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Talenti's comparison theorem on Finsler manifolds with nonnegative Ricci curvature

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Abstract. We establish a Talenti-type comparison theorem for the Dirichlet problem associated with Poisson's equation on complete non-compact Finsler manifolds having nonnegative Ricci curvature and Euclidean volume growth. The proof relies on anisotropic symmetrization arguments and leverages the sharp isoperimetric inequality recently established by Manini [*Preprint, arXiv:2212.05130, 2022*]. In addition, we characterize the rigidity of the comparison principle under the additional assumption that the reversibility constant of the Finsler manifold is finite. As application, we prove a Faber-Krahn inequality for the first Dirichlet eigenvalue of the Finsler-Laplacian.

1 Introduction

Talenti's comparison theorem [23] is a fundamental result that establishes a relationship between the solutions of two elliptic boundary value problems: the Poisson equation with Dirichlet boundary condition and a so-called 'symmetrized' problem of similar kind. More precisely, given a bounded domain

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$\Omega \subset \mathbb{R}^n$ and a nonnegative function $f \in L^2(\Omega)$, one might consider the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

and its 'symmetrized' counterpart

$$\begin{cases} -\Delta v = f^\star & \text{in } \Omega^\star, \\ v = 0 & \text{on } \partial\Omega^\star, \end{cases} \tag{2}$$

where $\Omega^\star$ denotes the Euclidean open ball centered at the origin and having the same Lebesgue measure as Ω, while $f^\star : \Omega^\star \to \mathbb{R}$ is the Schwarz rearrangement of f, see Kesavan [13, Chapter 1].

According to Talenti [23], if u and v are the weak solutions of the problems (1) and (2), respectively, then one has that

$$u^\star(x) \leq v(x), \quad \text{a.e. } x \in \Omega^\star,$$

where $u^\star : \Omega^\star \to \mathbb{R}$ is the Schwarz rearrangement of u.

The key ingredient of Talenti's proof is the classical technique known as Schwarz symmetrization, which turns out to be an invaluable method in addressing numerous isoperimetric and variational problems in the Euclidean space. For example, with the help of this symmetrization procedure, Talenti's comparison principle has been extended to several boundary value problems, see e.g., Alvino, Ferone and Trombetti [3], Alvino, Lions and Trombetti [2], Alvino, Nitsch and Trombetti [3], and Talenti [24]. For a comprehensive introduction to Talenti's technique and its countless applications, we refer to Kesavan [13] and references therein.

Recently, there has been an increasing endeavor to study similar comparison results on complete Riemannian manifolds having Ricci curvature bounded from below, see Chen and Li [8], Chen, Li and Wei [9], Colladay, Langford and McDonald [11], and Mondino and Vedovato [16].

In particular, in 2023 Chen and Li [8] extended Talenti's original comparison result to complete noncompact Riemannian manifolds having nonnegative Ricci curvature and Euclidean volume growth. In their proof, they applied a Schwarz-type symmetrization method 'from the manifold (M, g) to the Euclidean space $(\mathbb{R}^n, |\cdot|)$', obtaining a comparison result between the solution of the Dirichlet problem

$$\begin{cases} -\Delta_g u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

defined in (M, g), where $\Omega \subset M$ is a bounded domain, $f \in L^2(\Omega)$ is a nonnegative function and Δ_g denotes the Laplace-Beltrami operator induced by the Riemannian metric g, and the solution of the 'symmetrized' problem

$$\begin{cases} -\Delta v = f^\star & \text{in } \Omega^\star, \\ v = 0 & \text{on } \partial\Omega^\star, \end{cases} \tag{4}$$

which is defined on the Euclidean symmetric rearrangement of Ω, namely $\Omega^\star \subset \mathbb{R}^n$.

More precisely, for the given bounded domain $\Omega \subset M$ from (3), one can consider the Euclidean open ball $\Omega^\star \subset \mathbb{R}^n$, which is centered at the origin and $\text{Vol}_g(\Omega) = \mathsf{AVR}_g \text{Vol}_e(\Omega^\star)$. Here, Vol_g and Vol_e stand for the Riemannian volume induced by the metric g and the canonical Euclidean volume, while AVR_g denotes the positive asymptotic volume ratio of (M, g). Furthermore, $f^\star : \Omega^\star \mapsto [0, \infty)$ stands for the Euclidean rearrangement function of f.

Then, due to Chen and Li [8, Theorem 1.1], if u and v are the weak solutions of problems (3) and (4), respectively, then $u^\star(x) \leq v(x)$, a.e. $x \in \Omega^\star$, where $u^\star : \Omega^\star \mapsto \mathbb{R}$ is the Euclidean rearrangement of u. Moreover, equality holds for a.e. $x \in \Omega^\star$ if and only if (M, g) is isometric to the Euclidean space $\mathbb{R}^n$ endowed with its canonical metric and Ω is isoperimetric to the Euclidean ball $\Omega^\star$.

In these types of symmetrization results, a crucial element lies in the application of the sharp isoperimetric inequality. This inequality was recently established for Riemannian manifolds having nonnegative Ricci curvature and Euclidean volume growth by Brendle [6] and, alternatively, by Balogh and Kristály [4], facilitating the application of symmetrization arguments on these curved spaces.

The sharp isoperimetric inequality was newly extended to potentially non-reversible Finsler manifolds with nonnegative n-Ricci curvature and Euclidean volume growth by Manini [15], who also characterized the inequality's rigidity property. This breakthrough enables the utilization of rearrangement arguments within the broader framework of Finsler geometry. However, in order to fully integrate the general Finslerian context, the rearrangement needs to be performed 'from the Finsler manifold to a Minkowski normed space', laying the foundation for a so-called anisotropic (or convex) symmetrization.

In light of these results, the main objective of the present paper is to extend the Talenti-type comparison result of Chen and Li [8] to complete Finsler manifolds having nonnegative n-Ricci curvature and Euclidean volume growth. This is achieved by the adaptation of the usual Euclidean rearrangement technique to the Finslerian context and the application of the sharp isoperimetric inequality available on Finsler manifolds. Moreover, we also prove a rigidity property of the comparison principle in the spirit of Manini [15].

2 Main results

To present our findings, let (M, F) be a noncompact, complete $n(\geq 2)$-dimensional Finsler manifold with $\mathsf{Ric}_n \geq 0$, equipped with the induced Finsler metric $d_F : M \times M \to \mathbb{R}$ and the Busemann-Hausdorff volume form $d\nu_F$. Let $r_F \in [1, \infty]$ denote the reversibility constant of (M, F), see Section 3.

The asymptotic volume ratio of (M, F) is expressed as

$$\mathsf{AVR}_F = \lim_{r\to\infty} \frac{\mathrm{Vol}_F(B_F(x, r))}{\omega_n r^n},$$

where $B_F(x, r) = \{z \in M : d_F(x, z) < r\}$ denotes the forward geodesic ball centered at a fixed $x \in M$ with radius $r > 0$, $\omega_n = \pi^{\frac{n}{2}}/\Gamma(1 + \frac{n}{2})$ denotes the volume of the n-dimensional Euclidean open unit ball, and $\mathrm{Vol}_F(S) = \int_S d\nu_F$, for any measurable set $S \subset M$. Since $\mathsf{Ric}_n \geq 0$, due to the Bishop-Gromov volume comparison theorem (see Shen [22, Theorem 1.1]), we have that $\mathsf{AVR}_F \in [0, 1]$.

We suppose that (M, F) exhibits Euclidean volume growth, i.e., $\mathsf{AVR}_F > 0$.

On a bounded domain $\Omega \subset M$, we consider the Dirichlet problem

$$\begin{cases} -\Delta_F u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}$}$$

where Δ_F is the Finsler-Laplace operator defined on (M, F) and $f \in L^2(\Omega)$ is a nonnegative function.

Now let $(\mathbb{R}^n, H)$ be a reversible Finsler manifold equipped with the canonical volume form $d\nu_H$, such that H is a normalized Minkowski norm, i.e., the set

$$W_H(1) := \{x \in \mathbb{R}^n : H(x) < 1\}$$

has measure $\mathrm{Vol}_H(W_H(1)) = \mathrm{Vol}_e(W_H(1)) = \omega_n$.

We consider the anisotropic rearrangement of $\Omega \subset M$ with respect to (w.r.t.) the norm H, which is defined as a Wulff-shape

$$\Omega_H^\star = \{x \in \mathbb{R}^n : H(x) < R\}$$

for some $R > 0$ such that $\mathrm{Vol}_F(\Omega) = \mathsf{AVR}_F \mathrm{Vol}_H(\Omega_H^\star)$.

Our main result is a Talenti-type comparison principle concerning the solution of problem ($\mathcal{P}$) and the Dirichlet problem

$$\begin{cases} -\Delta_{H^*} v = f_H^\star & \text{in } \Omega_H^\star, \\ v = 0 & \text{on } \partial\Omega_H^\star, \end{cases} \tag{$\mathcal{P}^\star$}$$

where $H^* : \mathbb{R}^n \to \mathbb{R}$,

$$H^*(x) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{\langle \xi, x \rangle}{H(\xi)}$$

is the dual norm (i.e., the polar transform) of H, Δ_{H^*} is the Finsler-Laplace operator associated to H^*, and $f_H^\star : \Omega_H^\star \to [0, \infty)$ is the anisotropic rearrangement of f w.r.t. the norm H. Namely,

$$f_H^\star(x) = w\left(\mathsf{AVR}_F \omega_n H(x)^n\right)$$

for some nonincreasing function $w : [0, \mathrm{Vol}_F(\Omega)] \to [0, \infty)$, such that for every $t \geq 0$,

$$\mathrm{Vol}_F\big(\{x \in \Omega : f(x) > t\}\big) = \mathsf{AVR}_F \cdot \mathrm{Vol}_H\big(\{x \in \Omega_H^\star : f_H^\star(x) > t\}\big)$$

holds true, see Section 4.

Specifically, we have the following theorem.

Theorem 1 *Let (M, F) be a noncompact, complete n-dimensional Finsler manifold with $\mathsf{Ric}_n \geq 0$, $n \geq 2$ and $\mathsf{AVR}_F > 0$. Assume that $\Omega \subset M$ is a bounded domain and $f \in L^2(\Omega)$ is a nonnegative function. Let $H : \mathbb{R}^n \to [0, \infty)$ be an absolutely homogeneous, normalized Minkowski norm, and $H^* : \mathbb{R}^n \to [0, \infty)$ be its dual norm. Finally, let $\Omega_H^\star$ and $f_H^\star$ be the anisotropic rearrangement w.r.t. H of the set Ω and the function f. If $u : \Omega \to \mathbb{R}$ and $v : \Omega_H^\star \to \mathbb{R}$ are the weak solutions to problems ($\mathcal{P}$) and ($\mathcal{P}^\star$), respectively, then we have that*

$$u_H^\star(x) \leq v(x), \qquad a.e.\ x \in \Omega_H^\star, \tag{5}$$

where $u_H^\star : \Omega_H^\star \to \mathbb{R}$ is the anisotropic rearrangement of u w.r.t. the norm H.

If, in addition, we suppose that $r_F < \infty$ and for all $x_1, x_2 \notin \partial M$ and for all geodesics $\gamma : [0, 1] \to M$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$, it holds that $\gamma(t) \notin \partial M$, for all $t \in [0, 1]$, then we have the following rigidity property.

If equality holds in (5), then there exists (a unique) $x_0 \in M$ such that, up to a negligible set, $\Omega = B_F(x_0, r)$ with $r = \left(\frac{\mathrm{Vol}_F(\Omega)}{\mathsf{AVR}_F \omega_n}\right)^{\frac{1}{n}}$.

Moreover, the Busemann-Hausdorff measure $\mathrm{d}\nu_F$ *has the following representation:*

$$\mathrm{d}\nu_F = \int_{\partial B_F(x_0,r)} \mathfrak{m}_\alpha \mathfrak{q}(\mathrm{d}\alpha), \quad \mathfrak{q} \in \mathcal{P}\left(\partial B_F(x_0, r)\right), \quad \mathfrak{m}_\alpha \in \mathcal{M}_+(M),$$

where $\mathfrak{m}_\alpha$ *is concentrated on the geodesic ray from* x_0 *through* α*, and* $\mathfrak{m}_\alpha$ *can be identified with* $n\omega_n \mathsf{AVR}_F t^{n-1}\mathcal{L}^1 \llcorner_{[0,\infty)}$.

The proof relies on an anisotropic rearrangement argument similar to the one outlined in Kristály, Mester and Mezei [14], along with the sharp and rigid isoperimetric inequality due to Manini [15]. Although the general strategy of the Schwarz-type symmetrization is well-established (see Talenti [23] or Chen and Li [8]), a meticulous adaptation is necessary in order to fully characterize the Finslerian setting.

As a consequence of Theorem 1, we derive the following Faber-Krahn-type inequality concerning the first Dirichlet eigenvalue of the Finsler-Laplacian Δ_F. For similar eigenvalue comparison results, refer to the works of Ge and Shen [12] and Yin and He [25]. Here, we introduce an alternative approach by using anisotropic rearrangement and the Talenti comparison result. In addition, we also provide a characterization of the equality case, which follows directly from Theorem 1.

Theorem 2 *Let* (M, F) *be a noncompact, complete* n*-dimensional Finsler manifold with* $\mathsf{Ric}_n \geq 0$*,* $n \geq 2$ *and* $\mathsf{AVR}_F > 0$*. Assume that* $\Omega \subset M$ *is a bounded domain. Let* $H : \mathbb{R}^n \to [0, \infty)$ *be an absolutely homogeneous, normalized Minkowski norm,* $H^* : \mathbb{R}^n \to [0, \infty)$ *its dual norm, and* $\Omega_H^\star \subset \mathbb{R}^n$ *the anisotropic rearrangement of* Ω *w.r.t. the norm* H.

Let us consider the eigenvalue problem

$$\begin{cases} -\Delta_F u = \lambda_1(\Omega) u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{EP}$}$$

where $\lambda_1(\Omega)$ *denotes the first Dirichlet eigenvalue of the Finsler-Laplacian* Δ_F.

Then, we have that

$$\lambda_1\left(\Omega_H^\star\right) \leq \lambda_1(\Omega), \tag{6}$$

where $\lambda_1\left(\Omega_H^\star\right)$ *is the first eigenvalue associated with the eigenvalue problem*

$$\begin{cases} -\Delta_{H^*} v = \lambda_1(\Omega_H^\star) v, & \text{in } \Omega_H^\star, \\ v = 0, & \text{on } \partial\Omega_H^\star. \end{cases} \tag{$\mathcal{EP}^\star$}$$

If, in addition, we suppose that $r_F < \infty$ *and for all* $x_1, x_2 \notin \partial M$ *and for all geodesics* $\gamma : [0,1] \to M$ *with* $\gamma(0) = x_1$ *and* $\gamma(1) = x_2$*, it holds that* $\gamma(t) \notin \partial M$*, for all* $t \in [0,1]$*, then we have the following rigidity property.*

If equality holds in (6), then there exists (a unique) $x_0 \in M$ *such that, up to a negligible set,* $\Omega = B_F(x_0, r)$ *with* $r = \left(\frac{\mathrm{Vol}_F(\Omega)}{\mathsf{AVR}_F \omega_n}\right)^{\frac{1}{n}}$. *Moreover, the Busemann-Hausdorff measure* dv_F *has the following representation:*

$$dv_F = \int_{\partial B_F(x_0,r)} \mathfrak{m}_\alpha \mathfrak{q}(d\alpha), \quad \mathfrak{q} \in \mathcal{P}\left(\partial B_F(x_0, r)\right), \quad \mathfrak{m}_\alpha \in \mathcal{M}_+(M),$$

where $\mathfrak{m}_\alpha$ *is concentrated on the geodesic ray from* x_0 *through* α*, and* $\mathfrak{m}_\alpha$ *can be identified with* $n\omega_n \mathsf{AVR}_F t^{n-1} \mathcal{L}^1 \llcorner_{[0,\infty)}$.

The paper is organized as follows. In Section 3 we briefly present the fundamental notions of Finsler geometry that are used throughout the paper. Section 4 recalls the sharp isoperimetric inequality due to Manini [15], then presents the anisotropic rearrangement method applied in our arguments. Finally, in Section 5 we present the proof of Theorem 1 and 2.

3 Preliminaries on Finsler geometry

This section summarizes the fundamental notions of Finsler geometry necessary for our further developments. For a comprehensive presentation of the subject, see Bao, Chern and Shen [5], Ohta and Sturm [18] and Shen [21].

Let M be a connected n-dimensional differentiable manifold and $TM = \cup_{x \in M}\left\{(x,y) : y \in T_x M\right\}$ the tangent bundle of M.

The pair (M, F) is called a Finsler manifold if $F : TM \to [0,\infty)$ is a continuous function such that

(i) F is C^∞ on $TM \setminus \{0\}$;

(ii) $F(x, \lambda y) = \lambda F(x,y)$ for all $\lambda \geq 0$ and all $(x,y) \in TM$;

(iii) the $n \times n$ Hessian matrix $\left(g_{ij}(x,y)\right) := \left(\frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j} F^2(x,y)\right)$ is positive definite for all $(x,y) \in TM \setminus \{0\}$.

Note that in general, $F(x,y) \neq F(x,-y)$. If (M,F) is a Finsler manifold such that $F(x, \lambda y) = |\lambda| F(x,y)$, for every $\lambda \in \mathbb{R}$ and $(x,y) \in TM$, we say that the Finsler manifold is reversible. Otherwise, (M,F) is called nonreversible.

The reversibility constant of (M, F) is defined by the number

$$r_F = \sup_{x \in M} \sup_{y \in T_x M \setminus \{0\}} \frac{F(x, y)}{F(x, -y)} \in [1, \infty],$$

measuring how much the manifold deviates from being reversible, see Rademacher [19]. Specifically, $r_F = 1$ if and only if (M, F) is a reversible Finsler manifold.

A smooth curve $\gamma : [a, b] \to M$ is called a geodesic if its velocity field $\dot{\gamma}$ is parallel along the curve, i.e., $D_{\dot{\gamma}}\dot{\gamma} = 0$, where D denotes the covariant derivative induced by the Chern connection, see Bao, Chern and Shen [5, Chapter 2]. (M, F) is said to be complete if every geodesic $\gamma : [a, b] \to M$ can be extended to a geodesic defined on $\mathbb{R}$.

The Finslerian distance function $d_F : M \times M \to [0, \infty)$ is defined by

$$d_F(x_1, x_2) = \inf_{\gamma \in \Gamma(x_1, x_2)} \int_a^b F\big(\gamma(t), \dot{\gamma}(t)\big) dt,$$

where $\Gamma(x_1, x_2)$ denotes the set of all piecewise differentiable curves $\gamma : [a, b] \to M$ such that $\gamma(a) = x_1$ and $\gamma(b) = x_2$. Clearly, $d_F(x_1, x_2) = 0$ if and only if $x_1 = x_2$, and d_F verifies the triangle inequality. However, in general, d_F is not symmetric. In fact, we have that d_F is symmetric if and only if (M, F) is a reversible Finsler manifold.

For a point $x \in M$ and a number $r > 0$, the forward open geodesic ball with center x and radius r is defined as

$$B_F(x, r) = \{z \in M : d_F(x, z) < r\}.$$

For a fixed point $x \in M$ let $y, v \in T_x M$ be two linearly independent tangent vectors. The flag curvature is defined as

$$K^y(y, v) = \frac{g_y(R(y, v)v, y)}{g_y(y, y) g_y(v, v) - g_y(y, v)^2},$$

where g is the fundamental tensor induced by the Hessian matrices (g_{ij}) and R is the Chern curvature tensor, see Bao, Chern and Shen [5, Chapter 3].

The Ricci curvature at the point $x \in M$ and in the direction $y \in T_x M$ is defined by

$$\mathrm{Ric}_x(y) = F^2(x, y) \sum_{i=1}^{n-1} K^y(y, e_i),$$

where $\{e_1, \cdots, e_{n-1}, \frac{1}{F(x,y)}y\}$ is an orthonormal basis of T_xM with respect to g.

The density function $\sigma_F : M \to [0, \infty)$ is defined by

$$\sigma_F(x) = \frac{\omega_n}{\mathrm{Vol}_e(B(x,1))},$$

where $\omega_n = \pi^{\frac{n}{2}}/\Gamma(1 + \frac{n}{2})$ is the volume of the n-dimensional Euclidean open unit ball, Vol_e denotes in the sequel the canonical Euclidean volume, and

$$B(x,1) = \left\{(y^i) \in \mathbb{R}^n : \ F\left(x, \sum_{i=1}^n y^i \frac{\partial}{\partial x^i}\right) < 1\right\} \subset \mathbb{R}^n.$$

The canonical Busemann-Hausdorff volume form on (M, F) is defined as

$$dv_F(x) = \sigma_F(x) dx^1 \wedge \cdots \wedge dx^n,$$

see Shen [21, Section 2.2]. Note that in the following we may omit the parameter x for the sake of brevity. The Finslerian volume of a measurable set $\Omega \subset M$ is given by $\mathrm{Vol}_F(\Omega) = \int_\Omega dv_F$.

The mean distortion of (M, F) is defined by $\mu : TM \setminus \{0\} \to (0, \infty)$,

$$\mu(x,y) = \frac{\sqrt{\det\left[g_{ij}(x,y)\right]}}{\sigma_F(x)},$$

while the mean covariation is given by $\mathbf{S} : TM \setminus \{0\} \to \mathbb{R}$,

$$\mathbf{S}(x,y) = \frac{d}{dt}\Big(\ln\, \mu\big(\gamma(t), \dot{\gamma}(t)\big)\Big)\Big|_{t=0},$$

where γ is the geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = y$.

We say that (M, F) has nonnegative n-Ricci curvature, denoted by $\mathsf{Ric}_n \geq 0$, if $\mathrm{Ric}_x(y) \geq 0$ for all $(x, y) \in TM$ and the mean covariation $\mathbf{S}$ is identically zero. Examples of Finsler manifolds with vanishing mean covariation include the so-called Berwald spaces, which, in particular, contain both Riemannian manifolds and Minkowski spaces, see Shen [22].

As previously introduced, the asymptotic volume ratio of (M, F) is defined by

$$\mathsf{AVR}_F = \lim_{r \to \infty} \frac{\mathrm{Vol}_F(B_F(x,r))}{\omega_n r^n},$$

where $x \in M$ is arbitrarily fixed. Note that AVR_F is well-defined, being independent of the choice of the point $x \in M$.

On the one hand, an n-dimensional Finsler manifold (M, F) equipped with the canonical volume form $d\nu_F$ satisfies the condition that for every $x \in M$,

$$\lim_{r \searrow 0} \frac{\mathrm{Vol}_F(B_F(x,r))}{\omega_n r^n} = 1.$$

On the other hand, if (M, F) is a complete Finsler manifold having $\mathsf{Ric}_n \geq 0$, the Bishop-Gromov volume comparison principle asserts that the function $r \mapsto \frac{\mathrm{Vol}_F(B_F(x,r))}{r^n}$ is nonincreasing on $(0, \infty)$, see Shen [22, Theorem 1.1]. Consequently, it follows that if $\mathsf{Ric}_n \geq 0$, then $\mathsf{AVR}_F \in [0, 1]$.

The polar transform $F^* : T^*M \to [0, \infty)$ is defined as the dual metric of F, namely

$$F^*(x, \alpha) = \sup_{y \in T_xM \setminus \{0\}} \frac{\alpha(y)}{F(x, y)},$$

where $T^*M = \cup_{x \in M} \{(x, \alpha) : \alpha \in T_x^*M\}$ is the cotangent bundle of M and T_x^*M is the dual space of T_xM.

The Legendre transform is defined by $J^* : T^*M \to TM$,

$$J^*(x, \alpha) = \sum_{i=1}^{n} \frac{\partial}{\partial \alpha^i} \left(\frac{1}{2} F^{*2}(x, \alpha) \right) \frac{\partial}{\partial x^i},$$

for every $\alpha = \sum_{i=1}^n \alpha^i dx^i \in T_x^*M$. Note that if $J^*(x, \alpha) = (x, y)$, then

$$F(x, y) = F^*(x, \alpha) \quad \text{and} \quad \alpha(y) = F^*(x, \alpha) F(x, y).$$

Let $u : M \to \mathbb{R}$ be a differentiable function in the distributional sense. The gradient of u is defined as $\nabla_F u(x) = J^*(x, Du(x))$, where $Du(x) \in T_x^*M$ denotes the (distributional) derivative of u at the point $x \in M$.

Using the properties of the Legendre transform, it follows that

$$F(x, \nabla_F u(x)) = F^*(x, Du(x)) \quad \text{and} \quad Du(x)\big(\nabla_F u(x)\big) = F^*(x, Du(x))^2.$$

In local coordinates, one has that

$$Du(x) = \sum_{i=1}^{n} \frac{\partial u}{\partial x^i}(x) dx^i \quad \text{and} \quad \nabla_F u(x) = \sum_{i,j=1}^{n} g^*_{ij}(x, Du(x)) \frac{\partial u}{\partial x^i}(x) \frac{\partial}{\partial x^j},$$

where (g^*_{ij}) is the Hessian matrix $\left(g^*_{ij}(x, \alpha)\right) = \left(\frac{1}{2} \frac{\partial^2}{\partial \alpha^i \partial \alpha^j} F^{*2}(x, \alpha)\right)$, see Ohta and Sturm [18, Lemma 1.1]. In general, the gradient operator ∇_F is not linear.

Given a vector field V on M, the divergence of V is defined in local coordinates as $\mathrm{div}V(x) = \frac{1}{\sigma_F(x)} \sum_{i=1}^n \frac{\partial}{\partial x^i}(\sigma_F(x)V^i(x))$.

The Finsler-Laplace operator is defined by

$$\Delta_F u(x) = \mathrm{div}\left(\nabla_F u(x)\right).$$

Note that Δ_F is usually not linear. However, in the particular case when $(M, F) = (M, g)$ is a Riemannian manifold, Δ_F coincides with the usual Laplace-Beltrami operator Δ_g.

The divergence theorem implies that

$$\int_M \varphi(x)\Delta_F u(x)\, dv_F = -\int_M D\varphi(x)\big(\nabla_F u(x)\big)\, dv_F, \tag{7}$$

for all $\varphi \in C_0^\infty(M)$, see Ohta and Sturm [18].

In the specific case when $(\mathbb{R}^n, H)$ is a reversible Finsler manifold, then H is actually a smooth, absolutely homogeneous norm on $\mathbb{R}^n$. Consequently, the polar transform of H is in fact its dual norm $H^* : \mathbb{R}^n \to [0, \infty)$,

$$H^*(x) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{\langle \xi, x\rangle}{H(\xi)}.$$

In this case, the Finsler-Laplace operator Δ_{H^*} associated with the norm H^* is given by

$$\Delta_{H^*} u = \mathrm{div}\left(H^*(\nabla u)\nabla_\xi H^*(\nabla u)\right),$$

where ∇_ξ stands for the gradient operator with respect to the variables $\xi \in \mathbb{R}^n$.

Due to Cianchi and Salani [10, Lemma 3.1], we have the following relation between the norms H and H^*:

$$H^*\left(\nabla_\xi H(\xi)\right) = 1, \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{8}$$

Finally, let $\Omega \subset M$ be an open subset. The Sobolev space on Ω associated with the Finsler structure F and the Busemann-Hausdorff measure dv_F is defined by

$$W_F^{1,2}(\Omega) = \left\{ u \in W_{loc}^{1,2}(\Omega) : \int_\Omega F^*(x, Du(x))^2\, dv_F < +\infty \right\},$$

while $W_{0,F}^{1,2}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_F^{1,2}(\Omega)} = \left(\int_\Omega |u(x)|^2\, dv_F + \int_\Omega F^*(x, Du(x))^2\, dv_F \right)^{\frac{1}{2}}.$$

4 Anisotropic symmetrization

In the following, let (M, F) be a noncompact, complete $n(\geq 2)$-dimensional Finsler manifold having $\mathsf{Ric}_n \geq 0$, equipped with the induced Finsler metric d_F and the Busemann-Hausdorff volume form dv_F. In this case, (M, d_F, dv_F) is a metric measure space which satisfies the $\mathsf{CD}(0, n)$ condition, see Ohta [17].

We further suppose that (M, F) has Euclidean volume growth, i.e., $\mathsf{AVR}_F > 0$. For this geometric setting, the sharp isoperimetric inequality has been recently established by Manini [15, Theorem 1.3]. In particular, for every bounded open set $\Omega \subset M$, one has the following isoperimetric inequality:

$$\mathcal{P}_F(\partial\Omega) \geq n\omega_n^{\frac{1}{n}} \mathsf{AVR}_F^{\frac{1}{n}} \operatorname{Vol}_F(\Omega)^{\frac{n-1}{n}}. \tag{9}$$

Here $\mathcal{P}_F(\partial\Omega)$ denotes the anisotropic perimeter of Ω, defined as $\mathcal{P}_F(\partial\Omega) = \int_{\partial\Omega} d\sigma_F$, where $d\sigma_F$ is the $(n-1)$-dimensional Lebesgue measure induced by dv_F. It is noteworthy that inequality (9) holds true in the general Finslerian setting, unrestricted by any reversibility assumption regarding the Finsler structure F.

Moreover, due to Manini's rigidity result [15, Theorem 1.4], the equality in (9) can be characterized by introducing the additional assumption that the reversibility constant r_F of (M, F) is finite. More precisely, one has the following theorem.

Theorem 3 ([15, Theorem 1.4]) *Let $(M, F, \mathfrak{m})$ be a Finsler manifold (possibly with boundary) satisfying the $\mathsf{CD}(0, n)$ condition for some $n > 1$, such that $\mathsf{AVR}_F > 0$, $r_F < \infty$ and all closed forward geodesic balls are compact. Assume that for all $x_1, x_2 \notin \partial M$ and for all geodesics $\gamma : [0, 1] \to M$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$, it holds that $\gamma(t) \notin \partial M$, for every $t \in [0, 1]$. Let $\Omega \subset X$ be a bounded Borel set that saturates the isoperimetric inequality*

$$\mathcal{P}(\partial\Omega) \geq n\omega_n^{\frac{1}{n}} \mathsf{AVR}_F^{\frac{1}{n}} \mathfrak{m}(\Omega)^{\frac{n-1}{n}}.$$

Then there exists (a unique) $x \in M$ such that, up to a negligible set,

$$\Omega = B_F(x, r) \quad \textit{with} \quad r = \left(\frac{\mathfrak{m}(\Omega)}{\mathsf{AVR}_F \omega_n}\right)^{\frac{1}{n}}.$$

Moreover, the measure $\mathfrak{m}$ has the following representation:

$$\mathfrak{m} = \int_{\partial B_F(x,r)} \mathfrak{m}_\alpha \mathfrak{q}(d\alpha), \quad \mathfrak{q} \in \mathcal{P}\left(\partial B_F(x, r)\right), \quad \mathfrak{m}_\alpha \in \mathcal{M}_+(M),$$

with $\mathfrak{m}_\alpha$ *concentrated on the geodesic ray from* x *through* α*, and* $\mathfrak{m}_\alpha$ *can be identified with* $n\omega_n \mathsf{AVR}_F t^{n-1}\mathcal{L}^1 \llcorner_{[0,\infty)}$.

In particular, if the measure chosen on (M, F) is the Busemann-Hausdorff measure $d\nu_F$, then, by Theorem 3, an extremizer set of the isoperimetric inequality (9) satisfies that

$$\Omega = B_F(x, r), \quad \text{where} \quad r = \left(\frac{\mathrm{Vol}_F(\Omega)}{\mathsf{AVR}_F \omega_n}\right)^{\frac{1}{n}}. \tag{10}$$

In order to leverage Manini's results, we adapt the classical Schwarz symmetrization technique (see e.g., Kesavan [13]) to accommodate the Finslerian context. For similar concepts of anisotropic (or convex) rearrangements, we refer to Alvino, Ferone, Trombetti and Lions [1], Kristály, Mester and Mezei [14] and Schaftingen [20].

Let us consider a reversible Finsler manifold $(\mathbb{R}^n, H)$ endowed with the canonical volume form $d\nu_H$. In addition, we assume, without loss of generality, that the set

$$\mathcal{W}_H(1) \coloneqq \{x \in \mathbb{R}^n : H(x) < 1\}$$

has measure $\mathrm{Vol}_e(\mathcal{W}_H(1)) = \omega_n$, i.e., H is a normalized Minkowski norm. Consequently, we have that the density function of $(\mathbb{R}^n, H)$, $\sigma_H = 1$ is constant, which yields $d\nu_H(x) = dx$. Accordingly, it turns out that $\mathrm{Vol}_H(\mathcal{W}_H(1)) = \mathrm{Vol}_e(\mathcal{W}_H(1)) = \omega_n$ and $\mathsf{AVR}_H = 1$.

Our goal is to apply a so-called anisotropic (or convex) rearrangement technique 'from the Finsler manifold (M, F) to the Minkowski normed space $(\mathbb{R}^n, H)$'.

In the following, let $\Omega \subset M$ be a bounded domain.

The anisotropic rearrangement of Ω w.r.t. the normalized Minkowski norm H is a Wulff-shape

$$\Omega_H^\star = \{z \in \mathbb{R}^n : H(z) < R\} \eqqcolon \mathcal{W}_H(R),$$

where $R > 0$ is determined such that

$$\mathrm{Vol}_F(\Omega) = \mathsf{AVR}_F \mathrm{Vol}_H(\Omega_H^\star) = \mathsf{AVR}_F \mathrm{Vol}_e(\Omega_H^\star).$$

Remark 1 *Clearly, in the particular case when* H *is the standard Euclidean norm* $|\cdot|$*, the anisotropic rearrangement of* Ω *w.r.t.* $|\cdot|$ *is precisely the usual Euclidean symmetric rearrangement, which is an* n*-dimensional open Euclidean* *ball centered at the origin and having radius*

$$\mathsf{R} = \left(\frac{\mathrm{Vol}_{\mathsf{F}}(\Omega)}{\mathsf{AVR}_{\mathsf{F}}\omega_n} \right)^{\frac{1}{n}}.$$

In this case, if $\Omega \subset M$ is an extremizer of the isoperimetric inequality (9), it turns out that the metric balls Ω and $\Omega^\star_{|\cdot|}$ have equal radii, see (10).

Now let $u : \Omega \to \mathbb{R}$ be a nonnegative, measurable function.

The distribution function of $u : \Omega \to [0, \infty)$ is defined by the function $\mu_u : [0, \infty) \to [0, \mathrm{Vol}_F(\Omega)]$,

$$\mu_u(t) = \mathrm{Vol}_F\big(\{x \in \Omega : u(x) > t\}\big).$$

It can be seen that μ_u is decreasing and $\mu_u(t) = 0$, for all $t \geq \operatorname{ess\,sup}_\Omega u$.

The decreasing rearrangement of u is defined by $u^\sharp : [0, \mathrm{Vol}_F(\Omega)] \to [0, \infty)$,

$$u^\sharp(s) = \begin{cases} \operatorname{ess\,sup}_\Omega u, & \text{if } s = 0, \\ \inf\{t : \mu_u(t) \leq s\}, & \text{if } 0 < s \leq \mathrm{Vol}_F(\Omega). \end{cases}$$

Finally, the anisotropic rearrangement of u w.r.t. the normalized Minkowski norm H is given by $u^\star_H : \Omega^\star_H \to [0, \infty)$,

$$u^\star_H(x) = u^\sharp(\mathsf{AVR}_F \omega_n H(x)^n). \tag{11}$$

By the previous definition, it follows that for every $t \geq 0$, one has that

$$\mathrm{Vol}_F\big(\{x \in \Omega : u(x) > t\}\big) = \mathsf{AVR}_F \cdot \mathrm{Vol}_H\big(\{x \in \Omega^\star_H : u^\star_H(x) > t\}\big),$$

which implies that $\mu_u(t) = \mathsf{AVR}_F \cdot \mu_{u^\star_H}(t)$, for all $t \geq 0$.

By the layer cake representation, it follows that

$$\|u\|_{L^p(\Omega)} = \|u^\sharp\|_{L^p(0,\mathrm{Vol}_F(\Omega))} = \mathsf{AVR}_F^{\frac{1}{p}} \|u^\star_H\|_{L^p(\Omega^\star_H)},$$

for every $p \in [1, \infty]$.

Similarly to Kesavan [13, Theorem 1.2.2], one can prove the following Hardy-Littlewood-Pólya-type inequality: if $u, g \in L^2(\Omega)$ are nonnegative functions, then

$$\int_\Omega u(x)g(x)\,dv_F \leq \int_0^{\mathrm{Vol}_F(\Omega)} u^\sharp(s)g^\sharp(s)\,ds = \int_{\Omega^\star_H} u^\star_H(x)g^\star_H(x)\,dv_H.$$

In particular, one has that

$$\int_{\{x\in\Omega:u(x)>t\}} g(x)dv_F \leq \int_0^{\mu_u(t)} g^\sharp(s)ds, \tag{12}$$

for any $t \in [0,\infty)$ fixed.

Remark 2 *For a fixed function* $u : \Omega \to [0,\infty)$*, one can define multiple anisotropic rearrangements of* u *w.r.t. various Minkowski norms. However, by definition, these all will be equimeasurable in the following sense: if* $u^\star_{H_1}$ *and* $u^\star_{H_2}$ *are the anisotropic rearrangements of* u *w.r.t. two different absolutely homogeneous, normalized Minkowski norms* H_1 *and* H_2*, then for any* $p \in [1,\infty]$*,*

$$\|u^\star_{H_1}\|_{L^p(\Omega^\star_{H_1})} = \|u^\star_{H_2}\|_{L^p(\Omega^\star_{H_2})}.$$

In the particular case when $(M,F) = (M,g)$ *is a Riemannian manifold and* $H(x) = |x|$ *is the standard Euclidean norm,* $u^\star_{|\cdot|}$ *turns out to be the classical radially symmetric rearrangement of* u*. This type of rearrangement is employed by Chen and Li* [8] *in their comparison result on Riemannian manifolds. Therefore, our findings effectively extend the results of Chen and Li* [8] *to the broader, Finslerian framework.*

In the Finslerian case, however, it is indicated to substitute the classical Euclidean symmetrization with the anisotropic rearrangement (11)*. This choice is motivated by the fact that the minimizers of the isoperimetric inequality* (9)*, when analyzed within a Minkowski space* $(\mathbb{R}^n, H)$*, correspond to Wulff-shapes associated with* H *(up to translations), see Cabré, Ros-Oton, and Serra* [7, Theorem 1.2] *or Manini* [15, Theorem 1.5]*.*

5 Proof of Theorems 1&2

This section contains the proof of the Talenti comparison principle from Theorem 1 and the Faber-Krahn inequality from Theorem 2. The key ingredients are the anisotropic rearrangement technique presented in Section 4 and the sharp isoperimetric inequality (9). For the characterization of the equality case, we use the rigidity result from Theorem 3.

Proof of Theorem 1.

Step 1. We start by studying the solution of the Dirichlet problem defined on (M, F), namely,

$$\begin{cases} -\Delta_F u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\mathcal{P}$}$$

Let $u : \Omega \subset M \to \mathbb{R}$ be the weak solution of $(\mathcal{P})$. Since $f \in L^2(\Omega)$ is a nonnegative function, by the maximum principle, it follows that u is nonnegative on Ω.

We consider the distribution function $\mu_u : [0, \infty) \to [0, \mathrm{Vol}_F(\Omega)]$ of u, and for any $0 \leq t \leq \operatorname{ess\,sup}_\Omega u$ fixed, we define the level sets

$$\Omega_t = \{x \in \Omega : u(x) > t\} \quad \text{and} \quad \Gamma_t = \{x \in \Omega : u(x) = t\}.$$

Note that by Sard's theorem, we have that $\Gamma_t = \partial\Omega_t$.

Then, we have that $\mu_u(t) = \mathrm{Vol}_F(\Omega_t)$ and $\mathcal{P}_F(\Gamma_t) = -\mu_u'(t)$, see Section 4.

Applying the co-area formula given by Shen [21, Theorem 3.3.1], we can prove that

$$\int_{\Gamma_t} F^*(x, Du(x)) d\sigma_F = -\frac{d}{dt} \int_{\Omega_t} F^*(x, Du(x))^2 dv_F \tag{13}$$

and

$$\int_{\Gamma_t} \frac{1}{F^*(x, Du(x))} d\sigma_F = -\frac{d}{dt} \int_{\Omega_t} dv_F = -\mu_u'(t) \ . \tag{14}$$

Since u is the weak solution of $(\mathcal{P})$, by (7) we have that

$$\int_\Omega D\varphi(x)\big(\nabla_F u(x)\big)\, dv_F = \int_\Omega f(x)\varphi(x)\, dv_F, \tag{15}$$

for every test function $\varphi \in W^{1,2}_{0,F}(\Omega)$.

For a fixed $t > 0$ and $h > 0$, we define the function

$$\varphi_h(x) = \begin{cases} 0, & \text{if} \quad 0 \leq u(x) \leq t, \\ \frac{u(x)-t}{h}, & \text{if} \quad t < u(x) \leq t+h, \\ 1, & \text{if} \quad u(x) > t+h. \end{cases}$$

By choosing φ_h as test funcion in (15) and taking the limit $h \to 0$, we obtain

$$-\frac{d}{dt} \int_{\Omega_t} F^*(x, Du(x))^2 dv_F = \int_{\Omega_t} f(x) dv_F. \tag{16}$$

Let $f^\sharp : [0, \mathrm{Vol}_F(\Omega)] \to [0, \infty)$ be the decreasing rearrangement of f. Combining (13), (16) and the Hardy-Littlewood-Pólya-type inequality (12), it follows that

$$\int_{\Gamma_t} F^*(x, Du(x)) d\sigma_F = \int_{\Omega_t} f(x) d\nu_F \le \int_0^{\mu_u(t)} f^\sharp(s) ds. \tag{17}$$

By applying the isoperimetric inequality (9) to the set Ω_t, then using the Cauchy-Schwarz inequality and relations (14) and (17), we obtain that

$$\begin{aligned} n^2 (\omega_n \mathsf{AVR}_F)^{\frac{2}{n}} \mathrm{Vol}_F(\Omega_t)^{2-\frac{2}{n}} &\le \mathcal{P}_F(\Gamma_t)^2 = \left(\int_{\Gamma_t} d\sigma_F \right)^2 \\ &\le \int_{\Gamma_t} \frac{1}{F^*(x, Du(x))} d\sigma_F \cdot \int_{\Gamma_t} F^*(x, Du(x)) d\sigma_F \\ &\le -\mu_u'(t) \int_0^{\mu_u(t)} f^\sharp(s) ds. \end{aligned} \tag{18}$$

Hence, since $\mu_u(t) = \mathrm{Vol}_F(\Omega_t)$, we have that

$$1 \le n^{-2} (\omega_n \mathsf{AVR}_F)^{-\frac{2}{n}} \mu_u(t)^{\frac{2}{n}-2} \left(-\mu_u'(t) \right) \int_0^{\mu_u(t)} f^\sharp(s) ds.$$

Integrating from 0 to t and applying a change of variables yields

$$\begin{aligned} t &\le n^{-2} (\omega_n \mathsf{AVR}_F)^{-\frac{2}{n}} \int_0^t \mu_u(\tau)^{\frac{2}{n}-2} \left(-\mu_u'(\tau) \right) \int_0^{\mu_u(\tau)} f^\sharp(s) ds d\tau \\ &= n^{-2} (\omega_n \mathsf{AVR}_F)^{-\frac{2}{n}} \int_{\mu_u(t)}^{\mathrm{Vol}_F(\Omega)} \eta^{\frac{2}{n}-2} \int_0^{\eta} f^\sharp(s) ds d\eta. \end{aligned}$$

Using the definition of the decreasing rearrangement $u^\sharp : [0, \mathrm{Vol}_F(\Omega)] \to [0, \infty)$ of u, we obtain that

$$u^\sharp(\xi) \le n^{-2} (\omega_n \mathsf{AVR}_F)^{-\frac{2}{n}} \int_{\xi}^{\mathrm{Vol}_F(\Omega)} \eta^{\frac{2}{n}-2} \int_0^{\eta} f^\sharp(s) ds d\eta. \tag{19}$$

Step 2. Now we turn to the Dirichlet problem defined on $(\mathbb{R}^n, H)$, namely,

$$\begin{cases} -\Delta_{H^*} v = f_H^\star & \text{in } \Omega_H^\star \\ v = 0 & \text{on } \partial \Omega_H^\star, \end{cases} \tag{$\mathcal{P}^\star$}$$

where $\Omega_H^\star \subset \mathbb{R}^n$ is a Wulff-shape such that

$$\mathrm{Vol}_F(\Omega) = \mathsf{AVR}_F \mathrm{Vol}_H(\Omega_H^\star), \tag{20}$$

while $f_H^\star : \Omega_H^\star \to [0, \infty)$ is the anisotropic rearrangement of f w.r.t. the norm H, i.e.,

$$f_H^\star(x) = f^\sharp(\mathsf{AVR}_F \omega_n H(x)^n),$$

where $f^\sharp$ is the decreasing rearrangement of f.

We can associate to $(\mathcal{P}^\star)$ the energy functional $\mathcal{E} : W_{0,H}^{1,2}(\Omega_H^\star) \to \mathbb{R}$, defined as

$$\mathcal{E}(v) = \frac{1}{2}\int_{\Omega_H^\star} H^*(\nabla v(x))^2 dv_H - \int_{\Omega_H^\star} f_H^\star(x)v(x)\, dv_H.$$

Let $v : \Omega_H^\star \to \mathbb{R}$ be the weak solution of problem $(\mathcal{P}^\star)$. By the maximum principle, we have that v is nonnegative on $\Omega_H^\star$.

Since the anisotropic rearrangements $\Omega_H^\star$ and $f_H^\star$ are constructed w.r.t. the Minkowski norm H, we can suppose (by abuse of notation) that the solution of $(\mathcal{P}^\star)$ satisfies

$$v(x) = v(H(x)) \text{ on } \Omega_H^\star.$$

Then, by relation (8), we have that

$$H^*(\nabla v(x)) = H^*\big(v'(H(x))\nabla H(x)\big) = -v'(H(x))H^*(\nabla H(x)) = -v'(H(x)).$$

Consequently, we obtain that

$$\begin{aligned}\mathcal{E}(v) &= \frac{1}{2}\int_{\Omega_H^\star} v'(H(x))^2 dv_H - \int_{\Omega_H^\star} f^\sharp(\mathsf{AVR}_F \omega_n H(x)^n)v(H(x))\, dv_H \\ &= n\omega_n \left\{\frac{1}{2}\int_0^R v'(\rho)^2\rho^{n-1}d\rho - \int_0^R f^\sharp(\mathsf{AVR}_F\omega_n\rho^n)v(\rho)\rho^{n-1}\, d\rho\right\},\end{aligned}$$

where $R > 0$ is determined such that the Wulff-shape $\Omega_H^\star = W_H(R)$ satisfies (20).

Since v is the critical point of $\mathcal{E}$, it follows that v satisfies the ordinary differential equation

$$(v'(\rho)\rho^{n-1})' + f^\sharp(\mathsf{AVR}_F\omega_n\rho^n)\rho^{n-1} = 0, \tag{21}$$

together with the boundary conditions

$$v(R) = v'(0) = 0.$$

Integrating (21) from 0 to r and applying a change of variables yields

$$-r^{n-1}v'(r) = \int_0^r f^\sharp(\mathsf{AVR}_F\omega_n\rho^n)\rho^{n-1}d\rho$$

$$= (\mathsf{AVR}_F n\omega_n)^{-1} \int_0^{\mathsf{AVR}_F \omega_n r^n} f^\sharp(s)ds.$$

Then, integrating from r to R and using a change of variable again yields that

$$\begin{aligned} v(r) &= (\mathsf{AVR}_F n\omega_n)^{-1} \int_r^R \rho^{1-n} \int_0^{\mathsf{AVR}_F \omega_n \rho^n} f^\sharp(s)dsd\rho \\ &= n^{-2}(\omega_n \mathsf{AVR}_F)^{-\frac{2}{n}} \int_{\mathsf{AVR}_F \omega_n r^n}^{\mathsf{AVR}_F \mathrm{Vol}_H(\Omega_H^\star)} \eta^{\frac{2}{n}-2} \int_0^\eta f^\sharp(s)dsd\eta. \end{aligned}$$

Hence, we obtain that $v = v_H^\star$ and

$$v^\sharp(\xi) = n^{-2}(\omega_n \mathsf{AVR}_F)^{-\frac{2}{n}} \int_{\mathsf{AVR}_F \xi}^{\mathsf{AVR}_F \mathrm{Vol}_H(\Omega_H^\star)} \eta^{\frac{2}{n}-2} \int_0^\eta f^\sharp(s)dsd\eta.$$

where $v^\sharp : [0, \mathrm{Vol}_H(\Omega_H^\star)] \to [0, \infty)$ is the decreasing rearrangement of v.

Step 3. Using relations (19) and (20), we obtain that

$$u^\sharp(\mathsf{AVR}_F \xi) \leq v^\sharp(\xi), \text{ a.e. } \xi \in [0, \mathrm{Vol}_H(\Omega_H^\star)]. \tag{22}$$

Keeping in mind the definitions of the anisotropic rearrangements $u_H^\star$ and $v_H^\star = v$ (see (11)), it follows that

$$u_H^\star(x) = u^\sharp(\mathsf{AVR}_F \omega_n H(x)^n) \leq v^\sharp(\omega_n H(x)^n) = v_H^\star(x),$$

a.e. $x \in \Omega_H^\star$, which concludes the proof of inequality (5).

Step 4. If $u_H^\star(x) = v(x)$, for a.e. $x \in \Omega_H^\star$, it follows that we have equality in (22), which in turn implies that equality holds in (19), as well. Consequently, we obtain that equality is achieved in the isoperimetric inequality (18) for every level set Ω_t. In particular, for $t = 0$ we have that

$$\mathcal{P}_F(\partial\Omega) = n\omega_n^{\frac{1}{n}} \mathsf{AVR}_F^{\frac{1}{n}} \mathrm{Vol}_F(\Omega)^{\frac{n-1}{n}}.$$

Therefore, we can apply Theorem 3, which completes the proof.

Proof of Theorem 2.

Let $u : \Omega \subset M \to \mathbb{R}$ be the eigenfunction associated with the first eigenvalue $\lambda_1(\Omega)$ of $(\mathcal{EP})$, and consider the anisotropic rearrangement function of u w.r.t. the norm H, i.e., $u_H^\star : \Omega_H^\star \subset \mathbb{R}^n \to \mathbb{R}$.

Given $\lambda_1(\Omega)$ and $u_H^\star$, we can define the Dirichlet problem

$$\begin{cases} -\Delta_{H^*} v = \lambda_1(\Omega) u_H^\star, & \text{in } \Omega_H^\star, \\ v = 0, & \text{on } \partial\Omega_H^\star. \end{cases} \tag{23}$$

If $v : \Omega_H^\star \to \mathbb{R}$ is a solution of problem (23), then, by Theorem 1, it follows that

$$u_H^\star(x) \leq v(x), \quad a.e.\ x \in \Omega_H^\star. \tag{24}$$

Consequently, we have that

$$\int_{\Omega_H^\star} u_H^\star(x) v(x) dv_H \leq \int_{\Omega_H^\star} v(x)^2 dv_H. \tag{25}$$

Multiplying by v the equation from (23), then integrating on $\Omega_H^\star$ and using relation (7), we obtain that

$$\int_{\Omega_H^\star} H^*(\nabla v(x))^2 dv_H = \lambda_1(\Omega) \int_{\Omega_H^\star} u_H^\star(x) v(x) dv_H.$$

Therefore, by applying (25) and the variational characterization of the first eigenvalue of problem ($\mathcal{EP}^\star$) (see Shen [21, page 176]), it follows that

$$\lambda_1(\Omega) = \frac{\displaystyle\int_{\Omega_H^\star} H^*(\nabla v(x))^2 dv_H}{\displaystyle\int_{\Omega_H^\star} u_H^\star(x) v(x) dv_H} \geq \frac{\displaystyle\int_{\Omega_H^\star} H^*(\nabla v(x))^2 dv_H}{\displaystyle\int_{\Omega_H^\star} v(x)^2 dv_H} \geq \lambda_1\left(\Omega_H^\star\right),$$

which completes the proof of (6).

If equality holds in (6), then we have equalities in all of the above inequalities. In particular, we have equality in (24). Thus we can apply the rigidity result of Theorem 1, which concludes the proof. □

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