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A generalized (ψ, φ)- weak contraction in metric spaces

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Abstract. In this paper, we introduce weakly generalized (ψ, φ)-weak quasi contraction for four self-maps and establish a common fixed point theorem using weak compatible property.

1 Introduction

In 1997, Alber and Guerre-Delabrier [2] defined the concept of weak contraction as a generalization of contraction and established the existence of fixed points for a self-map in Hilbert space. In 2001, Rhoades [9] extended this concept to metric spaces. A mapping $T : X \to X$ is said to be a *weak contraction* if there exists a function $\varphi : [0, \infty) \to [0, \infty)$, $\varphi(t) > 0$ for all $t > 0$ and $\varphi(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \; \forall \, x, y \in X. \tag{1}$$

As weak contractions are defined through φ, these are referred as φ-*weak contraction*.

Rhoades [9] established that every φ-weak contraction has a unique fixed point in complete metric space when φ is continuous.

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Theorem 1 *Let* (X, d) *be a nonempty complete metric space, and let* $T : X \to X$ *be a* φ*-weak contraction on X. If* $\varphi(t) > 0$*, for all* $t > 0$ *and* $\varphi(0) = 0$*, then T has a unique fixed point.*

Afterwards, Dutta and Choudhury [4] generalized the concept of weak contraction and proved the following theorem.

Theorem 2 *[4] Let* (X, d) *be a nonempty complete metric space, and let T be a self-map on X, satisfying*

$$\psi(d(Tx, Ty)) \le \psi(d(x, y)) - \varphi(d(x, y)) \tag{2}$$

for each $x, y, \in X$*, where,* $\psi, \varphi : \mathfrak{R}^+ \to \mathfrak{R}^+$ *are both continuous and non-decreasing function with* $\psi(t) = \varphi(t) = 0$ *iff* $t = 0$*. Then T has a unique fixed point in X.*

Throughout this paper, we denote
$\Psi = \{\psi : \mathfrak{R}^+ \to \mathfrak{R}^+$(i) ψ is continuous (ii) ψ is non-decreasing (iii) $\psi(t) = 0 \Leftrightarrow t = 0\}$
$\Phi = \{\varphi : \mathfrak{R}^+ \to \mathfrak{R}^+$ (i) lower semi-continuous for all $t > 0$ and φ is discontinuous at $t = 0$ with $\varphi(0) = 0$ $\}$.
In fact, the function Ψ is called the altering distance function and it was introduced by Khan, Swaleh and Sessa [7].

In 2009, Doric [3] introduced generalized (ψ, φ)-weak contraction for a pair of self-maps as follows.

Definition 1 *[3] Let* (X, d) *be a metric space. Let S and T be self-maps in X. If there exist* $\psi \in \Psi$ *and* $\varphi \in \Phi$ *such that*

$$\psi(d(Sx, Ty)) \le \psi(M(x, y)) - \varphi(M(x, y)) \tag{3}$$

for each $x, y, \in X$*, where*

$$M(x, y) = \max\left\{d(x, y), d(Tx, x), d(Sy, y), \frac{1}{2}[d(y, Tx) + d(x, Sy)]\right\}$$

then we say that S and T satisfy generalized (ψ, φ)*-weak contraction condition.*

Theorem 3 *[3] Let* (X, d) *be a nonempty metric space. Let S and T be self-maps of X, satisfying generalized* (ψ, φ)*-weak contraction condition. Then S and T have a unique common fixed point in X.*

In 2010, Abbas and Doric [1] extended the concept of generalized (ψ, φ)-weak contraction for a pair of self-maps to four self-maps in the following way.

Definition 2 *[1] Let (X, d) be a metric space. Let A, B, S and T be self-maps in X. If there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\psi(d(Ax, By)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \tag{4}$$

for each $x, y, \in X$, where

$$M(x, y) = \max\left\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{1}{2}[d(Sx, By) + d(Ax, Ty)]\right\},$$

then we say that A, B, S and T satisfy generalized (ψ, φ)-weak contraction condition.

Theorem 4 *[1] Let (X, d) be a complete metric space and A, B, S and T be self-maps of X satisfying generalized (ψ, φ)-weak contraction condition. Suppose that $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and that the pairs (A, S) and (B, T) are weakly compatible. Then A, B, S and T have a unique common fixed point in X, provided one of the range spaces $A(X), B(X), S(X)$ and $T(X)$ are closed in X.*

In 2015, P.P. Murthy et al, [8] extended the concept of generalized (ψ, φ)-weak contraction condition in a complete metric space by using a weaker condition than the (1.2) in complete metric space.

Theorem 5 *[8] Let (X, d) be a complete metric space, and A, B, S and $T : X \to X$be a continuous mapping satisfying*

$$\psi(d(Ax, By)) \leq \psi(M(x, y)) - \varphi(N(x, y)), \tag{5}$$

for all $x, y, \in X$, with $x \neq y$, for some $\psi \in \Psi$ and $\varphi \in \Phi$

$$M(x, y) = \max\left\{d(Sx, Ty), \frac{1}{2}[d(Sx, Ax)+d(Ty, By)], \frac{1}{2}[d(Sx, By)+d(Ty, Ax)]\right\},$$

and

$$N(x, y) = \min\left\{d(Sx, Ty), \frac{1}{2}[d(Sx, Ax)+d(Ty, By)], \frac{1}{2}[d(Sx, By)+d(Ty, Ax)]\right\},$$

$$A(X) \subseteq T(X) \text{ and } B(X) \subseteq S(X) \tag{6}$$

$$(A, S) \text{ and } (B, T) \text{ are weak compatible pairs.} \tag{7}$$

Then A, B, S and T have a unique common fixed point in X.

Definition 3 *[5] (i) Let S and T be mappings of a metric space* (X, d) *into itself. The mappings S and T are said to be compatible*

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}$ *is a sequence in X such that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z,$$

for some $z \in X$*.*

Definition 4 *[6] (i) A pair of self-mapping S and T of a metric space* (X, d) *is said to be weakly compatible if they commute at their coincidence points i.e if* $Ax = Bx$ *for some* $x \in X$*, then* $ABx = BAx$*,*
(ii) be occasionally weakly compatible (owc) [10] if $TSx = STx$ *for some* $x \in X$*.*

Remark. Every compatible map are weakly compatible but the converse is not true [6].

In this paper, we introduce weakly generalized (ψ, φ)-weak quasi- contraction condition and establish a common fixed point theorem by using weakly compatible pairs in metric space.

Definition 5 *Let* (X, d) *be a metric space, and A, B, S and* $T : X \to X$*be a mappings satisfying*

$$\psi(d(Ax, By)) \leq \psi(M(x, y)) - \varphi(N(x, y)), \tag{8}$$

for all $x, y, \in X$*, with* $x \neq y$*, for some* $\psi \in \Psi$ *and* $\varphi \in \Phi$

$$M(x, y) = \max\left\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}[d(Sx, By) + d(Ty, Ax)]\right\},$$

and

$$N(x, y) = \min\left\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}[d(Sx, By) + d(Ty, Ax)]\right\}.$$

Then we say that A, B, S and T satisfy weakly generalized (ψ, φ)*-weak quasi contraction condition.*

Remark. If ψ and φ in (5) satisfy '(ψ, φ) is non-decreasing' then the inequality (5) implies that inequality (8). But its converse need not be true. The following example shows that there exist maps A, B, S and T which are weakly generalized (ψ, φ)-weak quasi-contraction condition, but they do not satisfy the condition (5).

Example 1 *Let* $X = [0, 2)$ *be endowed with the Euclidean metric* $d(x, y) = |x - y|$*, and let A, B, S and* $T \to X$ *be defined by*

$$A(X) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases} \quad B(X) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{4} & \text{if } x \neq 0 \end{cases}$$

$$S(X) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{3}{2} & \text{if } x \neq 0 \end{cases} \quad T(X) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{5}{4} & \text{if } x \neq 0 \end{cases}$$

where $x, y \in X$*, defined as* $\psi \in \Psi$ *and* $\varphi \in \Phi$*, by*

$$\psi(t) = \frac{t}{2} \text{ and } \varphi(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{t}{16} & \text{if } t > 0 \end{cases}$$

In particular, $x \neq 0$ and $y \neq 0$, the inequality (5) does not hold

$$\psi(d(Ax, By)) = \psi(\frac{3}{4}) \leq \psi(\frac{3}{4}) - \varphi(\frac{1}{4})$$

$$\frac{3}{8} \leq \frac{3}{8} - \frac{1}{64}.$$

But, these mappings satisfy the condition (8) in all possible cases.

2 Fixed point theorems in metric space

Before stating the main result we prove the following lemma.

Lemma 1 *Let* (X, d) *be a metric space, and A, B, S and* $T : X \to X$*be a mapping satisfying the condition* (6) *and* (7)*, weakly generalized* (ψ, φ)*- weak quasi contraction condition. Then the sequence* $\{y_n\}$ *is a Cauchy sequence.*

Proof. Let $x_0 \in X$, from (6), there exists a point $x_1 \in X$ such that $y_0 = Ax_0 = Tx_1$, for this x_1, there exists a point $x_2 \in X$ such that $y_0 = Bx_1 = Sx_2$. In general $\{y_n\}$ is defined by

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \tag{9}$$

$$y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}. \tag{10}$$

Now, we suppose that

$$y_{2n} \neq y_{2n+1} \ \forall n \tag{11}$$

For this suppose that $x = x_{2n}, y = x_{2n+1}$ in (8), we have

$$\begin{aligned}\psi(d(Ax_{2n}, Bx_{2n+1})) &\leq \psi(M(x_{2n}, x_{2n+1})) - \phi(N(x_{2n}, x_{2n+1})) \\ &= \psi\big(max\{d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, Ax_{2n}), d(Tx_{2n+1}, Bx_{2n+1}), \\ &\frac{1}{2}[d(Sx_{2n}, Bx_{2n+1}) + d(Tx_{2n+1}, Ax_{2n})]\}\big) - \varphi(N(x_{2n}, x_{2n+1})).\end{aligned} \tag{12}$$

Using (9), (10) in (12), then we get

$$\begin{aligned}\psi(d(y_{2n}, y_{2n+1})) &\leq \psi\big(max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ &\frac{1}{2}[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})]\}\big) - \varphi(N(x_{2n}, x_{2n+1})), \\ &\leq \psi\big(max\{d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n}), \\ &\frac{1}{2}[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]\}\big) - \varphi(N(x_{2n}, x_{2n+1})).\end{aligned} \tag{13}$$

If $y_{2n+1} \neq y_{2n+2}$ $\forall n$ then taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in (8), and applying the above process, then we get

$$\begin{aligned}\psi(d(y_{2n+2}, y_{2n+1})) &\leq \psi\big(max\{d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), \\ &\frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})]\}\big) - \varphi(N(x_{2n+1}, x_{2n+2})).\end{aligned} \tag{14}$$

From (13) and (14) for any n, then we have

$$\begin{aligned}\psi(d(y_n, y_{n+1})) \leq \psi\Big(&max\{d(y_{n-1}, y_n), d(y_{n+1}, y_n), \\ &\frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]\}\Big) - \varphi(N(x_n, x_{n+1})).\end{aligned} \tag{15}$$

If

$$d(y_{n-1}, y_n) < d(y_n, y_{n+1}). \tag{16}$$

Then inequality (15) reduces to

$$\psi(d(y_n, y_{n+1})) < \psi(d(y_n, y_{n+1})) - \varphi(N(x_n, x_{n+1})).$$

On taking liminf as $n \to \infty$ on both sides, then we have

$$\underline{\lim}_{n\to\infty}\psi(d(y_n, y_{n+1})) < \underline{\lim}_{n\to\infty}\psi(d(y_n, y_{n+1})) - \underline{\lim}_{n\to\infty}\varphi(N(x_n, x_{n+1})). \tag{17}$$

The right-hand side is positive due to the property of Φ, therefore inequality (17), change the form

$$\underline{\lim}_{n\to\infty}\psi(d(y_n, y_{n+1})) < \underline{\lim}_{n\to\infty}\psi(d(y_n, y_{n+1})),$$

a contradiction. From (15) we have

$$\begin{aligned}\psi(d(y_n, y_{n+1})) &\leq \psi(d(y_n, y_{n-1})) - \varphi(N(x_n, x_{n+1})) \\ &< \psi(d(y_n, y_{n-1})).\end{aligned} \tag{18}$$

Therefore by the property of ψ, we get

$$d(y_n, y_{n+1}) < d(y_n, y_{n-1}). \tag{19}$$

Hence, the sequence $\{d(y_n, y_{n+1})\}$ is a non increasing sequence of nonnegative real number and hence it converges to some real number r (say), $r \geq 0$. Suppose $r > 0$, on taking liminf as $n \to \infty$ on (18), we have

$$\underline{\lim}_{n\to\infty}\psi(d(y_n, y_{n+1})) < \underline{\lim}_{n\to\infty}\psi(d(y_n, y_{n-1}))$$

The right term $\underline{\lim}_{n\to\infty}\varphi(N(x_n, x_{n+1})) > 0$, due to the property of φ. Hence

$$\psi(r) < \psi(r),$$

a contradiction. Thus

$$\underline{\lim}_{n\to\infty}\psi d(y_n, y_{n+1}) = 0,$$

and then

$$\underline{\lim}_{n\to\infty}d(y_n, y_{n+1}) = 0. \tag{20}$$

Next, we prove that $\{y_n\}$ is a Cauchy sequence. It is enough to show that the sub-sequence $\{y_{2n}\}$ of $\{y_n\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and the sequence of natural numbers $\{2m(k)\}$ and $\{2n(k)\}$ such that $2m(k) > 2n(k) > 2k$ for $k \in N$ and

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon, \tag{21}$$

For each k, let $2m(k)$ be the least positive integer exceeding $2n(k)$ and satisfying (21). Then we have

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon \text{ and } d(y_{2n(k)}, y_{2m(k)-2}) \leq \epsilon. \tag{22}$$

We have

$$\begin{aligned}\epsilon < (d(y_{2m(k)}, y_{2n(k)})) &\leq (d(y_{2m(k)}, y_{2m(k)-1})) + d(y_{2m(k)-1}, y_{2m(k)-2}) \\ &\quad + d(y_{2m(k)-2}, y_{2n(k)}) \\ &\leq (d(y_{2m(k)}, y_{2m(k)-1})) + d(y_{2m(k)-1}, y_{2m(k)-2}) + \epsilon\end{aligned}$$

by taking the liminf as $k \to \infty$ and using (21), we get

$$\epsilon < \underline{\lim}_{k\to\infty}(d(y_{2m(k)}, y_{2n(k)})) \leq \epsilon.$$

Therefore

$$\underline{\lim}_{k\to\infty}(d(y_{2m(k)}, y_{2n(k)})) = \epsilon.$$

Using triangular inequality

$$|d(y_{2m(k)}, y_{2n(k)}) - d(y_{2m(k)-1}), y_{2n(k)+1})| \leq d(y_{2m(k)}, y_{2m(k)-1}) + d(y_{2n(k)}, y_{2n(k)+1}).$$

We take the limit $k \to \infty$, on both sides, we get

$$\lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)+1}) = \epsilon. \tag{23}$$

Again using triangular inequality

$$|d(y_{2m(k)}, y_{2n(k)}) - d(y_{2m(k)-1}, y_{2n(k)})| \leq d(y_{2m(k)}, y_{2m(k)-1}),$$

on taking $\lim_{k\to\infty}$, on both sides

$$\lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)}) = \epsilon. \tag{24}$$

Now consider

$$\begin{aligned}\psi(d(y_{2m(k)}, y_{2n(k)})) &\leq \psi d(y_{2n(k)}, y_{2n(k)+1}) + \psi(d(y_{2n(k)+1}, y_{2m(k)})) \\ &= \psi d(y_{2n(k)}, y_{2n(k)+1}) + \psi(d(Ax_{2m(k)}, Bx_{2n(k)+1}))\end{aligned} \tag{25}$$

Then

$$\begin{aligned}&\psi(d(Ax_{2m(k)}, Bx_{2n(k)+1}) \leq \psi(M(x_{2m(k)}, x_{2n(k)+1})) - \varphi(N(x_{2m(k)}, x_{2n(k)+1})) \\ &= \psi\big(max\{d(Sx_{2m(k)}, Tx_{2n(k)+1}), d(Sx_{2m(k)}, Ax_{2m(k)}), d(Tx_{2n(k)+1}), Bx_{2n(k)+1}), \\ &\frac{1}{2}[d(Sx_{2m(k)}, Bx_{2n(k)+1}) + d(Tx_{2n(k)+1}, Ax_{2m(k)})]\big) - \varphi(N(x_{2m(k)}, x_{2n(k)+1}))\end{aligned}$$

$$\begin{aligned}&= \psi\big(max\{d(y_{2m(k)-1}, y_{2n(k)}), d(y_{2m(k)-1}, y_{2m(k)}), d(y_{2n(k)}, y_{2n(k)+1}), \\ &\frac{1}{2}[d(y_{2m(k)-1}, y_{2n(k)-1}) + d(y_{2n(k)}, y_{2m(k)})]\}\big) - \varphi(N(x_{2m(k)}, x_{2n(k)+1}))\end{aligned} \tag{26}$$

Using (25) and (26) and taking $\underline{\lim}_{k\to\infty}$, on both side we get

$$\psi(\epsilon) \le \psi(\epsilon) - \underline{\lim}_{k\to\infty}\varphi(N(x_{2m(k)}, x_{2n(k)+1})).$$

We observe that the last term on the right-hand side of the above inequality is non-zero. Thus we arrive at a contradiction. Therefore $\{y_{2n}\}$ is a Cauchy so that $\{y_n\}$ is a Cauchy sequence □

Theorem 6 *Let* (X, d) *be a metric space, and A, B, S and* $T : X \to X$ *be a mapping satisfying the condition* (6), (7) *and weakly generalized* (ψ, φ) *weak quasi contraction condition. Then A, B, S and T have a unique common fixed point in X, provided any one of the ranges* $A(X), B(X), S(X), T(X)$ *is a closed subspace of X.*

Proof. Since $\{y_n\}$ is a Cauchy sequence and assumes that $S(X)$ is a closed subspace of X, $\{y_{2n}\}$ is sub-sequence of $\{y_n\}$, we get

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = z, \tag{27}$$

where $z \in X$. Since $\{y_n\}$ is a Cauchy sequence it follows that $\lim_{n\to\infty} y_n = z$, therefore

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} y_{2n+1} = z. \tag{28}$$

Consequently, the subsequence also converges to z in X.
Therefore

$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z \ \forall z \in X. \tag{29}$$

Since $S(X)$ is closed. Then, there exists a $u \in X$ such that

$$z = Su. \tag{30}$$

We claim that $Au = z$. Suppose not, putting $x = u$ and $y = x_{2n+1}$ then in inequality (8), we get

$$\begin{aligned}\psi(d(Au, Bx_{2n+1})) &\le \psi(M(u, x_{2n+1})) - \varphi(N(u, x_{2n+1}))\\ &= \psi\Big(\max\big\{d(Su, Tx_{2n+1}), d(Su, Au), d(Tx_{2n+1}, Bx_{2n+1}),\\ &\quad \frac{1}{2}[d(Su, Bx_{2n+1}) + d(Tx_{2n+1}, Au)]\big\}\Big) - \varphi(N(u, x_{2n+1}))\end{aligned}$$

on taking the $\underline{\lim}_{n\to\infty}$, we get

$$\psi(d(Au, z)) \leq \psi\left(\max\left\{d(z, z), d(z, Au), \frac{1}{2}[d(z, z) + d(z, Au)]\right\}\right) - \underline{\lim}_{n\to\infty}\varphi(N(u, x_{2n+1})) \tag{31}$$

we obtain that the last term on the right side of the inequality (31) is non-zero by the property of φ, then we get

$$\psi(d(Au, z)) < \psi(d(Au, z)) \tag{32}$$

a contradiction.

$$Au = z. \tag{33}$$

Therefore from (30) and (33), we get

$$Au = Su = z. \tag{34}$$

Since the pair (A, S) is weakly compatible, then we get

$$Au = Su \Rightarrow ASu = SAu \Rightarrow Az = Sz. \tag{35}$$

We shall show that z is a common fixed point of A and S.
If $Az \neq z$, then we take $x = z$ and $y = x_{2n+1}$ in (8), we have

$$\begin{aligned}\psi(d(Az, Bx_{2n+1})) &\leq \psi(M(z, x_{2n+1})) - \varphi(N(z, x_{2n+1})) \\ &= \psi\Big(\max\{d(Sz, Tx_{2n+1}), d(Sz, Az), d(Tx_{2n+1}, Bx_{2n+1}), \\ &\quad \frac{1}{2}[d(Sz, Bx_{2n+1}) + d(Tx_{2n+1}, Az)]\}\Big) - \varphi(N(z, x_{2n+1})),\end{aligned}$$

on taking $\underline{\lim}_{n\to\infty}$, we have

$$\underline{\lim}_{n\to\infty}\psi(d(Az, Bx_{2n+1})) \leq \underline{\lim}_{n\to\infty}\psi(M(z, x_{2n+1})) - \underline{\lim}_{n\to\infty}\varphi(N((z, x_{2n+1})). \tag{36}$$

It is clear that from the condition of φ right-hand side term

$$\underline{\lim}_{n\to\infty}\varphi(N((z, x_{2n+1}))$$

is non-zero, then we get

$$\psi(d(Az, z)) < \psi(d(Az, z))$$

a contradiction. Thus, we have

$$\psi(d(Az, z)) < \psi(d(Az, z)),$$

which implies that

$$Az = z. \tag{37}$$

From (35) and (37), we get

$$Az = Sz = z. \tag{38}$$

Since $A(X) \subset T(X)$, there is a point $v \in X$ such that $Az = Tv$.
Thus from (38), we have

$$Az = Sz = Tv = z. \tag{39}$$

Suppose that $Bv \neq z$. On taking $x = x_{2n}$ and $y = v$ in inequality (8), we have

$$\begin{aligned} \psi(d(Ax_{2n}, Bv)) &\leq \psi(M(x_{2n}, v)) - \varphi(N(x_{2n}, v)) \\ &= \psi\Big(max\big\{ d(Sx_{2n}, Tv), d(Sx_{2n}, Ax_{2n}), d(Tv, Bv), \\ &\frac{1}{2}[d(Sx_{2n}, Bv) + d(Tv, Ax_{2n})]\big\}\Big) - \varphi(N(x_{2n}, v)), \end{aligned} \tag{40}$$

on taking the liminf as $n \to \infty$ and using (39)

$$\begin{aligned} \underline{\lim}_{n\to\infty}\psi(d(Ax_{2n}, Bv)) \leq \underline{\lim}_{n\to\infty}\psi(M(z, Bv)) - \underline{\lim}_{n\to\infty}\varphi(N(x_{2n}, v)) \\ \underline{\lim}_{n\to\infty}\psi\Big(max\big\{ d(Sz, Tv), d(Sz, Av), d(Tv, Bv), \\ \frac{1}{2}[d(Sz, Bv) + d(Tv, Az)]\big\}\Big) - \underline{\lim}_{n\to\infty}\varphi(N(x_{2n}, v)), \end{aligned}$$

by the property of φ function, $\underline{\lim}_{n\to\infty}\varphi(N(x_{2n}, v))$ is positive, then we have

$$\psi(d(z, Bv)) < \psi(d(z, Bv)),$$

by monotone properties of ψ, we get

$$Bv = z. \tag{41}$$

From (39) and (41), we get

$$Az = Sz = Bv = Tv = z. \tag{42}$$

Since (B, T) is weakly compatible, then

$$\begin{aligned} z = Bv = Tv &\Rightarrow BTv = TBv \\ &\Rightarrow Bz = Tz. \end{aligned} \tag{43}$$

Finally, we have to show that z is a common fixed point of B and T.
Taking $x = x_{2n}$ and $y = z$ in inequality (8), then we have

$$\begin{aligned} \psi(d(Ax_{2n}, Bz)) &\leq \psi(M(x_{2n}, z)) - \varphi(N(x_{2n}, z)) \\ &= \psi\Big(max\{d(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n}), d(Tz, Bz), \\ &\frac{1}{2}[d(Sx_{2n}, Bz) + d(Tz, Ax_{2n})]\}\Big) - \varphi(N(x_{2n}, z)), \end{aligned} \tag{44}$$

on taking the liminf as $n \to \infty$, using (42) and (43)

$$\begin{aligned} \underline{\lim}_{n\to\infty}\psi(d(Ax_{2n}, Bz)) &\leq \underline{\lim}_{n\to\infty}\psi(M(x_{2n}, z)) - \underline{\lim}_{n\to\infty}\varphi(N(x_{2n}, z)) \\ &\leq \underline{\lim}_{n\to\infty}\psi\Big(max\{d(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n}), d(Tz, Bz), \\ &\frac{1}{2}[d(Sx_{2n}, Bz) + d(Tz, Ax_{2n})]\}\Big) - \underline{\lim}_{n\to\infty}\varphi(N(x_{2n}, z)), \end{aligned}$$

by the property of φ function, $\underline{\lim}_{n\to\infty}\varphi(N(x_{2n}, z))$ is positive, then we have

$$\psi(d(z, Bz)) < \psi(d(z, Bz)),$$

by monotone properties of ψ, we have

$$Bz = z. \tag{45}$$

By using (42), (43) and (45), we get

$$Az = Sz = Bz = Tz = z. \tag{46}$$

Hence A, B, S and T have a common fixed point in X.
Similarly, we can take $A(X), B(X), T(X)$ is a closed subspace of X.
Uniqueness follow easily from (5). □

Theorem 7 *Let* (X, d) *be a metric space, and A, B, S and* $T : X \to X$*be a mapping satisfying the condition weakly generalized* (ψ, φ)*- weak quasi contraction condition. And* (6)*, the pairs* (A, S) *and* (B, T) *satisfying occasionally weakly compatible. Then A, B, S and T have a unique common fixed point in X, provided any one of the ranges* $A(X), B(X), S(X), T(X)$ *is a closed subspace of X.*

We get the following corollaries.

Corollary 1 *Let* (X, d) *be a complete metric space, and A, B, S and* $T : X \to X$*be a continuous mapping satisfying* (6)

$$\psi(d(Ax, By)) \le \psi(M(x, y)) - \varphi(M(x, y)) \tag{47}$$

for all $x, y, \in X$*, with* $x \neq y$ *and*

$$M(x, y) = \max\left\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}(d(Sx, By) + d(Ty, Ax))\right\},$$

where $\psi \in \Psi$ *and* $\varphi \in \Phi$*. Then A, B, S and T have a unique common fixed point in X.*

Now, the following example is support of our main result.

Example 2 *Let* $X = [0, 3)$ *be endowed with the Euclidean metric* $d(x, y) = |x - y|$*, and let A, B, S and* $T \to X$ *be defined by*

$$A(X) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{x}{5} + 1 & \text{if} \quad x \neq 0 \end{cases} \quad B(X) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{x}{4} + 1 & \text{if} \quad x \neq 0 \end{cases}$$

$$S(X) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{x}{2} + 1 & \text{if} \quad x \neq 0 \end{cases} \quad T(X) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{2x}{3} + 1 & \text{if} \quad x \neq 0 \end{cases}$$

where $x, y \in X$

$$A(X) = \{0\} \cup \left[1, \frac{8}{5}\right) \subset \{0\} \cup [1, 3) = T(X)$$

and

$$B(X) = \{0\} \cup \left[1, \frac{7}{4}\right) \subset \{0\} \cup \left[0, \frac{5}{2}\right) = S(X).$$

Define $\psi(t)$ *and* φ *as follows:*

$$\psi(t) = t^2 \ \forall\, t \in \mathfrak{R}^+,$$

and

$$\varphi(t) = \begin{cases} 0 & \text{if} \quad t = 0 \\ 1 + \frac{t}{2} & \text{if} \quad t > 0 \end{cases}$$

Case 1: *If* $x = 0$ *and* $y = 0$

$$\psi(d(Ax, By)) = 0, \psi(M(x, y)) = 0, \varphi(N(x, y)) = 0,$$

hence equation(8) satisfied.
Case 2: *If* $x = 0$ *and* $y \neq 0$

$$\psi(d(Ax, By)) = \left(\frac{y}{4} + 1\right)^2,$$

and

$$M(x, y) = max\left\{\left|\frac{2y}{3} + 1\right|, 0, \left|\frac{2y}{3} - \frac{y}{4}\right|, \left|\frac{2y}{3} + 1\right|\right\}$$

$$M(x, y) = \left|\frac{2y}{3} + 1\right|,$$

and

$$N(x, y) = \left|\frac{2y}{3} - \frac{y}{4}\right|,$$

$$\psi(M(x, y)) - \varphi(N(x, y)) = \left(\frac{2y}{3} + 1\right)^2 - \left(1 + \frac{5y}{48}\right)$$

$$\psi(M(x, y)) - \varphi(N(x, y)) \geq \psi(d(Ax, By)).$$

Case 3: *If* $x \neq 0$ *and* $y = 0$

$$\psi(d(Ax, By)) = \left(\frac{x}{5} + 1\right)^2,$$

and

$$M(x, y) = max\left\{\left|\frac{2x}{5} + 1\right|, \left|\frac{x}{2} - \frac{x}{5}\right|, 0, \frac{1}{2}\left|\frac{x}{2} + \frac{x}{5}\right|\right\}$$

$$M(x, y) = \left|\frac{2x}{5} + 1\right|,$$

and

$$N(x, y) = \left|\frac{x}{2} - \frac{x}{5}\right|,$$

$$\psi(M(x, y)) - \varphi(N(x, y)) = \left(\frac{2x}{5} + 1\right)^2 - \left(1 + \frac{3x}{20}\right)$$

$$\psi(M(x, y)) - \varphi(N(x, y)) \geq \psi(d(Ax, By)).$$

Case 4: *If* $x \neq 0$ *and* $y \neq 0$

$$\psi(d(Ax, By)) = \left(\frac{x}{5} - \frac{y}{4}\right)^2,$$

and

$$M(x,y) = \max\left\{\left|\frac{2x}{5} - \frac{5y}{3}\right|, \left|\frac{x}{5}\right|, \left|\frac{5y}{12}\right|, \frac{1}{2}\left[\left|\frac{x}{2} - \frac{y}{4}\right| + \left|\frac{2y}{3} - \frac{x}{5}\right|\right]\right\}$$

$$M(x,y) = \left|\frac{5y}{12}\right|,$$

and

$$N(x,y) = \frac{1}{2}\left[\left|\frac{x}{2} - \frac{y}{4}\right| + \left|\frac{2y}{3} - \frac{x}{5}\right|\right],$$

$$\psi(M(x,y)) - \varphi(N(x,y)) = \left(\frac{5y}{12}\right)^2 - \left(1 + \frac{18x + 25y}{240}\right)$$

$$\psi(M(x,y)) - \varphi(N(x,y)) \geq \psi(d(Ax, By)).$$

Hence the inequality holds in each of the cases.

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