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Inequalities for rational functions with poles in the Half plane

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Abstract. In this paper we prove certain Bernstein-type inequalities for rational functions with poles in the right half plane. We also deduce some estimates for the maximum modulus of polar derivative of a polynomial on the imaginary axis in terms of the modulus of the polynomial.

1 Introduction

Let $\mathcal{P}_n$ denote the class of all complex polynomials $p(z) := \sum_{j=0}^{n} c_j z^j$ of degree at most n. For every $p \in \mathcal{P}_n$, the following inequality is due to Bernstein [4]:

$$\max_{|z|=1}|p'(z)| \leq n \max_{|z|=1}|p(z)|.$$

It was conjectured by Erdös and proved by Lax [6] that if all the zeros of p lie outside the open unit disk, then

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{2} \max_{|z|=1}|p(z)|.$$

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Later Turán [11] proved that if all the zeros of p lie inside the closed unit disk, then

$$\max_{|z|=1}|p'(z)| \geq \frac{n}{2}\max_{|z|=1}|p(z)|.$$

There have been many refinements and generalisations of the results of Lax and Turan (see [9], [10]). Li, Mohapatra and Rodriguez [7] extended the above inequalities to rational functions r with poles outside the closed unit disk and proved the following results:

Theorem 1 *Suppose* $r(z) = \dfrac{p(z)}{\prod_{j=1}^{n}(z-a_j)}$, *where* $p \in \mathcal{P}_n$ *and* $|a_j| > 1$, *for all* $1 \leq j \leq n$. *Then for* $|z| = 1$

$$|r'(z)| \leq |B'(z)|\max_{|z|=1}|r(z)|. \tag{1}$$

where $B(z) = \prod_{j=1}^{n}\left(\dfrac{1-\overline{a_j}z}{z-a_j}\right)$ *is the Blashke Product for unit disk.*

They also proved:

Theorem 2 *Suppose* $r(z) = \dfrac{p(z)}{\prod_{j=1}^{n}(z-a_j)}$, *where* $p \in \mathcal{P}_n$ *and* $|a_j| > 1$, *for all* $1 \leq j \leq n$*and all the zeroes of* r *lie outside open unit disk. Then for* $|z| = 1$

$$|r'(z)| \leq \frac{1}{2}|B'(z)|\max_{|z|=1}|r(z)|. \tag{2}$$

Theorem 3 *Suppose* $r(z) = \dfrac{p(z)}{\prod_{j=1}^{n}(z-a_j)}$, *where* $p \in \mathcal{P}_n$ *and* $|a_j| > 1$, *for all* $1 \leq j \leq n$*and all the zeroes of* r *lie inside closed unit disk. Then for* $|z| = 1$

$$|r'(z)| \geq \frac{1}{2}(|B'(z)| - (n-m)\max_{|z|=1}|r(z)|). \tag{3}$$

where m *is the number of zeros of* r.

Following the paper by Li, Mohapatra and Rodriguez [7], there have been many generalizations of Theorems 1, 2 and 3 (For details see [2], [3], [5], [8]). In all the cases, it is assumed that the poles of the rational function r are either inside or outside of the unit circle in the complex plane. In this paper, instead of assuming that the poles of r are inside/outside unit circle we consider the case when the poles are in the left/right half of the complex plane and derive the corresponding inequalities on the imaginary axis. So we derive these estimates on a line which is an unbounded set unlike the boundary of a disk. Further, we obtain certain estimates of the maximum modulus of the polar derivative $D_\zeta p(z)$ of a polynomial $p(z)$ in terms of the maximum modulus of $p(z)$ on the imaginary axis. We start with the following notations and definitions:

Let $\mathbb{I} := \{z \in \mathbb{C} : \Re(z) = 0\}$, $\mathbb{I}^+ := \{z \in \mathbb{C} : \Re(z) > 0\}$ and $\mathbb{I}^- := \{z \in \mathbb{C} : \Re(z) < 0\}$. For $a_j \in \mathbb{I}^+, j = 1, 2, \dots, n$, let

$$w(z) := \prod_{j=1}^{n} (z - a_j),$$

and $$\mathcal{R}_n = R_n(a_1, a_2, \dots, a_n) := \left\{ \frac{p(z)}{w(z)} : \quad p \in \mathcal{P}_n \right\}.$$

Thus $\mathcal{R}_n$ is the set of all rational functions with poles $a_1, a_2, \dots, a_n$ in the open right half plane and with finite limit at ∞. We define the corresponding Blashke product $B(z)$ for the half plane

$$B(z) := \prod_{j=1}^{n} \left(\frac{z + \overline{a_j}}{z - a_j} \right).$$

Clearly $B(z) \in \mathcal{R}_n$.

We also define for $p(z) = \sum_{j=0}^{n} c_j z^j$, the *conjugate transpose*(reciprocal) p^* of p as

$$p^*(z) := (-1)^n \overline{p(-\bar{z})} = \overline{c_n} z^n - \overline{c_{n-1}} z^{n-1} + \cdots + (-1)^n \overline{c_0}.$$

For $r(z) = \frac{p(z)}{w(z)} \in \mathcal{R}_n$, we define $r^*(z) := B(z)\overline{r(-\bar{z})}$. Note that if $r = \frac{p}{w} \in \mathcal{R}_n$, then $r^* = \frac{p^*}{w}$ and hence $r^* \in \mathcal{R}_n$. Further, we define the polar derivative $D_\zeta p(z)$ of a polynomial $p(z)$ with respect to ζ as

$$D_\zeta p(z) := np(z) - (z - \zeta)p'(z).$$

It is clear that $D_\zeta p(z)$ is a polynomial of degree atmost $n - 1$ and

$$\lim_{\zeta \to \infty} \left(\frac{D_\zeta p(z)}{\zeta} \right) = p'(z).$$

For details regarding Bernstein-type inequalities for polar derivatives on unit circle (see [1], [12]).

2 Main results

In this paper we assume that all the poles a_j, $j = 1, 2, \ldots, n$ lie in open right half plane $\mathbb{I}^+$. For the case when all the poles are in open left half plane $\mathbb{I}^-$, we obtain analogous results with suitable modifications. We first prove:

Theorem 4 *Let* i *be the imaginary unit, then* $B(z) = i$ *has exactly n simple roots, say* $t_1, t_2, \ldots, t_n$ *and all lie on the imaginary axis* $\mathbb{I}$. *Further, if* $r \in \mathcal{R}_n$ *and* $z \in \mathbb{I}$, *then*

$$r'(z)(B(z) - i) - B'(z)r(z) = (B(z) - i)^2 \sum_{k=1}^{n} \frac{u_k r(t_k)}{|z - t_k|^2}, \tag{4}$$

where

$$\frac{1}{u_k} = B'(t_k) = i \sum_{j=1}^{n} \frac{2\Re(a_j)}{|t_k - a_j|^2}, 1 \leq k \leq n. \tag{5}$$

Moreover for $z \in \mathbb{I}$

$$\frac{B'(z)}{B(z)} = \sum_{j=1}^{n} \frac{2\Re(a_j)}{|z - a_j|^2}. \tag{6}$$

From Theorem 1 we can deduce the following:

Corollary 1 *Let* t_k, $k = 1, 2, \ldots, n$ *be as defined in Theorem 4 and* s_k, $k = 1, 2, \ldots, n$ *be the roots of* $B(z) = -i$, *then for* $z \in \mathbb{I}$

$$|r'(z)| \leq \frac{1}{2}|B'(z)|[\max_{1 \leq k \leq n}|r(t_k)| + \max_{1 \leq k \leq n}|r(s_k)|]. \tag{7}$$

Corollary 1 immediately gives us the following:

Corollary 2 *If* $z \in \mathbb{I}$, *then*

$$\max_{z \in \mathbb{I}}|r'(z)| \leq |B'(z)| \max_{z \in \mathbb{I}}|r(z)| \tag{8}$$

The inequality is sharp in the sense that the equality holds if we take $r(z) = \lambda B(z)$ *for some* $\lambda \in \mathbb{C}$.

This is the Bernstein-type inequality for $\mathcal{R}_n$, the rational functions with all the poles in open right half plane and is identical to Theorem 1.

Theorem 5 *If* $r \in \mathcal{R}_n$ *and* $z \in \mathbb{I}$, *then*

$$|(r^*(z))'| - |r'(z)| \leq |B'(z)||r(z)|. \tag{9}$$

Theorem 6 *Suppose* $r \in \mathcal{R}_n$
(i) If r *has all its zeros in the closed left half plane* $\mathbb{I} \cup \mathbb{I}^-$, *then for* $z \in \mathbb{I}$

$$\Re\left(\frac{r'(z)}{r(z)}\right) \geq \frac{1}{2}|B'(z)|. \tag{10}$$

(ii) If r *has all its zeros in the closed right half plane* $\mathbb{I} \cup \mathbb{I}^+$, *then for* $z \in \mathbb{I}$

$$\Re\left(\frac{r'(z)}{r(z)}\right) \leq \frac{1}{2}|B'(z)|. \tag{11}$$

The inequalities are sharp and the equality holds if all the zeros of r *lie on the imaginary axis* $\mathbb{I}$.

If we set $a_1 = a_2 = \cdots = a_n = \alpha$ in Theorem 4, then we get the following estimates for the polar derivative of a polynomial $p \in \mathcal{P}_n$:

Theorem 7 *If* $p \in \mathcal{P}_n$ *and* $\alpha \in \mathbb{I}^+$, *then there exists a* $z_0 \in \mathbb{I}$ *such that*

$$|D_\alpha p(z)| \leq 2n\left|\frac{z-\alpha}{z_0-\alpha}\right|^n |p(z_0)| \quad \text{for } z \in \mathbb{I}. \tag{12}$$

Theorem 8 *If* $p \in \mathcal{P}_n$, *then for* $\alpha \in \mathbb{I}^+$

$$|D_\alpha p^*(z)| - |D_\alpha p(z)| \leq 2n|P(z)| \quad \text{for } z \in \mathbb{I}. \tag{13}$$

Theorem 9 *Suppose* $p \in \mathcal{P}_n$
(i) If p *has all its zeros in the closed left half plane* $\mathbb{I} \cup \mathbb{I}^-$, *then for* $\alpha \in \mathbb{I}^+$

$$\Re\left(\frac{D_\alpha p(z)}{(\alpha - z)p(z)}\right) \geq \frac{n\Re(\alpha)}{|z-\alpha|^2} \quad \text{for } z \in \mathbb{I}. \tag{14}$$

(ii) If p *has all its zeros in the closed right half plane* $\mathbb{I} \cup \mathbb{I}^+$, *then for* $\alpha \in \mathbb{I}^+$

$$\Re\left(\frac{D_\alpha p(z)}{(\alpha - z)p(z)}\right) \leq \frac{n\Re(\alpha)}{|z-\alpha|^2} \quad \text{for } z \in \mathbb{I}. \tag{15}$$

The inequalities are sharp and equality holds for a polynomial p *having all the zeros on the imaginary axis* $\mathbb{I}$.

Proofs:

Proof of the Theorem 4. Suppose

$$B(z) - i = 0. \tag{16}$$

Then $w^*(z) - iw(z) = 0$, which is clearly a polynomial of degree n and therefore it has n zeros.

We claim that

$$z \in \mathbb{I} \text{ if and only if } |B(z)| = 1. \qquad (*)$$

Indeed, we have $\left|\frac{z+\overline{a_j}}{z-a_j}\right|^2 - 1 = \frac{4\Re(z)\Re(a_j)}{|z-a_j|^2}$. Therefore if $\Re(z) = 0$, then $\left|\frac{z+\overline{a_j}}{z-a_j}\right| = 1$ for all $j = 1, 2, \cdots, n$ and we get $|B(z)| = \prod_{j=1}^{n}\left|\frac{z+\overline{a_j}}{z-a_j}\right| = 1$. Conversely if $|B(z)| = 1$, then $\Re(z) > 0$, gives us

$$\left|\frac{z+\overline{a_j}}{z-a_j}\right|^2 - 1 = \frac{4\Re(z)\Re(a_j)}{|z-\alpha|^2} > 0 \quad \text{for all} \quad j = 1, 2, \cdots, n.$$

This in particular gives $|B(z)| > 1$, a contradiction. There will be a similar contradiction, if we assume that $\Re(z) < 0$. Hence $z \in \mathbb{I}$.

By $(*)$ all the roots of (16) lie on $\mathbb{I}$ and $w(z) \neq 0$ on $\mathbb{I}$. So the n zeros of $w^*(z) - iw(z)$ are the n roots (say) $t_1, t_2, \cdots, t_n$ of (16), which lie on the imaginary axis. We show that all $t_k,\ k = 1, 2, \cdots, n$ are distinct. We have

$$B(z) = \frac{\prod_{j=1}^{n}(z+\overline{a_j})}{\prod_{j=1}^{n}(z-a_j)}.$$

Therefore

$$\begin{aligned}\frac{B'(z)}{B(z)} &= \sum_{j=1}^{n}\left(\frac{1}{z+\overline{a_j}} - \frac{1}{z-a_j}\right)\\ &= \sum_{j=1}^{n}\frac{2\Re(a_j)}{|z-a_j|^2} \quad \text{for} \quad z \in \mathbb{I}.\end{aligned}$$

This proves (6) and hence for all $t_k, k = 1, 2, \cdots, n$, we get

$$B'(t_k) = i\sum_{j=1}^{n}\frac{2\Re(a_j)}{|t_k-a_j|^2}.$$

Since $\Re(a_j) > 0$, for all $j = 1, 2, \cdots, n$, $B'(t_k)$ is a non-zero (purely imaginary) number for all $k = 1, 2, \cdots, n$. Hence $t_k,\ k = 1, 2, \cdots, n$ are all distinct roots of (16). Now let

$$\begin{aligned} q(z) &= w^*(z) - iw(z) \\ &= w(z)\left(B(z) - i\right) \\ &= a\prod_{k=1}^{n}(z - t_k), \quad a \neq 0. \end{aligned}$$

Then $q \in \mathcal{P}_n$. Now for $r(z) = \dfrac{p(z)}{w(z)} \in \mathcal{R}_n$, let $p(z) = cz^n + \ldots$ Then $p(z) - \dfrac{c}{a}q(z)$ is a polynomial of degree atmost $n-1$. Since $t_k, k = 1, 2, \cdots, n$ are n distinct numbers, by Lagrange's interpolation formula

$$p(z) - \frac{c}{a}q(z) = \sum_{k=1}^{n}\frac{(p(t_k) - \frac{c}{a}q(t_k))q(z)}{(z - t_k)q'(t_k)}.$$

This implies

$$\frac{p(z)}{q(z)} - \frac{c}{a} = \sum_{k=1}^{n}\frac{p(t_k)}{(z - t_k)q'(t_k)},$$

which on differentiation gives

$$\left(\frac{p(z)}{q(z)}\right)' = -\sum_{k=1}^{n}\frac{p(t_k)}{(z - t_k)^2 q'(t_k)}. \tag{17}$$

Now $p(z) = w(z)r(z)$ and $q(z) = w(z)(B(z) - i)$ gives $\dfrac{p(z)}{q(z)} = \dfrac{r(z)}{B(z) - i}$ and hence

$$\left(\frac{p(z)}{q(z)}\right)' = \frac{(B(z) - i)r'(z) - r(z)B'(z)}{(B(z) - i)^2}.$$

Also $p(t_k) = w(t_k)r(t_k)$ and

$$\begin{aligned} q'(t_k) &= w'(t_k)(B(t_k) - i) + w(t_k)B'(t_k) \\ &= w(t_k)B'(t_k). \end{aligned}$$

Therefore from (17), we have

$$\frac{(B(z) - i)r'(z) - r(z)B'(z)}{(B(z) - i)^2} = -\sum_{k=1}^{n}\frac{r(t_k)}{(z - t_k)^2 B'(t_k)}$$

$$= \sum_{k=1}^{n} \frac{r(t_k)}{|z-t_k|^2 B'(t_k)}, \quad \text{for} \quad z \in \mathbb{I}$$

Hence

$$(B(z)-i)r'(z) - r(z)B'(z) = (B(z)-i)^2 \sum_{k=1}^{n} \frac{u_k r(t_k)}{|z-t_k|^2} \tag{18}$$

where

$$\frac{1}{u_k} = B'(t_k) = i \sum_{j=1}^{n} \frac{2\Re(a_j)}{|t_k - a_j|^2}.$$

This proves (4) and (5).

Remark 1 *Note that* u_k, $(k = 1, 2, \cdots, n)$ *are purely imaginary numbers with negative imaginary part under our assumption* $\Re(a_j) > 0$ *for all* $j = 1, 2, \cdots, n$.

Proof of Corollary 1. By the same argument as in Theorem 4 applied to $B(z) = -i$ instead of $B(z) = i$, we get

$$(B(z)+i)r'(z) - r(z)B'(z) = (B(z)+i)^2 \sum_{k=1}^{n} \frac{v_k r(s_k)}{|z-s_k|^2}, \tag{19}$$

where

$$\frac{1}{v_k} = B'(t_k) = -i \sum_{j=1}^{n} \frac{2\Re(a_j)}{|s_k - a_j|^2}.$$

Subtracting (18) from (19) we have

$$2ir'(z) = (B(z)+i)^2 \sum_{k=1}^{n} \frac{v_k r(s_k)}{|z-s_k|^2} - (B(z)-i)^2 \sum_{k=1}^{n} \frac{u_k r(t_k)}{|z-t_k|^2}. \tag{20}$$

Taking $r(z) \equiv 1$ in (18) and (19) we get

$$B'(z) = -(B(z)-i)^2 \sum_{k=1}^{n} \frac{u_k}{|z-t_k|^2}$$

$$B'(z) = -(B(z)+i)^2 \sum_{k=1}^{n} \frac{v_k}{|z-s_k|^2}$$

and hence

$$|B'(z)| = |B(z) - i|^2 \left| \sum_{k=1}^{n} \frac{u_k}{|z - t_k|^2} \right| \tag{21}$$

$$|B'(z)| = |B(z) + i|^2 \left| \sum_{k=1}^{n} \frac{v_k}{|z - s_k|^2} \right|. \tag{22}$$

Now from (20)

$$|2r'(z)| \leq |(B(z) + i)|^2 \left| \sum_{k=1}^{n} \frac{v_k r(s_k)}{|z - s_k|^2} \right| + |(B(z) - i)|^2 \left| \sum_{k=1}^{n} \frac{u_k r(t_k)}{|z - t_k|^2} \right|.$$

Using (21) and (22), we get for $z \in \mathbb{I}$

$$|r'(z)| \leq \frac{1}{2}|B'(z)|[\max_{1\leq k\leq n}|r(t_k)| + \max_{1\leq k\leq n}|r(s_k)|]$$

Proof of Theorem 5. We have

$$r^*(z) = B(z)\overline{r(\overline{-z})}.$$

Therefore

$$\begin{aligned}(r^*(z))' &= B'(z)\overline{r(\overline{-z})} - B(z)\overline{r'(\overline{-z})}\\ &= B'(z)\overline{r(z)} - B(z)\overline{r'(z)} \quad \text{for } z \in \mathbb{I}\end{aligned}$$

This implies that

$$\begin{aligned}|(r^*(z))'| &\leq |B'(z)|\left|\overline{r(z)}\right| + |B(z)|\left|\overline{r'(z)}\right|\\ &= |B'(z)||r(z)| + |B(z)||r'(z)|.\end{aligned}$$

Since $|B(z)| = 1$ on imaginary axis, it follows that for $z \in \mathbb{I}$

$$|(r^*(z))'| - |r'(z)| \leq |B'(z)||r(z)|.$$

Proof of Theorem 6. Let $b_1, b_2, \dots, b_m, m \leq n.$ be the zeros of r.
(i) Suppose $\Re(b_j) \leq 0$ for all $j = 1, 2, \dots, m$. Then $p(z) = c\prod_{j=1}^{m}(z - b_j)$ with $c \neq 0$ and we have

$$r(z) = \frac{p(z)}{w(z)} = \frac{c\prod_{j=1}^{m}(z - b_j)}{\prod_{j=1}^{n}(z - a_j)}.$$

Taking logarithms on both sides and differentiating we get

$$\frac{r'(z)}{r(z)} = \sum_{j=1}^{m} \frac{1}{z-b_j} - \sum_{j=1}^{n} \frac{1}{z-a_j}. \tag{23}$$

Now for $\Re(z) = 0$

$$\Re\left(\frac{1}{z-b_j}\right) = \frac{-\Re(b_j)}{|z-b_j|^2} \geq 0 \quad \text{for all} \quad j = 1, 2, \ldots, m$$

and therefore

$$\sum_{j=1}^{m} \Re\left(\frac{1}{z-b_j}\right) \geq 0.$$

Hence from (23) and by using (6) we have

$$\begin{aligned}
\Re\left(\frac{r'(z)}{r(z)}\right) &= \sum_{j=1}^{m} \Re\left(\frac{1}{z-b_j}\right) - \sum_{j=1}^{n} \Re\left(\frac{1}{z-a_j}\right) \\
&\geq -\sum_{j=1}^{n} \Re\left(\frac{1}{z-a_j}\right) \\
&= -\sum_{j=1}^{n} \frac{\Re(z-a_j)}{|z-a_j|^2} \\
&= \sum_{j=1}^{n} \frac{\Re(a_j)}{|z-a_j|^2} \quad \text{for } \Re(z) = 0 \\
&= \frac{1}{2}\left|\frac{B'(z)}{B(z)}\right|.
\end{aligned}$$

Since $|B(z)| = 1$ for $z \in \mathbb{I}$, we conclude

$$\Re\left(\frac{r'(z)}{r(z)}\right) \geq \frac{1}{2}|B'(z)|.$$

(ii) Suppose $\Re(b_j) \geq 0$ for all $j = 1, 2, \ldots, m$. Then for $\Re(z) = 0$

$$\Re\left(\frac{1}{z-b_j}\right) = \frac{-\Re(b_j)}{|z-b_j|^2} \leq 0 \quad \text{for all} \quad j = 1, 2, \cdots, m.$$

This in particular gives

$$\sum_{j=1}^{m} \Re\left(\frac{1}{z-b_j}\right) \leq 0.$$

Thus as in part (i), we get for $\Re(z)=0$

$$\begin{aligned}
\Re\left(\frac{r'(z)}{r(z)}\right) &= \sum_{j=1}^{m} \Re\left(\frac{1}{z-b_j}\right) - \sum_{j=1}^{n} \Re\left(\frac{1}{z-a_j}\right) \\
&\leq -\sum_{j=1}^{n} \Re\left(\frac{1}{z-a_j}\right) \\
&= -\sum_{j=1}^{n} \frac{\Re(z-a_j)}{|z-a_j|^2} \\
&= \sum_{j=1}^{n} \frac{\Re(a_j)}{|z-a_j|^2} \\
&= \frac{1}{2}\left|\frac{B'(z)}{B(z)}\right|.
\end{aligned}$$

That is

$$\Re\left(\frac{r'(z)}{r(z)}\right) \leq \frac{1}{2}|B'(z)|,$$

Proof of Theorem 7. Let s_k and t_k, $k = 1, 2, \ldots, n$ be as defined in Corollary 1 and Let
$z_0 \in \{t_1, t_2, \ldots, t_n, s_1, s_2, \ldots, s_n\}$, be such that
$|r(z_0)| = max\{|r(t_1)|, |r(t_2)|, \ldots, |r(t_n)|, |r(s_1)|, |r(s_2)|, \ldots, |r(s_n)|\}$. By Corollary 1

$$|r'(z)| \leq |B'(z)||r(z_0)| \tag{24}$$

For $a_1 = a_2 = \ldots = a_n = \alpha$, $r(z) = \dfrac{p(z)}{(z-\alpha)^n}$ and $B(z) = \dfrac{(z+\overline{\alpha})^n}{(z-\alpha)^n}$

$$\begin{aligned}
\text{so that } r'(z) &= \left(\frac{p(z)}{(z-\alpha)^n}\right)' \\
&= \frac{(z-\alpha)^n p'(z) - p(z)n(z-\alpha)^{n-1}}{(z-\alpha)^{2n}} \\
&= \frac{1}{-(z-\alpha)^{n+1}} D_\alpha p(z).
\end{aligned}$$

Also from (6)

$$|B'(z)| = \frac{2n\Re(\alpha)}{|z-\alpha|^2} \quad \text{for } z \in \mathbb{I}.$$

Substituting in (24), we get for $z \in \mathbb{I}$

$$\begin{aligned}\left|\frac{1}{(z-\alpha)^{n+1}} D_\alpha p(z)\right| &\le \frac{2n\Re(\alpha)}{|z-\alpha|^2}\left|\frac{P(z_0)}{(z_0-\alpha)^n}\right| \\ &= \frac{2n\Re(\alpha - z)}{|z-\alpha|^2}\left|\frac{P(z_0)}{(z_0-\alpha)^n}\right|\end{aligned}$$

and hence

$$\left|\frac{1}{(z-\alpha)^{n+1}} D_\alpha p(z)\right| \le \frac{2n\Re(\alpha - z)}{|z-\alpha|^2}\left|\frac{P(z_0)}{(z_0-\alpha)^n}\right|.$$

Proof of Theorem 8. We have from Theorem 5 for every $z \in \mathbb{I}$

$$|(r^*(z))'| - |r'(z)| \le |B'(z)||r(z)| \tag{25}$$

Taking $r(z) = \dfrac{p(z)}{(z-\alpha)^n}$, so that

$$r'(z) = \frac{1}{-(z-\alpha)^{n+1}} D_\alpha p(z).$$

Also

$$r^*(z) = \frac{p^*(z)}{(z-\alpha)^n}$$

gives

$$(r^*(z))' = \frac{1}{-(z-\alpha)^{n+1}} D_\alpha p^*(z).$$

Further from (6), we have for $z \in \mathbb{I}$

$$|B'(z)| = \frac{2n\Re(\alpha)}{|z-\alpha|^2}.$$

Therefore from (25), for $z \in \mathbb{I}$

$$\left|\frac{1}{(z-\alpha)^{n+1}} D_\alpha p^*(z)\right| - \left|\frac{1}{(z-\alpha)^{n+1}} D_\alpha p(z)\right| \le \frac{2n\Re(\alpha)}{|z-\alpha|^2}\left|\frac{p(z)}{(z-\alpha)^n}\right|. \tag{26}$$

Also for $z \in \mathbb{I}$

$$\begin{aligned}\Re(\alpha) &= \Re(\alpha - z)\\ &\leq |\alpha - z|\\ &= |z-\alpha|.\end{aligned}$$

Thus from (26) we get

$$\left|\frac{1}{(z-\alpha)^{n+1}} D_\alpha p^*(z)\right| - \left|\frac{1}{(z-\alpha)^{n+1}} D_\alpha p(z)\right| \leq \frac{2n|z-\alpha|}{|z-\alpha|^2}\left|\frac{p(z)}{(z-\alpha)^n}\right| .$$

This gives

$$|D_\alpha p^*(z)| - |D_\alpha p(z)| \leq 2n|p(z)| \quad \text{for } z \in \mathbb{I}.$$

Proof of Theorem 9. Let $b_1, b_2, \ldots, b_m, m \leq n$, be the zeros of p.
(i) Suppose $\Re(b_j) \leq 0$ for all $j = 1, 2, \ldots, m$. Taking $r(z) = \dfrac{p(z)}{(z-\alpha)^n}$ in Theorem 6 (i), we get by using the fact that $r'(z) = \dfrac{-D_\alpha p(z)}{(z-\alpha)^{n+1}}$ and $|B'(z)| = \dfrac{2n\Re(\alpha)}{|z-\alpha|^2}$ for $z \in \mathbb{I}$,

$$\Re\left(\frac{D_\alpha p(z)}{(\alpha - z)p(z)}\right) \geq \frac{n\Re(\alpha)}{|z-\alpha|^2} \quad \text{for } z \in \mathbb{I}. \tag{27}$$

(ii) Suppose $\Re(b_j) \geq 0$ for all $j = 1, 2, \ldots, m$. Then taking $r(z) = \dfrac{p(z)}{w(z)}$ in Theorem 6 (ii).

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