(1-\alpha)D(G)Tr\Big) \\
&\quad + r_{v_i}\Big((1-\alpha)^2 D^2(G)\Big) \\
&= \alpha Tr_i r_{v_i}(D_\alpha(G)) + (1-\alpha)T_i \\
&\leq \alpha Tr_{\max} r_{v_i}(D_\alpha(G)) + (1-\alpha)T_{\max}.
\end{aligned}$$

So, we have

$$r_{v_i}\left(D_\alpha^2(G) - \alpha Tr_{\max} D_\alpha(G)\right) \leq (1-\alpha)T_{\max}.$$

Using Lemma 3, we get

$$\partial^2(G) - \alpha Tr_{\max}\partial(G) - (1-\alpha)T_{\max} \leq 0,$$

from this the result now follows. In order to get the equality, all inequalities in the above argument should be equalities. That is, $Tr_i = Tr_{\max}$ and $T_i = T_{\max}$ holds for any vertex v_i. So, by Lemma 3, it follows that G is a transmission regular graph.

Conversely, if G is transmission regular, then it is easy to check that the equality holds. □

The following gives a lower bound for the generalized distance spectral radius $\partial(G)$, in terms of the minimum transmission degree $Tr_{\min}$, the second minimum transmission degree $T_{\min}$ and the parameter α.

Theorem 5 *Let G be a connected graph of order $n \geq 2$ and let $\alpha \in [0,1)$. Let $Tr_{\min}$ and $T_{\min}$ be respectively the minimum transmission degree and the second minimum transmission degree of the graph G. Then*

$$\partial(G) \geq \frac{\alpha Tr_{\min} + \sqrt{\alpha^2 Tr_{\min}^2 + 4(1-\alpha)T_{\min}}}{2}.$$

Equality holds if and only if G is transmission regular.

Proof. Proceeding similar to Theorem 4, we arrive at

$$\begin{aligned} r_{v_i}(D_\alpha^2(G)) &= \alpha Tr_i r_{v_i}(D_\alpha(G)) + (1-\alpha)T_i \\ &\geq \alpha Tr_{\min} r_{v_i}(D_\alpha(G)) + (1-\alpha)T_{\min}. \end{aligned} \tag{7}$$

Since (7) is true for all v_i, in particular it is true for $v_{\min}$, where $v_{\min}$ is the vertex corresponding the row with minimum row sum. Therefore, from (7), we get

$$r_{v_{\min}}\Big(D_\alpha^2(G) - \alpha Tr_{\min} D_\alpha(G)\Big) - (1-\alpha)T_{\min} \geq 0.$$

Now, using Lemma 3, we get

$$\partial^2(G) - \alpha Tr_{\min}\partial(G) - (1-\alpha)T_{\min} \geq 0,$$

from this the result now follows. The equality case be discussed similarly as in Theorem 4. □

The following gives a lower bound for the generalized distance spectral radius $\partial(G)$, in terms of the order n, the maximum degree Δ and the parameter α.

Theorem 6 *Let G be a connected graph of order $n \geq 2$ and let $\alpha \in [0,1)$. If $\Delta = \Delta(G)$ is the maximum degree of the graph G, then*

$$\partial(G) \geq \frac{\alpha(2n-\Delta-2) + \sqrt{\alpha^2(2n-\Delta-2)^2 + 4(1-\alpha)(2n-2-\Delta)^2}}{2}, \tag{8}$$

with equality if and only if G is a regular graph with diameter less than or equal to 2.

Proof. Let G be a connected graph of order n and let $d_i = d(v_i)$ be the degree of the vertex v_i, for $1 \leq i \leq n$. It is well known that $Tr_i = Tr(v_i) \geq d_i + 2(n-1-d_i) = 2n-2-d_i$, for all $1 \leq i \leq n$, with equality if and only if G is a degree regular graph of diameter less than or equal to 2. Similar to the Theorem 4, we have

$$\begin{aligned} r_{v_i}(D_\alpha^2(G)) &= \alpha Tr_i r_{v_i}(D_\alpha(G)) + (1-\alpha)T_i \\ &\geq \alpha Tr_i r_{v_i}(D_\alpha(G)) + (1-\alpha)(2n-d_j-2)\sum_{j=1}^{n} d_{ij} \\ &\geq \alpha(2n-d_i-2)r_{v_i}(D_\alpha(G)) + (1-\alpha)(2n-2-\Delta)^2 \\ &\geq \alpha(2n-\Delta-2)r_{v_i}(D_\alpha(G)) + (1-\alpha)(2n-2-\Delta)^2, \end{aligned}$$

where we have used the fact that $Tr_i \geq 2n-2-d_i \geq 2n-2-\Delta$. Thus it follows that for each $v_i \in V(G)$, we have

$$r_{v_i}((D_\alpha)^2) \geq r_{v_i}(\alpha(2n-\Delta-2)D_\alpha) + (1-\alpha)(2n-2-\Delta)^2. \tag{9}$$

Since (9) is true for all v_i, in particular it is true for $v_{\min}$, where $v_{\min}$ is the vertex corresponding the row with minimum row sum. So, from (9), we get

$$r_{v_{\min}}\Big(D_\alpha^2(G) - \alpha(2n-2-\Delta)D_\alpha(G)\Big) - (1-\alpha)(2n-2-\Delta)^2 \geq 0.$$

Now, using Lemma 3, it follows that

$$\partial^2(G) - \alpha(2n-\Delta-2)\partial(G) - (1-\alpha)(2n-2-\Delta)^2 \geq 0,$$

which gives that

$$\partial(G) \geq \frac{\alpha(2n-\Delta-2) + \sqrt{\alpha^2(2n-\Delta-2)^2 + 4(1-\alpha)(2n-\Delta-2)^2}}{2}.$$

This proves the first part of the proof.

Suppose that equality holds in inequality (8), then all the inequalities hold as equalities in the above argument. Since the equality holds in $Tr_i \geq 2n-2-d_i \geq 2n-2-\Delta$ if G is Δ-regular graph of diameter less than or equal to 2 and equality holds in Lemma 3 if G is a transmission regular graph. It follows that equality holds in (8) if G is Δ-regular graph of diameter less than or equal to 2. Conversely, it is easily seen that $\partial(G) = \frac{\alpha(2n-\Delta-2)+\sqrt{\alpha^2(2n-\Delta-2)^2+4(1-\alpha)(2n-2-\Delta)^2}}{2}$ if G is a regular graph with diameter less than or equal to 2. □

We conclude this section with the following remark.

Remark 1 *As mentioned in the introduction that* $D_0(G) = D(G)$ *and* $2D_{\frac{1}{2}}(G) = D^Q(G)$, *it follows that from the bounds obtained in this section for* $\partial(G)$, *we can obtain the corresponding bounds for the distance spectral radius* $\rho_1^D(G)$ *and the distance signless Laplacian spectral radius* $\rho_1^Q(G)$ *by taking* $\alpha = 0$ *and* $\alpha = \frac{1}{2}$, *respectively.*

3 Lower bounds for the generalized distance spectral radius of a tree

In this section, we obtain a lower bound for the generalized distance spectral radius $\partial(G)$ of a tree, in terms of the order n, diameter d and the parameter α.

The following gives the generalized distance spectrum of the complete bipartite graph $K_{r,s}$, where $r + s = n$, and can be found in [20].

Lemma 4 *The generalized distance spectrum of complete bipartite graph* $K_{r,s}$ *consists of eigenvalue* $\alpha(2r + s) - 2$ *with multiplicity* $r - 1$, *the eigenvalue* $\alpha(2s + r) - 2$ *with multiplicity* $s - 1$ *and the remaining two eigenvalues as* x_1, x_2, *where* $x_1, x_2 = \frac{\alpha(s+r)+2(s+r)-4\pm\sqrt{(r^2+s^2)(\alpha-2)^2+2rs(\alpha^2-2)}}{2}$.

Suppose a graph G has a special kind of symmetry so that its associated matrix is written in the form

$$M = \begin{pmatrix} X & \beta & \cdots & \beta & \beta \\ \beta^t & B & \cdots & C & C \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \beta^t & C & \cdots & B & C \\ \beta^t & C & \cdots & C & B \end{pmatrix}, \tag{10}$$

where $X \in R^{t\times t}$, $\beta \in R^{t\times s}$ and $B, C \in R^{s\times s}$, such that $n = t + cs$, where c is the number of copies of B. Then the spectrum of this matrix can be obtained as the union of the spectrum of smaller matrices using the following technique given in [13]. In the statement of the following lemma, $\sigma^{[k]}(Y)$ indicates the multi-set formed by k copies of the spectrum of Y, denoted by $\sigma(Y)$.

Lemma 5 *Let* M *be a matrix of the form given in* (10), *with* $c \geq 1$ *copies of the block* B. *Then*

(i) $\sigma^{[c-1]}(B - C) \subseteq \sigma(M)$;

(ii) $\sigma(M)\backslash\sigma^{[c-1]}(B-C)=\sigma(M')$ *is the set of the remaining* $t+s$ *eigenvalues of* M, *where* $M'=\begin{pmatrix} X & \sqrt{c}.\beta \\ \sqrt{c}.\beta^t & B+(c-1)C \end{pmatrix}$.

Let $T_{a,b}$, with $a+b=n-2$ and $a\geq b\geq 1$ be the tree obtained by joining an edge between the root vertices of stars $K_{1,a}$ and $K_{1,b}$(the vertex of degree greater than one in a star is called root vertex). It is clear that a tree with diameter $d=3$ is always of the form $T_{a,b}$. The following gives the generalized distance spectrum of $T_{a,b}$.

Lemma 6 *The generalized distance spectrum of* $T_{a,b}$ *is*

$$\{\alpha(h_1+2)-2^{[b-1]},\alpha(h_2+2)-2^{[a-1]},x_1,x_2,x_3,x_4\},$$
$$h_1=2a+3b+1,h_2=2b+3a+1,$$

where $x_1\geq x_2\geq x_3\geq x_4$ *are the eigenvalues of the matrix*

$$M_2=\begin{pmatrix} \alpha(2a+b+1) & 1-\alpha & 2(1-\alpha)\sqrt{a} & (1-\alpha)\sqrt{b} \\ 1-\alpha & \alpha(2b+a+1) & (1-\alpha)\sqrt{a} & 2(1-\alpha)\sqrt{b} \\ 2(1-\alpha)\sqrt{a} & (1-\alpha)\sqrt{a} & \alpha h_1+2(1-\alpha)(a-1) & 3(1-\alpha)\sqrt{ab} \\ (1-\alpha)\sqrt{b} & 2(1-\alpha)\sqrt{b} & 3(1-\alpha)\sqrt{ab} & \alpha h_2+2(1-\alpha)(b-1) \end{pmatrix}.$$

Proof. Let $V(K_{1,b})=\{v_1,u_1,\dots,u_b\}$ and $V(K_{1,a})=\{v_2,w_1,\dots,w_a\}$. Then, the vertex set of $T_{a,b}$ is $V(T_{a,b})=\{v_1,v_2,u_1,\dots,u_b,w_1,\dots,w_a\}$. It is easy to see that $Tr(v_1)=2a+b+1$, $Tr(v_2)=2b+a+1$, $Tr(u_i)=2b+3a+1=h_2$ and $Tr(w_j)=2a+3b+1=h_1$, for $i=1,2,\dots,b$ and $j=1,2,\dots,a$. With this labeling, the generalized distance matrix of $T_{a,b}$ takes the form

$$D_\alpha(T_{a,b})=\begin{pmatrix} X & \beta & \beta & \cdots & \beta \\ \beta^t & \alpha h_1 & 2(1-\alpha) & \cdots & 2(1-\alpha) \\ \beta^t & 2(1-\alpha) & \alpha h_1 & \cdots & 2(1-\alpha) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \beta^t & 2(1-\alpha) & 2(1-\alpha) & \cdots & \alpha h_1 \end{pmatrix},\ \text{where } \beta=\begin{pmatrix} 2\\1\\3\\ \vdots \\ 3 \end{pmatrix}$$

and

$$X=\begin{pmatrix} \alpha(2a+b+1) & 1-\alpha & 1-\alpha & \cdots & 1-\alpha \\ 1-\alpha & \alpha(2b+a+1) & 2(1-\alpha) & \cdots & 2(1-\alpha) \\ 1-\alpha & 2(1-\alpha) & \alpha h_2 & \cdots & 2(1-\alpha) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1-\alpha & 2(1-\alpha) & 2(1-\alpha) & \cdots & \alpha h_2 \end{pmatrix}.$$

Using Lemma 5 with $B=[\alpha h_1]$, $C=[2(1-\alpha)]$ and $c=a$, it follows that $\sigma(D_\alpha(T_{a,b}))=\sigma^{[a-1]}(B-C)\cup\sigma(M_1)=\sigma^{[a-1]}([\alpha(h_1+2)-2])\cup\sigma(M_1)$, where

$M_1 = \begin{pmatrix} X & \sqrt{a}\beta \\ \sqrt{a}\beta & \alpha h_1 + 2(1-\alpha)(a-1) \end{pmatrix}$. Interchanging the third and last column of M_1 and then third and last row of the resulting matrix, we obtain a matrix similar to M_1. In the resulting matrix taking

$$X = \begin{pmatrix} \alpha(2a+b+1) & 1-\alpha & 2(1-\alpha)\sqrt{a} \\ 1-\alpha & \alpha(2b+a+1) & (1-\alpha)\sqrt{a} \\ 2(1-\alpha)\sqrt{a} & (1-\alpha)\sqrt{a} & \alpha h_1 + 2(1-\alpha)(a-1) \end{pmatrix},$$

$$\beta = \begin{pmatrix} 1-\alpha \\ 2(1-\alpha) \\ 3(1-\alpha)\sqrt{a} \end{pmatrix},$$

$B = [\alpha h_2]$, $C = [2(1-\alpha)]$ and $c = b$ in Lemma 5. It follows that $\sigma(M_1) = \sigma^{[b-1]}(B-C) \cup \sigma(M_2) = \sigma^{[b-1]}([\alpha(h_2+2)-2]) \cup \sigma(M_2)$, where M_2 is the matrix given in the statement. That completes the proof. □

The following gives a lower bound for the generalized distance spectral radius of a tree, in terms of the order n, the diameter d and the parameter α.

Theorem 7 *Let* T *be a tree of order* $n \geq 2$ *having diameter* d*. If* $d = 1$*, then* $\partial(T) = 1$*; if* $d = 2$*, then* $\partial(T) = \frac{(\alpha+2)n-4+\sqrt{\phi}}{2}$*,* $\phi = n^2\alpha^2 - (n^2+2-2n)4\alpha + 4(n^2-3n+3)$*; if* $d = 3$*, then* $\partial(T) = x_1$*, where* x_1 *is the largest eigenvalue of the matrix* M_2 *defined in Lemma 6. For* $d \geq 4$*, let* $P = v_1v_2\dots v_dv_{d+1}$ *be a diametral path of* G*, such that there are* a_1, a_2 *pendent vertices at* v_2, v_d*, respectively. Then*

$$\partial(T) \geq \max_{a_1,a_2}\left\{\frac{6n + d(d-7) + (a_1+a_2)(d-4) + 2 + \sqrt{\theta}}{2}\right\},$$

where $\theta = \alpha^2(a_2-a_1)^2(d-2)^2 + 4(1-\alpha)^2d^2$*.*

Proof. If T is a tree of diameter $d = 1$, then $T \cong K_2$ and so $\partial(T) = 1$. If T is a tree of diameter $d = 2$, then $T \cong K_{1,n-1}$ and so using Lemma 4, it follows that $\partial(T) = \frac{(\alpha+2)n-4+\sqrt{\phi}}{2}$, where $\phi = n^2\alpha^2 - (n^2+2-2n)4\alpha + 4(n^2-3n+3)$. If T is a tree of diameter $d = 3$, then $T \cong T_{a,b}$ and so using Lemma 6, it follows that $\partial(T) = x_1$, where x_1 is the largest eigenvalue of the matrix M_2 defined in Lemma 6. So, suppose that diameter of tree T is at least 4, then $n \geq 5$. Let $v_1v_2\dots v_{d+1}$ be a diametral path of T, and let a_1 and a_2 be the number of pendent neighbors of v_2 and v_d, respectively. We have

$$Tr(v_1) \geq 2(a_1-1) + 1 + 2 + \dots + (d-1) + da_2 + 3(n-a_1-a_2-d+1)$$

$$= 3n - a_1 + a_2(d-3) - 3d + 1 + \frac{d(d-1)}{2}.$$

Similarly,

$$Tr(v_{d+1}) \geq 3n - a_2 + a_1(d-3) - 3d + 1 + \frac{d(d-1)}{2}.$$

Let M be the principal submatrix of $D_\alpha(T)$ indexed by the vertices v_1 and v_{d+1}. Then

$$M = \begin{pmatrix} \alpha Tr(v_1) & (1-\alpha)d \\ (1-\alpha)d & \alpha Tr(v_{d+1}) \end{pmatrix},$$

thus

$$\begin{aligned} \partial(M) &= \frac{\alpha(Tr(v_1) + Tr(v_{d+1})) + \sqrt{\alpha^2(Tr(v_1) - Tr(v_{d+1}))^2 + 4(1-\alpha)^2 d^2}}{2} \\ &\geq \frac{\alpha(6n + d(d-7) + (a_1 + a_2)(d-4) + 2)) + \sqrt{\alpha^2(a_2 - a_1)^2(d-2)^2 + 4(1-\alpha)^2 d^2}}{2}. \end{aligned}$$

Now, by Interlacing Theorem [10], we have $\partial(T) \geq \partial(M)$. From this the result follows. That completes the proof. □

The following observation follows from Theorem 7.

Corollary 2 *Let T be a tree of order n having diameter $d \geq 4$. Then*

$$\partial(T) \geq \frac{1}{2}\Big(\alpha(6n + d^2 - 9d + 2) + 2d\Big).$$

Proof. Using $a_1, a_2 \geq 0$ in Theorem 7, the result follows. □

Taking $\alpha = 0$ in Theorem 7, we have the following observation, which gives a lower bound for the distance spectral radius $\rho^D(T)$ of a tree T.

Corollary 3 *Let T be a tree of order $n \geq 2$ having diameter d. If $d = 1$, then $\rho^D(T) = 1$; if $d = 2$, then $\rho^D(T) = n - 2 + \sqrt{n^2 - 3n + 3}$; if $d = 3$, then $\rho^D(T) = x_1$, where x_1 is the largest eigenvalue of the matrix M_2 (with $\alpha = 0$) defined in Lemma 6. For $d \geq 4$, let $P = v_1 v_2 \ldots v_d v_{d+1}$ be a diametral path of G, such that there are a_1, a_2 pendent vertices at v_2, v_d, respectively. Then*

$$\rho^D(T) \geq \max_{a_1, a_2} \left\{ \frac{6n + d(d-5) + (a_1 + a_2)(d-4) + 2}{2} \right\}.$$

Taking $\alpha = \frac{1}{2}$ in Theorem 7 and using the fact $2\partial(T) = \rho_1^Q(T)$, we have the following observation, which gives a lower bound for the distance signless Laplacian spectral radius $\rho^Q(T)$ of a tree T.

Corollary 4 *Let T be a tree of order $n \geq 2$ having diameter d. If $d = 1$, then $\rho^Q(T) = 1$; if $d = 2$, then $\rho^Q(T) = \frac{5n-8+\sqrt{9n^2-32n+32}}{2}$; if $d = 3$, then $\rho^Q(T) = 2x_1$, where x_1 is the largest eigenvalue of the matrix M_2 (with $\alpha = \frac{1}{2}$) defined in Lemma 6. For $d \geq 4$, let $P = v_1 v_2 \ldots v_d v_{d+1}$ be a diametral path of G, such that there are a_1, a_2 pendent vertices at v_2, v_d, respectively. Then*

$$\rho^Q(T) \geq \max_{a_1,a_2} \left\{6n + d(d-7) + (a_1 + a_2)(d-4) + 2 + \sqrt{t}\right\},$$

where $t = \frac{(a_2-a_1)^2}{4} + 2d^2 - 4d + 4$.

4 Lower bounds for the generalized distance spectral radius for a graph with given edge connectivity

In this section, we obtain a lower bound for the generalized distance spectral radius $\partial(G)$ for the family of graphs with fixed edge connectivity, in terms of the order n and the parameter α.

The *edge connectivity* of a connected graph is the minimum number of edges whose removal disconnects the graph. Let $G(n, r)$ be the set of all connected graphs of order n and edge connectivity r. It is clear that, $G(n, n-1) = K_n$. It is well known that $\partial(K_n) = n - 1$, therefore we will consider $r \leq n - 2$.

The following gives a lower bound for the generalized distance spectral radius of a graph belonging to the family $G(n, r)$, in terms of the order n, the edge connectivity r and the parameter α.

Theorem 8 *Let $G \in G(n, r)$ with $1 \leq r \leq n-2$ and $\frac{1}{2} \leq \alpha \leq 1$. If the degree of every vertex of G is greater than r, then*

$$\partial(G) \geq \frac{\alpha(4n - 2r - 2) + \sqrt{4\alpha^2(n_2 - n_1)^2 + 36(1-\alpha)^2}}{2},$$

where n_1 and n_2 are the cardinalities of the components of graph obtained from G by deleting r edges.

Proof. Let $G \in G(n, r)$, then every vertex of G is of degree greater or equal to r. Let us suppose that every vertex of G has degree at least $r + 1$. Let E_c be an edge cut of G with r edges. Let G_1 and G_2 be the two components of $G - E_c$ (the graph obtained from G by deleting the edges from E_c). Let $n_i = |V(G_i)|$ for $i = 1, 2$. We claim that $\min\{n_1, n_2\} \geq r + 2$. Suppose that

$\min\{n_1, n_2\} \leq r+1$. Without loss of generality, we assume that $n_2 \geq n_1$. Then we have $n_1 \leq r+1$. If $n_1 = r+1$, then there exists a vertex of G_1 which is not incident with any edge in E_c, and thus its degree in G is at most $n_1 - 1 = r$, which is a contradiction. On the other hand, if $n_1 \leq r$, then there exists a vertex of G_1 whose degree in G is at most $n_1 - 1 + \frac{r}{n_1} \leq (n_1 - 1)\frac{r}{n_1} + \frac{r}{n_1} = r$, again a contradiction. Therefore, we must have $\min\{n_1, n_2\} \geq r+2$. Thus, there exists a vertex u of G_1 (v of G_2, respectively) which is not adjacent to any vertex of G_2 (G_1, respectively).

Let G' be the graph obtained from G by adding all possible edges in G_1 and G_2. Then $G' - E_c = K_{n_1} \cup K_{n_2}$. Obviously, $G' \in G(n, r)$. Let t be the number of vertices of G' which are at a distance of 2 from u. Note that $t \leq r$. Since the diameter of G' is 3, we have

$$\begin{aligned} Tr_{G'}(u) &= n_1 - 1 + 2t + 3(n_2 - t) = n_1 + 3n_2 - 1 - t \\ &\geq n_1 + 3n_2 - 1 - r. \end{aligned}$$

Similarly,

$$Tr_{G'}(v) \geq n_2 + 3n_1 - 1 - r.$$

Let M be the principal submatrix of $D_\alpha(G')$ indexed by u and v. Then

$$M = \begin{pmatrix} \alpha Tr_{G'}(u) & 3(1-\alpha) \\ 3(1-\alpha) & \alpha Tr_{G'}(v) \end{pmatrix},$$

thus

$$\begin{aligned} \partial(M) &= \frac{\alpha(Tr(u) + Tr(v)) + \sqrt{\alpha^2(Tr(u) - Tr(v))^2 + 36(1-\alpha)^2}}{2} \\ &\geq \frac{\alpha(4n - 2r - 2) + \sqrt{4\alpha^2(n_2 - n_1)^2 + 36(1-\alpha)^2}}{2}. \end{aligned}$$

Now, using Lemma 2 and Interlacing Theorem [10], we have $\partial(G) \geq \partial(G') \geq \partial(M)$. From this the result follows. □

The following observation follows from Theorem 8.

Corollary 5 *Let $G \in G(n, r)$ with $1 \leq r \leq n-2$ and $\frac{1}{2} \leq \alpha \leq 1$. If the degree of every vertex of G is greater than r, then*

$$\partial(G) \geq \alpha(2n - r - 1) + 3(1-\alpha).$$

Proof. Using $(n_2 - n_1)^2 \geq 0$ in Theorem 8, the result follows. □

Taking $\alpha = \frac{1}{2}$ in Theorem 8 and using the fact $2\partial(G) = \rho_1^Q(G)$, we have the following observation, which gives a lower bound for the distance signless Laplacian spectral radius $\rho^Q(G)$ of a graph $G \in G(n, r)$.

Corollary 6 *Let $G \in G(n, r)$ with $1 \leq r \leq n-2$. If the degree of every vertex of G is greater than r, then*

$$\rho_1^Q(G) \geq 2n - r - 1 + \sqrt{(n_2 - n_1)^2 + 9},$$

where n_1 and n_2 are the cardinalities of the components of graph obtained from G by deleting r edges.

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