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Some generalisations and minimax approximants of D'Aurizio trigonometric inequalities

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Abstract. In this paper, we generalise J. Sándor's results on D'Aurizio's trigonometric inequalities using stratified families of functions.

1 Introduction and preliminaries

In this paper, we give some generalisations of the following results of József Sándor [1] concerning D'Aurizio's trigonometric inequalities:

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Theorem 1 (*J. Sándor*) *For* $0 < |x| < \pi/2$

$$1 - \frac{4}{\pi^2}x^2 < \frac{\cos x}{\cos \frac{x}{2}} < 1 - \frac{3}{8}x^2 \tag{1}$$

holds.

Theorem 2 (*J. Sándor*) *For* $0 < |x| < \pi/2$

$$2 - \frac{1}{4}x^2 < \frac{\sin x}{\sin \frac{x}{2}} < 2 - \frac{4(2-\sqrt{2})}{\pi^2}x^2 \tag{2}$$

holds.

The improved results are obtained using concepts presented in [2]. In this section, the important theorems from [2], which are necessary for further proofs, are listed.

Let

$$\varphi_p(x) : (a, b) \longrightarrow \mathbb{R}$$

be a family of functions with a variable $x \in (a, b)$ and a parameter $p \in \mathbb{R}^+$. In this paper, we call $\sup\limits_{x \in (a,b)} |\varphi_p(x)|$ an error and denote it by:

$$d^{(p)} = \sup_{x \in (a,b)} |\varphi_p(x)| . \tag{3}$$

In [2], the conditions for the existence of the unique value p_0 of the parameter, for which an infimum of an error (as a positive real number) is attained, are explored. Such infimum is denoted by:

$$d_0 = \inf_{p \in \mathbb{R}^+} \sup_{x \in (a,b)} |\varphi_p(x)| . \tag{4}$$

For such a value p_0, the function $\varphi_{p_0}(x)$ is called *the minimax approximant* on (a, b).

A family of functions $\varphi_p(x)$ is increasingly stratified if $p' > p'' \iff \varphi_{p'}(x) > \varphi_{p''}(x)$ for any $x \in (a, b)$ and, conversely, it is decreasingly stratified if $p' > p'' \iff \varphi_{p'}(x) < \varphi_{p''}(x)$ for any $x \in (a, b)$ $(p', p'' \in \mathbb{R}^+)$.

Based on Theorem 1 and Theorem 1' from [2], we can conclude that for stratified families of functions, the following theorem is true:

Theorem 3 *Let $\varphi_p(x)$ be an increasingly (decreasingly) stratified family of functions (for $p \in \mathbb{R}^+$) that are continuous with respect to $x \in (a, b)$ for each $p \in \mathbb{R}^+$, and let c, d be in $\mathbb{R}^+$ such that $c < d$. If:*

(a) *$\varphi_c(x) < 0$ ($\varphi_c(x) > 0$) and $\varphi_d(x) > 0$ ($\varphi_d(x) < 0$) for all $x \in (a, b)$, and at the endpoints $\varphi_c(a+) = \varphi_d(a+) = 0$, $\varphi_c(b-) = 0$ ($\varphi_d(b-) = 0$) and $\varphi_d(b-) \in \mathbb{R}^+$ ($\varphi_c(b-) \in \mathbb{R}^+$) hold;*

(b) *the functions $\varphi_p(x)$ are continuous with respect to $p \in (c, d)$ for each $x \in (a, b)$ and $\varphi_p(b-)$ is continuous with respect to $p \in (c, d)$ too;*

(c) *for all $p \in (c, d)$, there exists a right neighbourhood of point a in which $\varphi_p(x) < 0$ holds and a left neighbourhood of point b in which $\varphi_p(x) > 0$ holds;*

(d) *for all $p \in (c, d)$, the function $\varphi_p(x)$ has exactly one extremum $t^{(p)}$ on (a, b), which is minimum;*

then there exists exactly one solution p_0, for $p \in \mathbb{R}^+$, of the following equation

$$|\varphi_p(t^{(p)})| = \varphi_p(b-)$$

and for $d_0 = |\varphi_{p_0}(t^{(p_0)})| = \varphi_{p_0}(b-)$ we have

$$d_0 = \inf_{p \in \mathbb{R}^+} \sup_{x \in (a, b)} |\varphi_p(x)| .$$

Remark 1 *Theorem 1 in [2] considers the case of an increasingly stratified family of functions, while Theorem 1′ is analogous and considers the case of a decreasingly stratified family of functions. In this paper, both Theorems are unified in Theorem 3 and improved. Specifically, in condition (c), we have added that there exists a left neighbourhood of point b in which $\varphi_p(x) > 0$ holds. Although the theorems in [2] were correct, this addition uniquely defines the function $\varphi_c(x)$ in Theorem 1 and the function $\varphi_d(x)$ in Theorem 1′ from the paper [2].*

Exploring the fulfillment of the conditions for Theorem 3 is often reduced to the following statement [2]:

Theorem 4 (Nike theorem) *Let $f : (0, c) \longrightarrow \mathbb{R}$ be m times differentiable function (for some $m \geq 2$, $m \in \mathbb{N}$) satisfying the following conditions:*

(a) $f^{(m)}(x)>0$ *for* $x\in(0,c)$;

(b) *there is a right neighbourhood of zero in which the following inequalities are true*:

$$f<0,\ f'<0,\ \ldots,\ f^{(m-1)}<0;$$

(c) *there is a left neighbourhood of* c *in which the following inequalities are true*:

$$f>0,\ f'>0,\ \ldots,\ f^{(m-1)}>0.$$

Then the function f *has exactly one zero* $x_0\in(0,c)$, *and* $f(x)<0$ *for* $x\in(0,x_0)$ *and* $f(x)>0$ *for* $x\in(x_0,c)$. *Also, the function* f *has exactly one local minimum* t *on the interval* $(0,c)$. *More precisely, there is exactly one point* $t\in(0,c)$ $\big($*in fact* $t\in(0,x_0)\big)$ *such that* $f(t)<0$ *is the smallest value of the function* f *on the interval* $(0,c)$ *and particularly on* $(0,x_0)$.

2 Main results

In this section, some generalisations of Theorems 1 and 2 are given.

Generalisation of Theorem 1

First, we give some auxiliary results.

Lemma 1 *The family of functions*

$$\varphi_p(x)=1-\frac{\cos x}{\cos\frac{x}{2}}-p\,x^2 \qquad \Big(\textit{for } x\in(0,\pi/2)\Big)$$

is decreasingly stratified with respect to parameter $p\in\mathbb{R}^+$.

The family of functions $\varphi_p(x)$, introduced in the previous lemma, is formed based on the double inequality from Theorem 1 for parameter values $p=\frac{4}{\pi^2}$ and $p=\frac{3}{8}$, as will be discussed in the following analysis. With that aim, we introduce the function

$$g(x)=\frac{-2\cos^2\frac{x}{2}+\cos\frac{x}{2}+1}{x^2\cos\frac{x}{2}} \qquad \Big(\text{for } x\in(0,\pi/2)\Big)$$

which is strictly increasing, while $g(0+) = \dfrac{3}{8}$ and $g(\pi/2) = \dfrac{4}{\pi^2}$ hold [1]. Obviously,

$$\varphi_p(x) = 0 \Leftrightarrow p = g(x)\,.$$

Now we give the main results for the first generalisation:

Statement 1 *Let*

$$A = \frac{3}{8} = 0.375 \quad \textit{and} \quad B = \frac{4}{\pi^2} = 0.40528\ldots\,.$$

$(\mathbf{i})$ *If* $p \in (0, A]$*, then*

$$x \in \left(0, \frac{\pi}{2}\right) \implies \frac{\cos x}{\cos \dfrac{x}{2}} < 1 - A\,x^2 < 1 - p\,x^2\,.$$

$(\mathbf{ii})$ *If* $p \in (A, B)$*, then* $\varphi_p(x)$ *has exactly one zero* $x_0^{(p)}$ *on* $(0, \pi/2)$*. Also,*

$$x \in \left(0, x_0^{(p)}\right) \implies \frac{\cos x}{\cos \dfrac{x}{2}} > 1 - p\,x^2$$

and

$$x \in \left(x_0^{(p)}, \frac{\pi}{2}\right) \implies \frac{\cos x}{\cos \dfrac{x}{2}} < 1 - p\,x^2$$

hold.

$(\mathbf{iii})$ *If* $p \in [B, \infty)$*, then*

$$x \in \left(0, \frac{\pi}{2}\right) \implies \frac{\cos x}{\cos \dfrac{x}{2}} > 1 - B\,x^2 > 1 - p\,x^2\,.$$

Proof. The function $g(x)$ is increasing, continuous and surjection on (A, B), see [1]. It is obvious that

$$g(x) - p = \frac{\varphi_p(x)}{x^2}$$

holds. Therefore, $g(x) \neq p$ (i.e. $\varphi_p(x) \neq 0$) holds on $(0, \frac{\pi}{2})$ if $p \in (0, A]$ or $p \in [B, +\infty)$. We can easily see that $\varphi_A(\pi/2) > 0$ and $\varphi_B(\pi/3) < 0$. Hence, $\varphi_A(x) > 0$ for $x \in (0, \frac{\pi}{2})$ and $\varphi_B(x) < 0$ for $x \in (0, \frac{\pi}{2})$. Then (i) and (iii) follow from the decreasing stratification of the family $\varphi_p(x)$. Furthermore, for $p \in (A, B)$, the equation $g(x) = p$ has exactly one solution, which we denote by $x_0^{(p)}$, while $g(x) < p$ for $x \in (0, x_0^{(p)})$ and $g(x) > p$ for $x \in (x_0^{(p)}, \frac{\pi}{2})$. Hence, (ii) is true. □

Corollary 1 *For any* $0 < x < \pi/2$

$$1 - \frac{4}{\pi^2}x^2 < \frac{\cos x}{\cos \frac{x}{2}} < 1 - \frac{3}{8}x^2$$

holds, with the best possible constants $A = \dfrac{3}{8} = 0.375$ *and* $B = \dfrac{4}{\pi^2} = 0.40528\ldots$.

Statement 2 *Let*

$$\varphi_p(x) = 1 - \frac{\cos x}{\cos \frac{x}{2}} - p\,x^2 \;\; \textit{for } x \in (0, \tfrac{\pi}{2}) \textit{ and } p \in \mathbb{R}^+.$$

$(\mathbf{i})$ *For* $p \in (A, B)$, *there exists only one extremum of this function on* $(0, \frac{\pi}{2})$ *at* $t^{(p)}$ *and that extremum is minimum.*

$(\mathbf{ii})$ *There is exactly one solution to the equation*

$$\left|\varphi_p\left(t^{(p)}\right)\right| = \varphi_p\left(\frac{\pi}{2}-\right)$$

with the respect to parameter $p \in (A, B)$, *which can be determined numerically as*

$$p_0 = 0.39916\ldots$$

For the value

$$d_0 = \left|\varphi_{p_0}\left(t^{(p_0)}\right)\right| = \varphi_{p_0}\left(\frac{\pi}{2}-\right) = 0.015109\ldots,$$

the following result

$$d_0 = \inf_{p \in \mathbb{R}^+} \sup_{x \in (0,\pi/2)} |\varphi_p(x)|$$

holds.

$(\mathbf{iii})$ *The minimax approximant of the family* $\varphi_p(x)$ *is*

$$\varphi_{p_0}(x) = 1 - \frac{\cos x}{\cos \frac{x}{2}} - p_0\,x^2,$$

which determines the corresponding minimax approximation

$$\frac{\cos x}{\cos \frac{x}{2}} \approx 1 - 0.39916\,x^2 .$$

Proof. For $p \in (A, B)$, functions $\varphi_p(x)$ fulfill the conditions of Theorem 4 (Nike theorem):

(a) For $m = 3$

$$\varphi_p'''(x) = \frac{d^3\varphi_p}{d\,x^3} = \frac{1}{8}\frac{\left(6 - 2\cos^4\frac{x}{2} - \cos^2\frac{x}{2}\right)\sin\frac{x}{2}}{\cos^4\frac{x}{2}} > 0 \qquad \big(x \in (0, \pi/2)\big).$$

(b) Based on the Taylor expansions of the functions $\varphi_p(x)$ around $x = 0$:

$$\varphi_p(x) = \left(\frac{3}{8} - p\right)x^2 + \frac{1}{128}x^4 + o(x^4), \tag{5}$$

there exists a right neighbourhood $\mathcal{U}_0$ of the point 0 such that

$$\varphi_p(x), \varphi_p'(x) = \frac{d\varphi_p}{d\,x}, \varphi_p''(x) = \frac{d^2\varphi_p}{d\,x^2} < 0 \qquad \big(x \in \mathcal{U}_0\big).$$

(c) Based on the Taylor expansions of the functions $\varphi_p(x)$ around $x = \dfrac{\pi}{2}$:

$$\begin{aligned}\varphi_p(x) = {} & \left(1 - \frac{p\,\pi^2}{4}\right) + \left(-p\,\pi + \sqrt{2}\right)\left(x - \frac{\pi}{2}\right) + \\ & + \left(\frac{\sqrt{2}}{2} - p\right)\left(x - \frac{\pi}{2}\right)^2 + \frac{5\sqrt{2}}{24}\left(x - \frac{\pi}{2}\right)^3 + o\left(\left(x - \frac{\pi}{2}\right)^3\right),\end{aligned} \tag{6}$$

there exists a left neighbourhood $\mathcal{U}_{\pi/2}$ of the point $\pi/2$ such that

$$\varphi_p(x), \varphi_p'(x) = \frac{d\varphi_p}{d\,x}, \varphi_p''(x) = \frac{d^2\varphi_p}{d\,x^2} > 0 \qquad \big(x \in \mathcal{U}_{\pi/2}\big).$$

Based on Theorem 3, for $p \in (A, B)$, we can conclude that each function $\varphi_p(x)$ has exactly one extremum $t^{(p)}$, which is minimum, on $(0, \frac{\pi}{2})$ $\big($and thus exactly one zero $x_0^{(p)}$ on $(0, \frac{\pi}{2})\big)$.

The family of functions $\varphi_p(x)$, for values $p \in (A, B)$, fulfills the conditions of Theorem 3, thereby there exists a minimax approximant. Numerical determination of the minimax approximant and the error can be calculated in Maple in the manner we present here. Let $f(x, p) := \varphi_p(x)$ and $F(x, p) := \varphi_p'(x)$. With Maple code

```
fsolve({F(x,p) = 0, abs(f(x,p)) = f(π/2,p)}, {x = 0..π/2, p = A..B});
```

we have numerical values

$$\{p = 0.399161163, x = 1.069252853\}.$$

For the value $p_0 = 0.39916\ldots$, we have the minimax approximant of the family

$$\varphi_{p_0}(x) = 1 - \frac{\cos x}{\cos \frac{x}{2}} - p_0\, x^2$$

and numerical value of minimax error

$$d_0 = f(\pi/2, p_0) = 0.015109\ldots. \qquad \square$$

Figure 1 illustrates the stratified family of functions from Lemma 1 for $p \in \mathbb{R}^+$.

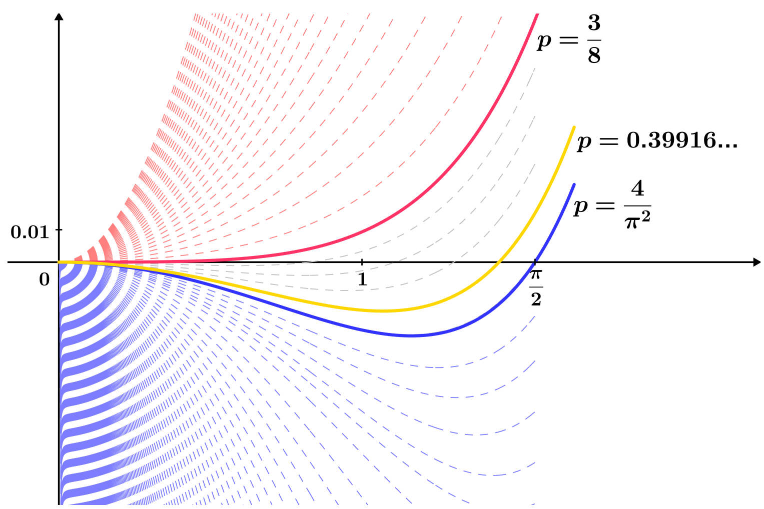

Figure 1: Stratified family of functions from Lemma 1

Generalisation of Theorem 2

First, we give some auxiliary results.

Lemma 2 *The family of functions*

$$\varphi_p(x) = -2 + \frac{\sin x}{\sin \frac{x}{2}} + p\, x^2 \qquad \Big(\textit{for } x \in (0, \pi/2)\Big)$$

is increasingly stratified with respect to parameter $p \in \mathbb{R}^+$.

The family of functions $\varphi_p(x)$, introduced in the previous lemma, is formed based on the double inequality from Theorem 2 for parameter values $p = \frac{1}{4}$ and $p = \frac{8-4\sqrt{2}}{\pi^2}$, as will be discussed in the following analysis. With that aim, we introduce the function

$$g(x) = \frac{2\left(1-\cos\frac{x}{2}\right)}{x^2} \qquad \Big(\text{for } x \in (0, \pi/2)\Big)$$

which is strictly decreasing, while $g(0+) = \dfrac{1}{4}$ and $g(\pi/2-) = \dfrac{8-4\sqrt{2}}{\pi^2}$ hold. Further,

$$\varphi_p(x) = 0 \Leftrightarrow p = g(x)\,.$$

holds, as in the previous case.

Now we give the main results for the second generalisation:

Statement 3 *Let*

$$A = \frac{8-4\sqrt{2}}{\pi^2} = 0.23741\ldots \quad \textit{and} \quad B = \frac{1}{4} = 0.25.$$

(i) *If* $p \in (0, A]$, *then*

$$x \in \left(0, \frac{\pi}{2}\right) \implies \frac{\sin x}{\sin\frac{x}{2}} < 2 - A\,x^2 < 2 - p\,x^2\,.$$

(ii) *If* $p \in (A, B)$, *then* $\varphi_p(x)$ *has exactly one zero* $x_0^{(p)}$ *on* $(0, \frac{\pi}{2})$. *Also,*

$$x \in \left(0, x_0^{(p)}\right) \implies \frac{\sin x}{\sin\frac{x}{2}} < 2 - p\,x^2$$

and

$$x \in \left(x_0^{(p)}, \frac{\pi}{2}\right) \implies \frac{\sin x}{\sin\frac{x}{2}} > 2 - p\,x^2$$

hold.

(iii) *If* $p \in [B, \infty)$, *then*

$$x \in \left(0, \frac{\pi}{2}\right) \implies \frac{\sin x}{\sin\frac{x}{2}} > 2 - B\,x^2 > 2 - p\,x^2\,.$$

Proof. The function $g(x)$ is increasing, continuous and surjection on (A, B), and

$$g(x) - p = -\frac{\varphi_p(x)}{x^2}$$

holds. Hence, $\varphi_p(x) \neq 0$ holds on $(0, \frac{\pi}{2})$ if $p \in (0, A]$ or $p \in [B, +\infty)$. It can be checked that $\varphi_A(\pi/3) < 0$ and $\varphi_B(\pi/2) > 0$, which means that $\varphi_A(x) < 0$ for $x \in (0, \frac{\pi}{2})$ and $\varphi_B(x) > 0$ for $x \in (0, \frac{\pi}{2})$. Then (i) and (iii) follow from the increasing stratification of the family $\varphi_p(x)$. For $p \in (A, B)$, the equation $g(x) = p$ has exactly one solution which we denote by $x_0^{(p)}$, while $g(x) > p$ for $x \in (0, x_0^{(p)})$ and $g(x) < p$ for $x \in (x_0^{(p)}, \frac{\pi}{2})$. Hence, (ii) holds. □

Corollary 2 *For any* $0 < x < \pi/2$

$$2 - \frac{1}{4}x^2 < \frac{\sin x}{\sin \frac{x}{2}} < 2 - \frac{4(2-\sqrt{2})}{\pi^2}x^2.$$

holds, with the best possible constants $A = \dfrac{8-4\sqrt{2}}{\pi^2} = 0.23741\ldots$ *and* $B = \dfrac{1}{4} = 0.25.$

Statement 4 *Let*

$$\varphi_p(x) = -2 + \frac{\sin x}{\sin \frac{x}{2}} + p\,x^2 \ \text{ for } x \in (0, \tfrac{\pi}{2}) \text{ and } p \in \mathbb{R}^+.$$

$(\mathbf{i})$ *For* $p \in (A, B)$, *there exists only one extremum of this function on* $(0, \frac{\pi}{2})$ *at* $t^{(p)}$ *and that extremum is minimum.*
$(\mathbf{ii})$ *There is exactly one solution to the equation*

$$\left|\varphi_p\left(t^{(p)}\right)\right| = \varphi_p\left(\frac{\pi}{2}-\right),$$

where $t^{(p)}$ *is a unique local minimum of* $\varphi_p(x)$, *by parameter* $p \in (A, B)$, *which we determine numerically as*

$$p_0 = 0.23955\ldots$$

For the value

$$d_0 = \left|\varphi_{p_0}\left(t^{(p_0)}\right)\right| = \varphi_{p_0}\left(\frac{\pi}{2}-\right) = 0.0052842\ldots,$$

the following result

$$d_0 = \inf_{p\in\mathbb{R}^+} \sup_{x\in(0,\pi/2)} |\varphi_p(x)|$$

holds.

(**iii**) *The minimax approximant of the family* $\varphi_p(x)$ *is*

$$\varphi_{p_0}(x) = -2 + \frac{\sin x}{\sin \frac{x}{2}} + p_0\, x^2,$$

which determines the corresponding minimax approximation

$$\frac{\sin x}{\sin \frac{x}{2}} \approx 2 - 0.23955\, x^2\,.$$

Proof. For $p \in (A, B)$, functions $\varphi_p(x)$ fulfill the conditions of Theorem 4 (Nike theorem):

(a) For $m = 3$

$$\varphi_p'''(x) = \frac{d^3\varphi_p}{d\,x^3} = \frac{1}{4}\sin\frac{x}{2} > 0 \qquad \big(x\in(0,\pi/2)\big).$$

(b) Based on the Taylor expansion of the functions $\varphi_p(x)$ around $x{=}0$:

$$\varphi_p(x) = \left(-\frac{1}{4} + p\right)x^2 + \frac{1}{192}x^4 + o(x^4) \tag{7}$$

there exists a right neighbourhood $\mathcal{U}_0$ of the point 0 such that

$$\varphi_p(x), \varphi_p'(x) = \frac{d\varphi_p}{d\,x}, \varphi_p''(x) = \frac{d^2\varphi_p}{d\,x^2} < 0 \qquad \big(x\in\mathcal{U}_0\big).$$

(c) Based on the Taylor expansion of the functions $\varphi_p(x)$ around $x{=}\dfrac{\pi}{2}$:

$$\begin{aligned}\varphi_p(x) = {} & \left(-2 + \sqrt{2} + \frac{p\,\pi^2}{4}\right) + \left(p\,\pi - \frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{2}\right) + \\ & + \left(p - \frac{\sqrt{2}}{8}\right)\left(x - \frac{\pi}{2}\right)^2 + \frac{\sqrt{2}}{48}\left(x - \frac{\pi}{2}\right)^3 + o\left(\left(x - \frac{\pi}{2}\right)^3\right)\end{aligned} \tag{8}$$

there exists a left neighbourhood $\mathcal{U}_{\pi/2}$ of the point $\pi/2$ such that it is

$$\varphi_p(x), \varphi'_p(x) = \frac{d\varphi_p}{dx}, \varphi''_p(x) = \frac{d^2\varphi_p}{dx^2} > 0 \qquad \big(x \in \mathcal{U}_{\pi/2}\big).$$

Based on Theorem 3, for $p \in (A, B)$, we can conclude that functions $\varphi_p(x)$ has exactly one extremum $t^{(p)}$, which is minimum, on $(0, \frac{\pi}{2})$ $\big($and thus exactly one zero $x_0^{(p)}$ on $(0, \frac{\pi}{2})\big)$.

The family of functions $\varphi_p(x)$, for values $p \in (A, B)$, fulfills the conditions of Theorem 3, thereby there exists a minimax approximant. Numerical determination of the minimax approximant and the error can be calculated in Maple in the manner we present here. Let $f(x, p) := \varphi_p(x)$ and $F(x, p) := \varphi'_p(x)$. With Maple code

```
fsolve({F(x,p) = 0, abs(f(x,p)) = f(π/2,p)}, {x = 0..π/2, p = A..B});
```

we have numerical values

$$\{p = 0.2395519170, x = 1.007887451\}.$$

For the value $p_0 = 0.23955\ldots$, we have the minimax approximant of the family

$$\varphi_{p_0}(x) = -2 + \frac{\sin x}{\sin \frac{x}{2}} + p_0\, x^2$$

and numerical value of minimax error

$$d_0 = f(\pi/2, p_0) = 0.0052842\ldots.$$

□

Figure 2 illustrates the stratified family of functions from Lemma 2 for $p \in \mathbb{R}^+$.

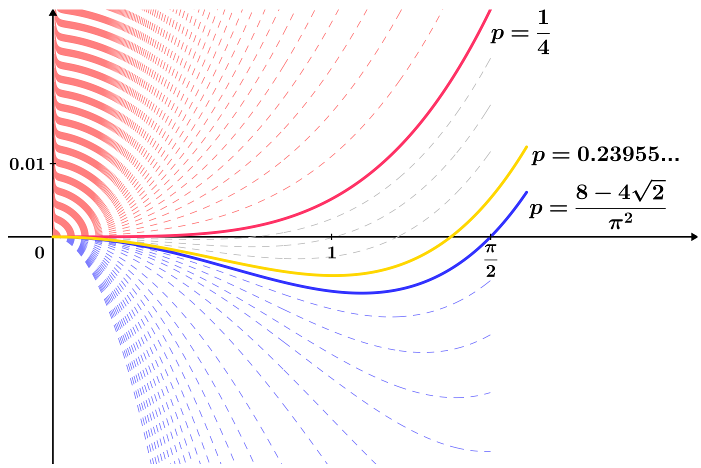

Figure 2: Stratified family of functions from Lemma 2

3 Conclusion

This paper specifies the results of J. Sándor [1] related to D'Aurizio's trigonometric inequality [8] using concepts from the paper [2]. Additionally, Theorems 1 and 1' from [2] were improved. Let us emphasize that the paper [2] presents one method for possible improvements of existing results in the Theory of analytic inequalities in terms of determining the corresponding minimax approximants for many inequalities from reviewed papers [6], [7], and books [3]-[5]. The concept of stratification is used in recent research to improve and generalise some inequalities, see [11]-[14], and can be used to improve many more from [3]-[5], [10], [15]-[21]. In further papers, the subject of our studies will be to determine the appropriate minimax approximants for papers [9] and [10] relating to the generalizations of D'Aurizio's trigonometric inequalities.

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