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A unified study of the Fourier series involving the Aleph-function of two variables

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Abstract. In 2016, authors have studied Fourier series involving the Aleph-function. In this paper, we make an application of integrals involving sine function, exponential function, the product of Kampé de Fériet functions and the Aleph-function of two variables to evaluate Fourier series. We also develop a multiple integral involving the Aleph-function of two variables to make its application to derive a multiple exponential Fourier series. Some interesting particular cases and remarks are also given.

1 Introduction and Preliminaries

Recently, I-function of two variables, [18], has been studied as a generalization of the H-function of two variables developed by Gupta and Mittal [4] (see also [13]). Singh and Joshi [20] investigated certain double integrals involving the H-function of two variables. These integrals are of a highly general nature and can be specialized to derive numerous known and new integral formulas,

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which hold significant importance in mathematical analysis and are potentially useful in solving various boundary value problems. Srivastava and Singh [25] extended these results to the I-function of two variables as determined by Sharma and Mishra [18].

More recently, Kumar [7] has introduced the Aleph-function of two variables (see also [19]), which is an extension of the I-function of two variables by Sharma and Mishra [18]. The Aleph-function of two variables also generalizes the Aleph-function of one variable introduced by Südland et al. [26]. Systematic studies on the Aleph-function of one variable have been conducted by Ram and Kumar [14], Kumar et al. [8, 9, 10, 11], Saxena et al. [16, 17] and others.

The Aleph-function of two variables is defined using a double Mellin-Barnes type integral as follows:

$$\begin{aligned} \aleph(z_1, z_2) &= \aleph^{0,\mathbf{n}:U}_{P_i,Q_i,\tau_i;r:V}\left(\begin{array}{c} z_1 \\ z_2 \end{array}\middle|\begin{array}{c} \mathbb{A} \\ \mathbb{B} \end{array}\right) \\ &= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\theta(s_1, s_2)\prod_{j=1}^{2}\phi_j(s_j)\; z_1^{s_1}\, z_2^{s_2}\, ds_1\, ds_2, \end{aligned} \tag{1}$$

where:

$$\begin{aligned} \mathbb{A} = &(a_j; \alpha_j, A_j)_{1,\mathbf{n}}, [\tau_i(a_{ji}; \alpha_{ji}, A_{ji})]_{\mathbf{n}+1,P_i}; (c_j, \gamma_j)_{1,n_1}, \left[\tau_{i'}\left(c_{ji'}, \gamma_{ji'}\right)\right]_{n_1+1,P_{i'}}; \\ &(e_j, E_j)_{1,n_2}, \left[\tau_{i''}\left(e_{ji''}, E_{ji''}\right)\right]_{n_2+1,P_{i''}}, \end{aligned} \tag{2}$$

$$\begin{aligned} \mathbb{B} = &[\tau_i(b_{ji}; \beta_{ji}, B_{ji})]_{1,Q_i}; (d_j, \delta_j)_{1,m_1}, \left[\tau_{i'}\left(d_{ji'}, \delta_{ji'}\right)\right]_{m_1+1,Q_{i'}}; \\ &(f_j, F_j)_{1,m_2}, \left[\tau_{i''}\left(f_{ji''}, F_{ji''}\right)\right]_{m_2+1,Q_{i''}}, \end{aligned} \tag{3}$$

$$U = m_1, n_1 : m_2, n_2, \tag{4}$$

$$V = P_{i'}, Q_{i'}, \tau_{i'}; r' : P_{i''}, Q_{i''}, \tau_{i''}; r''. \tag{5}$$

$\theta(s_1, s_2)$ and $\phi_j(s_j)$ are defined by K. Sharma [19] (see also, [7]). The conditions for the existence of equation (1) are provided as follows:

$$\Omega = \tau_i\sum_{j=1}^{P_i}\alpha_{ji} - \tau_i\sum_{j=1}^{Q_i}\beta_{ji} + \tau_{i'}\sum_{j=1}^{P_{i'}}\gamma_{ji} - \tau_{i'}\sum_{j=1}^{Q_{i'}}\delta_{ji'} < 0, \tag{6}$$

$$\Delta = \tau_i\sum_{j=1}^{P_i}A_{ji} - \tau_i\sum_{j=1}^{Q_i}B_{ji} + \tau_{i''}\sum_{j=1}^{P_{i''}}E_{ji''} - \tau_{i''}\sum_{j=1}^{Q_{i''}}F_{ji''} < 0, \tag{7}$$

The conditions for absolute convergence of double Mellin-Barnes type contour integral (1) are as follows:

$$|\arg(z_1)| < \frac{\pi}{2}\Theta \quad \text{and} \quad |\arg(z_2)| < \frac{\pi}{2}\Lambda,$$

where

$$\begin{aligned}\Theta = \sum_{j=1}^{n}\alpha_j - \tau_i\sum_{j=n+1}^{P_i}\alpha_{ji} - \tau_i\sum_{j=1}^{Q_i}\beta_{ji} + \sum_{j=1}^{n_1}\gamma_j - \tau_{i'}\sum_{j=n_1+1}^{P_{i'}}\gamma_{ji'}\\ + \sum_{j=1}^{n_2}E_j - \tau_{i''}\sum_{j=n_2+1}^{P_{i''}}\gamma_{ji''} > 0,\end{aligned} \tag{8}$$

and

$$\begin{aligned}\Lambda = \sum_{j=1}^{n}A_j - \tau_i\sum_{j=n+1}^{P_i}A_{ji} - \tau_i\sum_{j=1}^{Q_i}B_{ji} + \sum_{j=1}^{m_1}\delta_j - \tau_{i'}\sum_{j=m_1+1}^{Q_{i'}}\delta_{ji'}\\ + \sum_{j=1}^{m_2}F_j - \tau_{i''}\sum_{j=m_2+1}^{Q_{i''}}F_{ji''} > 0.\end{aligned} \tag{9}$$

Remark 1 *If $\tau_i, \tau_{i'}, \tau_{i''} \to 1$, the Aleph-function of two variables reduces to the* I*-function of two variables due to Sharma and Mishra [18].*

Remark 2 *If $\tau_i, \tau_{i'}, \tau_{i''} \to 1$ and $r = r' = r'' = 1$, the Aleph-function reduces to the* H*-function of two variables introduced by Gupta and Mittal [4] (see also, [13]).*

The Kampé de Fériet hypergeometric function is represented as follows [1].

$$K^{E;F;F'}_{G;H;H'}\left(\begin{array}{c} x \\ y \end{array} \middle| \begin{array}{c} (e);(f);(f') \\ (g);(h);(h') \end{array}\right) = \sum_{r,t=0}^{\infty} \frac{\prod_{k=1}^{E}(e_k)_{r+t}\prod_{k=1}^{F}(f_k)_r\prod_{k=1}^{F'}(f'_k)_t}{\prod_{k=1}^{G}(g_k)_{r+t}\prod_{k=1}^{H}(h_k)_r\prod_{k=1}^{H'}(h'_k)_t}\frac{x^r y^t}{r!t!} \tag{10}$$

For further details, see Appell and Kampé de Fériet [1]. For brevity, we shall use the following notations.

$$\epsilon = \frac{\prod_{k=1}^{E}(e_k)_{r+t}\prod_{k=1}^{F}(f_k)_r\prod_{k=1}^{F'}(f'_k)_t}{\prod_{k=1}^{G}(g_k)_{r+t}\prod_{k=1}^{H}(h_k)_r\prod_{k=1}^{H'}(h'_k)_t}, \tag{11}$$

$$\epsilon_1 = \frac{\prod_{k_1=1}^{E_1} (e_{1k_1})_{r_1+t_1} \prod_{k_1=1}^{F_1} (f_{1k_1})_{r_1} \prod_{k_1=1}^{F_1'} \left(f'_{1k_1}\right)_{t_1}}{\prod_{k_1=1}^{G_1} (g_{1k_1})_{r_1+t_1} \prod_{k_1=1}^{H_1} (h_{1k_1})_{r_1} \prod_{k_1=1}^{H_1'} \left(h'_{1k_1}\right)_{t_1}}, \tag{12}$$

$$\epsilon_n = \frac{\prod_{k_n=1}^{E_n} (e_{nk_n})_{r_n+t_n} \prod_{k_n=1}^{F_n} (f_{nk_n})_{r_n} \prod_{k_n=1}^{F_n'} \left(f'_{nk_n}\right)_{t_n}}{\prod_{k_n=1}^{G_n} (g_{nk_n})_{r_n+t_n} \prod_{k_n=1}^{H_n} (h_{nk_n})_{r_n} \prod_{k_n=1}^{H_n'} \left(h'_{nk_n}\right)_{t_n}}. \tag{13}$$

Mishra [12] has evaluated the following integral:

Lemma 1

$$\int_0^\pi (\sin x)^{w-1} e^{imx} {}_pF_q \left(\begin{matrix} (\alpha_p); \\ (\beta_q); \end{matrix} \; C (\sin x)^{2h} \right) dx = \frac{\pi e^{im\pi/2}}{2^{w-1}} \sum_{r=0}^{\infty} \frac{(\alpha_p)_r \, C^r \, \Gamma(w+2hr)}{(\beta_q)_r \, 4^{hr} \, \Gamma\left(\frac{w+2hr\pm m+1}{2}\right) r!} \tag{14}$$

where $(\alpha)_p$ *denotes* $\alpha_1, \cdots, \alpha_p$*;* $\Gamma(a \pm b)$ *represents* $\Gamma(a+b), \Gamma(a-b)$*;* h *is a positive integer:* $p < q$ *and* $\mathrm{Re}(w) > 0$*. We have the following results:*

$$\int_0^\pi e^{i(m-n)x} dx = \pi\delta_{m,n}; \quad \int_0^\pi e^{imx} \sin nx \, dx = i\frac{\pi}{2}\delta_{m,n} \tag{15}$$

where $\delta_{m,n} = 1$ *if* $m = n, 0$ *else.*

$$\int_0^\pi e^{imx} \cos nx \, dx = \pi \, \epsilon_{m,n}, \tag{16}$$

where $\epsilon_{m,n} = \frac{1}{2}$ *if* $m = n \neq 0, 1$ *if* $m = n = 0,\ 0$ *else.*

2 Main results

The integrals to be evaluate are:

Theorem 1

$$\int_0^\pi (\sin x)^{w-1} \, e^{imx} \, K_{G;H;H'}^{E;F;F'} \left(\begin{matrix} \alpha (\sin x)^{2\rho} \\ \beta (\sin x)^{2\gamma} \end{matrix} \,\middle|\, \begin{matrix} (e);(f);(f') \\ (g);(h);(h') \end{matrix} \right) \aleph \left(\begin{matrix} z_1 (\sin x)^{\sigma_1} \\ z_2 (\sin x)^{\sigma_2} \end{matrix} \right) dx$$

$$= \frac{\pi \, e^{im\pi/2}}{2^{w-1}} \sum_{r,t=0}^{\infty} E \, \frac{\alpha^r \, \beta^t}{4^{(\rho r+\gamma t)} \, r! \, t!}$$

$$\aleph_{p_i+1,q_i+2,\tau_i;r:V}^{0,\mathbf{n}+1:U} \left[\begin{matrix} 4^{-\sigma_1} z_1 \\ 4^{-\sigma_2} z_2 \end{matrix} \,\middle|\, \begin{matrix} (1-w-2\rho r-2\gamma t; 2\sigma_1, 2\sigma_2), \mathbb{A} \\ \left(\frac{1-w-2\rho r-2\gamma t\pm m}{2}; \sigma_1, \sigma_2\right), \mathbb{B} \end{matrix} \right], \tag{17}$$

provided that $\Re(w) > 0, \rho > 0, \gamma > 0, \sigma_1 > 0, \sigma_2 > 0$, $|\arg z_1| < \frac{\pi}{2}\Theta$ *and* $|\arg z_2| < \frac{\pi}{2}\Lambda$, *where* Θ *and* Λ *are defined respectively by* (8) *and* (9).

Theorem 2

$$\int_0^{\pi}\cdots\int_0^{\pi}\prod_{j=1}^{n}(\sin x)^{w_j-1}\, e^{im_jx_j}\, K^{E_j;F_j;F'_j}_{G_j;H_j;H'_j}\left(\begin{array}{c}\alpha_j(\sin x_j)^{2\rho_j}\\ \beta_j(\sin x_j)^{2\gamma_j}\end{array}\middle|\begin{array}{c}(e_j),(f_j),\left(f'_j\right)\\ (g_j),(h_j),\left(h'_j\right)\end{array}\right)$$
$$\times\ \aleph\left(\begin{array}{c}z_1\prod_{j=1}^{n}(\sin x_j)^{\sigma'_j}\\ z_2\prod_{j=1}^{n}(\sin x_j)^{\sigma''_j}\end{array}\right)dx_1\cdots dx_r$$
$$=\sum_{r_1,t_1,\cdots,r_n,t_n=0}^{\infty}\prod_{j=1}^{n}E_j\,\frac{\pi e^{im_j\pi/2}}{2^{w_j-1}}\,\frac{\alpha_j^{r_j}\,\beta_j^{t_j}}{4^{(\rho_jr_j+\gamma_jt_j)}\,r_j!\,t_j!}$$
$$\times\ \aleph^{0,\mathbf{n}+n:U}_{p_i+n,q_i+2n,\tau_i;r:V}\left[\begin{array}{c}z_14^{-\sum_{j=1}^{n}\sigma'_j}\\ z_24^{-\sum_{j=1}^{n}\sigma''_j}\end{array}\middle|\begin{array}{c}\left(1-w_j-2\rho_jr_j-2\gamma_jt_j;2\sigma'_j,2\sigma''_j\right)_{1,n},\mathbb{A}\\ \left(\frac{1-w_j-2\rho r_j-2\gamma t_j\pm m_j}{2};\sigma'_j,\sigma''_j\right)_{1,n},\mathbb{B}\end{array}\right], \tag{18}$$

provided that $\Re(w_j) > 0, \rho_j > 0, \gamma_j > 0, \sigma'_j > 0, \sigma''_j > 0$ *for* $j = 1,\cdots,n$, $|\arg z_1| < \frac{\pi}{2}\Theta$ *and* $|\arg z_2| < \frac{\pi}{2}\Lambda$.

Proof. To prove (17), we express the Aleph-function of two variables into the Mellin-Barnes contour integral with the help of (1) and the Kampé de Fériet function in double series with the help of (10). Now, we change the order of integration and summation, which is permissible under the conditions stated with the integral and we evaluate the inner integral with the help of 1. Now Interpreting the Mellin-Barnes contour integral in Aleph-function of two variables, we obtain the desired result (17). The integral (18) is obtained by the similar procedure. □

3 Exponential Fourier series

In this section, we give the exponential Fourier series of the product of Kampé de Fériet hypergeometric function and the Aleph-function of two variables by using the orthogonality property of exponential function.
Let

$$f^{(1)}(x)=(\sin x)^{w-1}\,K^{E;F;F'}_{G;H;H'}\left(\begin{array}{c}\alpha(\sin x)^{2\rho}\\ \beta(\sin x)^{2\gamma}\end{array}\middle|\begin{array}{c}(e);(f);(f')\\ (g);(h);(h')\end{array}\right)\aleph\left(\begin{array}{c}z_1(\sin x)^{\sigma_1}\\ z_2(\sin x)^{\sigma_2}\end{array}\right)$$

$$= \sum_{p=-\infty}^{\infty} A_p \, e^{-ipx} \tag{19}$$

$f(x)$ is a continuous function and bounded variation with interval $(0, \pi)$. Now, multiplied by e^{imx} both sides in (19) and integrating it with respect x from 0 to π and then making an appeal to (15) and (17), we get

$$A_p = \frac{e^{ip\pi/2}}{2^{w-1}} \sum_{r,t=0}^{\infty} E \, \frac{\alpha^r \beta^t}{4^{(\rho r+\gamma t)} r!\, t!} \aleph^{0,\mathbf{n}+1:U}_{p_i+1,q_i+2,\tau_i;r:V} \left(\begin{array}{c} 4^{-\sigma_1} z_1 \\ 4^{-\sigma_2} z_2 \end{array} \middle| \begin{array}{c} (1-w-2\rho r-2\gamma t; 2\sigma_1, 2\sigma_2)\,,\, \mathbb{A} \\ \left(\frac{1-w-2\rho r-2\gamma t \pm m}{2}; \sigma_1, \sigma_2\right)\,,\, \mathbb{B} \end{array} \right). \tag{20}$$

Using (19) and (20), we obtain the following exponential Fourier series:

Theorem 3

$$(\sin x)^{w-1} \; K^{E;F;F'}_{G;H;H'} \left(\begin{array}{c} \alpha\,(\sin x)^{2\rho} \\ \beta\,(\sin x)^{2\gamma} \end{array} \middle| \begin{array}{c} (e);(f);(f') \\ (g);(h);(h') \end{array} \right) \aleph \left(\begin{array}{c} z_1\,(\sin x)^{\sigma_1} \\ z_2\,(\sin x)^{\sigma_2} \end{array} \right)$$

$$= \sum_{p=-\infty}^{\infty} \sum_{r,t=0}^{\infty} E e^{ip(\pi/2-x)} \tag{21}$$

$$\times \frac{\alpha^r \beta^t}{4^{(\rho r+\gamma t)} \, r!\, t!} \aleph^{0,\mathbf{n}+1:U}_{p_i+1,q_i+2,\tau_i;r:V} \left(\begin{array}{c} 4^{-\sigma_1} z_1 \\ 4^{-\sigma_2} z_2 \end{array} \middle| \begin{array}{c} (1-w-2\rho r-2\gamma t; 2\sigma_1, 2\sigma_2)\,,\, \mathbb{A} \\ \left(\frac{1-w-2\rho r-2\gamma t \pm m}{2}; \sigma_1, \sigma_2\right)\,,\, \mathbb{B} \end{array} \right),$$

under the same conditions as (17).

4 Cosine Fourier series

In this section, we obtain the cosine Fourier series of the product of Kampé de Fériet hypergeometric function and the Aleph-function of two variables by using the orthogonality property (15) and (16).

$$f^{(2)}(x) = (\sin x)^{w-1} \; K^{E;F;F'}_{G;H;H'} \left(\begin{array}{c} \alpha\,(\sin x)^{2\rho} \\ \beta\,(\sin x)^{2\gamma} \end{array} \middle| \begin{array}{c} (e);(f);(f') \\ (g);(h);(h') \end{array} \right) \aleph \left(\begin{array}{c} z_1\,(\sin x)^{\sigma_1} \\ z_2\,(\sin x)^{\sigma_2} \end{array} \right)$$

$$= \frac{B_0}{2} + \sum_{p=1}^{\infty} B_p \, \cos px. \tag{22}$$

Integrating it with respect x from 0 to π, we have

$$\frac{B_0}{2} = \frac{1}{\pi^{\frac{1}{2}}} \sum_{r,t=0}^{\infty} E \frac{\alpha^r \beta^t}{r!\, t!} \frac{B_0}{2}$$
$$\aleph^{0,\mathbf{n}+1:U}_{p_i+1,q_i+2,\tau_i;r:V} \left(\begin{matrix} z_1 \\ z_2 \end{matrix} \,\middle|\, \begin{matrix} \left(1 - \frac{w}{2} - 2\rho r - 2\gamma t; 2\sigma', 2\sigma''\right), \mathbb{A} \\ \left(\frac{1-w}{2} - \rho r - \gamma t; \sigma', \sigma''\right), \mathbb{B} \end{matrix} \right). \tag{23}$$

Multiplying both sides in (22) by e^{imx} and integrating it withrespect x from 0 to π and use the equations (15), (16) and (17), we obtain

$$B_p = \frac{e^{ip\pi/2}}{2^{w-1}} \sum_{r,t=0}^{\infty} E \frac{\alpha^r \beta^t}{4^{(\rho r + \gamma t)} r!\, t!}$$
$$\aleph^{0,\mathbf{n}+1:U}_{p_i+1,q_i+2,\tau_i;r:V} \left(\begin{matrix} 4^{-\sigma_1} z_1 \\ 4^{-\sigma_2} z_2 \end{matrix} \,\middle|\, \begin{matrix} (1 - w - 2\rho r - 2\gamma t; 2\sigma_1, 2\sigma_2), \mathbb{A} \\ \left(\frac{1-w-2\rho r-2\gamma t \pm m}{2}; \sigma_1, \sigma_2\right), \mathbb{B} \end{matrix} \right). \tag{24}$$

Using the equations (22), (23) and (24), we obtain the following cosine Fourier series:

Theorem 4

$$(\sin x)^{w-1}\; K^{E;F;F'}_{G;H;H'} \left(\begin{matrix} \alpha\,(\sin x)^{2\rho} \\ \beta\,(\sin x)^{2\gamma} \end{matrix} \,\middle|\, \begin{matrix} (e);(f);(f') \\ (g);(h);(h') \end{matrix} \right) \aleph \left(\begin{matrix} z_1\,(\sin x)^{\sigma_1} \\ z_2\,(\sin x)^{\sigma_2} \end{matrix} \right)$$
$$= \frac{1}{\pi^{\frac{1}{2}}} \sum_{r,t=0}^{\infty} E \frac{\alpha^r \beta^t}{r!\, t!} \frac{B_0}{2} \aleph^{0,\mathbf{n}+1:U}_{p_i+1,q_i+1,\tau_i;r:V} \left(\begin{matrix} z_1 \\ z_2 \end{matrix} \,\middle|\, \begin{matrix} \left(1 - \frac{w}{2} - 2\rho r - 2\gamma t; 2\sigma', 2\sigma''\right), \mathbb{A} \\ \left(\frac{1-w}{2} - \rho r - \gamma t; \sigma', \sigma''\right), \mathbb{B} \end{matrix} \right)$$
$$+ \sum_{p=1}^{\infty} \sum_{r,t=0}^{\infty} E\, e^{ip\pi/2} \cos px \frac{\alpha^r \beta^t}{4^{(\rho r+\gamma t)} r!\, t!}$$
$$\times\; \aleph^{0,\mathbf{n}+1:U}_{p_i+1,q_i+2,\tau_i;r:V} \left(\begin{matrix} 4^{-\sigma_1} z_1 \\ 4^{-\sigma_2} z_2 \end{matrix} \,\middle|\, \begin{matrix} (1 - w - 2\rho r - 2\gamma t; 2\sigma_1, 2\sigma_2), \mathbb{A} \\ \left(\frac{1-w-2\rho r-2\gamma t \pm m}{2}; \sigma_1, \sigma_2\right), \mathbb{B} \end{matrix} \right) \tag{25}$$

under the same conditions as (17).

5 Sine Fourier series

In this section, we obtain the sine Fourier series of the product of Kampé de Fériet hypergeometric function and the Aleph-function of two variables by using the orthogonality property (15).

$$f^{(3)}(x) = (\sin x)^{w-1}\; K^{E;F;F'}_{G;H;H'} \left(\begin{matrix} \alpha(\sin x)^{2\rho} \\ \beta\,(\sin x)^{2\gamma} \end{matrix} \,\middle|\, \begin{matrix} (e);(f);(f') \\ (g);(h);(h') \end{matrix} \right) \aleph \left(\begin{matrix} z_1\,(\sin x)^{\sigma_1} \\ z_2\,(\sin x)^{\sigma_2} \end{matrix} \right)$$

$$= \sum_{p=-\infty}^{\infty} C_p \sin px. \tag{26}$$

Multiplying both sides in (26) e^{imx} and integrating it with respect x from 0 π and use the equations (15), and (17), we obtain

$$C_p = \frac{e^{ip\pi/2}}{i2^{w-2}} \sum_{r,t=0}^{\infty} E \frac{\alpha^r \beta^t}{4^{(\rho r+\gamma t)} r!\, t!}$$
$$\aleph^{0,\mathbf{n}+1:U}_{p_i+1,q_i+2,\tau_i;r:V} \left(\begin{array}{c} 4^{-\sigma_1} z_1 \\ 4^{-\sigma_2} z_2 \end{array} \middle| \begin{array}{c} (1-w-2\rho r-2\gamma t; 2\sigma_1, 2\sigma_2)\,,\, \mathbb{A} \\ \left(\frac{1-w-2\rho r-2\gamma t\pm m}{2}; \sigma_1, \sigma_2\right)\,,\, \mathbb{B} \end{array} \right). \tag{27}$$

Using the equations (26) and (27), we get the following sine Fourier series:

Theorem 5

$$f^{(3)}(x) = (\sin x)^{w-1}\, K^{E;F;F'}_{G;H;H'} \left(\begin{array}{c} \alpha\,(\sin x)^{2\rho} \\ \beta\,(\sin x)^{2\gamma} \end{array} \middle| \begin{array}{c} (e)\,;(f)\,;(f') \\ (g)\,;(h)\,;(h') \end{array} \right) \aleph \left(\begin{array}{c} z_1\,(\sin x)^{\sigma_1} \\ z_2\,(\sin x)^{\sigma_2} \end{array} \right)$$
$$= -2i \sum_{p=-\infty}^{\infty} \sum_{r,t=0}^{\infty} E\, e^{ip\pi/2} \sin px \frac{\alpha^r \beta^t}{4^{(\rho r+\gamma t)} r!\, t!}$$
$$\times\ \aleph^{0,\mathbf{n}+1:U}_{p_i+1,q_i+2,\tau_i;r:V} \left(\begin{array}{c} 4^{-\sigma_1} z_1 \\ 4^{-\sigma_2} z_2 \end{array} \middle| \begin{array}{c} (1-w-2\rho r-2\gamma t; 2\sigma_1, 2\sigma_2)\,,\, \mathbb{A} \\ \left(\frac{1-w-2\rho r-2\gamma t\pm m}{2}; \sigma_1, \sigma_2\right)\,,\, \mathbb{B} \end{array} \right), \tag{28}$$

under the same conditions as (17).

6 Multiple exponential Fourier series

In this section, we obtain the multiple exponential Fourier series of the product of Kampé de Fériet hypergeometric function and the Aleph-function of two variables.

$f(x_1, \cdots, x_n)$ is a function that is continuous and of bounded variation in the domain $\underbrace{(0,\pi) \times \cdots \times (0,\pi)}_{n}$.

$$f(x_1, \cdots, x_n) = \prod_{j=1}^{n} (\sin x)^{w_j-1}\, K^{E_j;F_j;F'_j}_{G_j;H_j;H'_j} \left(\begin{array}{c} \alpha_j\,(\sin x_j)^{2\rho_j} \\ \beta_j\,(\sin x_j)^{2\gamma_j} \end{array} \middle| \begin{array}{c} (e_j)\,,(f_j)\,,\left(f'_j\right) \\ (g_j)\,,(h_j)\,,\left(h'_j\right) \end{array} \right)$$
$$\times\ \aleph \left(\begin{array}{c} z_1 \prod_{j=1}^{n} (\sin x_j)^{\sigma'_j} \\ z_2 \prod_{j=1}^{n} (\sin x_j)^{\sigma''_j} \end{array} \right) = \sum_{p_1,\cdots,p_n=-\infty}^{\infty} A_{p_1,\cdots,p_n}\, e^{-i(p_1 x_1 + \cdots p_n x_n)}. \tag{29}$$

We fix $x_1, \cdots, x_{n-1}$ and multiplying both sides in (29) by $e^{im_n x_n}$ and integrating with respect to x_n from 0 to π, we obtain

$$\prod_{j=1}^{n-1} (\sin x)^{w_j-1} \; K_{G_j;H_j;H_j'}^{E_j;F_j;F_j'} \left(\begin{array}{c} \alpha_j (\sin x_j)^{2\rho_j} \\ \beta_j (\sin x_j)^{2\gamma_j} \end{array} \middle| \begin{array}{c} (e_j), (f_j), \left(f_j'\right) \\ (g_j), (h_j), \left(h_j'\right) \end{array} \right)$$
$$\aleph \left(\begin{array}{c} z_1 \prod_{j=1}^{n} (\sin x_j)^{\sigma_j'} \\ z_2 \prod_{j=1}^{n} (\sin x_j)^{\sigma_j''} \end{array} \right) \tag{30}$$
$$= \sum_{p_1=-\infty}^{\infty} \cdots \sum_{p_{n-1}=-\infty}^{\infty} e^{-i(p_1 x_1 + \cdots + p_n x_n)} + \sum_{p_n=-\infty}^{\infty} \int_0^{\pi} e^{i(m_n - p_n)} \, dx_n,$$

using the first relation of (15) and (17), from (30), we get

$$A_{p_1,\cdots,p_n} = \sum_{r_1,t_1,\cdots,r_n,t_n=0}^{\infty} \prod_{j=1}^{n} E_j \frac{e^{ip_j\pi/2}}{2^{w_j-1}} \frac{\alpha_j^{r_j} \beta_j^{t_j}}{4^{(\rho_j r_j + \gamma_j t_j)} r_j! \, t_j!}$$
$$\aleph_{p_i+n,q_i+2n,\tau_i;r:V}^{0,\mathbf{n}+n:U} \left(\begin{array}{c} z_1 4^{-\sum_{j=1}^{n} \sigma_j'} \\ z_2 4^{-\sum_{j=1}^{n} \sigma_j''} \end{array} \middle| \begin{array}{c} \left(1 - w_j - 2\rho_j r_j - 2\gamma_j t_j; 2\sigma_j', 2\sigma_j''\right)_{1,n}, \mathbb{A} \\ \left(\frac{1-w_j-2\rho r_j-2\gamma t_j \pm m_j}{2}; \sigma_j', \sigma_j''\right)_{1,n}, \mathbb{B} \end{array} \right). \tag{31}$$

Using (29) and (31), we obtain the multiple exponential Fourier series.

Theorem 6

$$\prod_{j=1}^{n} (\sin x)^{w_j-1} \; K_{G_j;H_j;H_j'}^{E_j;F_j;F_j'} \left(\begin{array}{c} \alpha_j (\sin x_j)^{2\rho_j} \\ \beta_j (\sin x_j)^{2\gamma_j} \end{array} \middle| \begin{array}{c} (e_j), (f_j), \left(f_j'\right) \\ (g_j), (h_j), \left(h_j'\right) \end{array} \right)$$
$$\aleph \left(\begin{array}{c} z_1 \prod_{j=1}^{n} (\sin x_j)^{\sigma_j'} \\ z_2 \prod_{j=1}^{n} (\sin x_j)^{\sigma_j''} \end{array} \right) \tag{32}$$
$$= \sum_{p_1,\cdots,p_n=-\infty}^{\infty} \sum_{r_1,t_1,\cdots,r_n,t_n=0}^{\infty} \prod_{j=1}^{n} E_j \frac{e^{ip_j(\pi/2-x)}}{2^{w_j-1}} \frac{\alpha_j^{r_j} \beta_j^{t_j}}{4^{(\rho_j r_j + \gamma_j t_j)} r_j! \, t_j!}$$
$$\times \; \aleph_{p_i+n,q_i+2n,\tau_i;r:V}^{0,\mathbf{n}+n:U} \left(\begin{array}{c} z_1 4^{-\sum_{j=1}^{n} \sigma_j'} \\ z_2 4^{-\sum_{j=1}^{n} \sigma_j''} \end{array} \middle| \begin{array}{c} \left(1 - w_j - 2\rho_j r_j - 2\gamma_j t_j; 2\sigma_j', 2\sigma_j''\right)_{1,n}, \mathbb{A} \\ \left(\frac{1-w_j-2\rho r_j-2\gamma t_j \pm m_j}{2}; \sigma_j', \sigma_j''\right)_{1,n}, \mathbb{B} \end{array} \right),$$

under the same conditions that (18).

7 Particular cases

By setting $\beta_1, \cdots, \beta_n = 0$ in equations (18) and (32), we respectively obtain the following multiple integral and multiple exponential Fourier series.

Corollary 1

$$\int_0^{\pi} \cdots \int_0^{\pi} \prod_{j=1}^{n} (\sin x)^{w_j - 1} e^{i m_j x_j} \; {}_{E_j+F_j}K_{G_j+H_j} \left(\alpha_j (\sin x_j)^{2\rho_j} \,\middle|\, \begin{matrix} (e_j), (f_j) \\ (g_j), (h_j) \end{matrix} \right)$$
$$\times \aleph \left(\begin{matrix} z_1 \prod_{j=1}^{n} (\sin x_j)^{\sigma'_j} \\ z_2 \prod_{j=1}^{n} (\sin x_j)^{\sigma''_j} \end{matrix} \right) dx_1 \cdots dx_r = \sum_{r_1, \cdots, r_n = 0}^{\infty} \prod_{j=1}^{n} \frac{e^{i p_j \pi/2}}{2^{w_j - 1}} \mathfrak{E}_j \frac{\alpha_j^{r_j}}{4^{\rho_j r_j} r_j!}$$
$$\times \aleph^{0, \mathbf{n}+n : U}_{p_i + n, q_i + 2n, \tau_i ; r : V} \left(\begin{matrix} z_1 4^{-\sum_{j=1}^{n} \sigma'_j} \\ z_2 4^{-\sum_{j=1}^{n} \sigma''_j} \end{matrix} \,\middle|\, \begin{matrix} \left(1 - w_j - 2\rho_j r_j ; 2\sigma'_j, 2\sigma''_j\right)_{1,n}, \mathbb{A} \\ \left(\frac{1 - w_j - 2\rho r_j \pm m_j}{2} ; \sigma'_j, \sigma''_j\right)_{1,n}, \mathbb{B} \end{matrix} \right), \tag{33}$$

under the same conditions that (18) *with* $\beta_1, \cdots, \beta_n = 0$, *and*

$$\mathfrak{E}_j = \frac{\prod_{k_j=1}^{E_j} \left(e_{jk_j}\right)_{r_j} \prod_{k_j=1}^{F_j} \left(f_{jk_j}\right)_{r_j}}{\prod_{k_j=1}^{G_j} \left(g_{jk_j}\right)_{r_j} \prod_{k_j=1}^{H_j} \left(h_{jk_j}\right)_{r_j}}, \qquad j = 1, \cdots, n.$$

Corollary 2

$$\prod_{j=1}^{n} (\sin x)^{w_j - 1} \; {}_{E_j+F_j}K_{G_j+H_j} \left(\alpha_j (\sin x_j)^{2\rho_j} \,\middle|\, \begin{matrix} (e_j), (f_j) \\ (g_j), (h_j) \end{matrix} \right) \aleph \left(\begin{matrix} z_1 \prod_{j=1}^{n} (\sin x_j)^{\sigma'_j} \\ z_2 \prod_{j=1}^{n} (\sin x_j)^{\sigma''_j} \end{matrix} \right)$$
$$= \sum_{p_1, \cdots, p_n = -\infty}^{\infty} \sum_{r_1, \cdots, r_n = 0}^{\infty} \prod_{j=1}^{n} E_j \frac{e^{i p_j (\pi/2 - x)}}{2^{w_j - 1}} \frac{\alpha_j^{r_j} \beta_j^{t_j}}{4^{(\rho_j r_j + \gamma_j t_j)} r_j! \, t_j!}$$
$$\times \aleph^{0, \mathbf{n}+n : U}_{p_i + n, q_i + 2n, \tau_i ; r : V} \left(\begin{matrix} z_1 4^{-\sum_{j=1}^{n} \sigma'_j} \\ z_2 4^{-\sum_{j=1}^{n} \sigma''_j} \end{matrix} \,\middle|\, \begin{matrix} \left(1 - w_j - 2\rho_j r_j ; 2\sigma'_j, 2\sigma''_j\right)_{1,n}, \mathbb{A} \\ \left(\frac{1 - w_j - 2\rho r_j \pm m_j}{2} ; \sigma'_j, \sigma''_j\right)_{1,n}, \mathbb{B} \end{matrix} \right), \tag{34}$$

under the same conditions that (18) *with* $\beta_1, \cdots, \beta_n = 0$.

If $\alpha_1 = \cdots = \alpha_n = 0$ in equation (33), we obtain the following multiple integral, defined as Corollary 3:

Corollary 3

$$\int_0^{\pi}\cdots\int_0^{\pi}\prod_{j=1}^{n}(\sin x)^{w_j-1}\ e^{im_jx_j}\,\aleph\begin{pmatrix} z_1\prod_{j=1}^{n}(\sin x_j)^{\sigma_j'} \\ z_2\prod_{j=1}^{n}(\sin x_j)^{\sigma_j''}\end{pmatrix}dx_1\cdots dx_r=\prod_{j=1}^{n}\frac{\pi e^{im_j\pi/2}}{2^{w_j-1}}$$

$$\times\ \aleph^{0,\mathbf{n}+n:U}_{p_i+n,q_i+2n,\tau_i;r:V}\left(\begin{array}{c} z_1 4^{-\sum_{j=1}^{n}\sigma_j'} \\ z_2 4^{-\sum_{j=1}^{n}\sigma_j''}\end{array}\middle|\begin{array}{c}\left(1-w_1;2\sigma_j',2\sigma_j''\right)_{1,n},\mathbb{A} \\ \left(\frac{1-w_1\pm m_j}{2};\sigma_j',\sigma_j''\right)_{1,n},\mathbb{B}\end{array}\right), \tag{35}$$

under the same conditions that (18) with $\beta_1,\cdots,\beta_n=0$ and $\alpha_1=\cdots=\alpha_n=0$.

Remark 3 *We can also obtain the similar formulas*
(i) *with the multivariable* H*-function defined by Srivastava and Panda [23, 24], for more details see also [3].*
(ii) *with the Aleph-function of one variable defined by Südland et al. [26], see Ayant and Kumar [2].*
(iii) *with the* $\bar{\mathrm{H}}$*-function defined by Inayat-Hussain [5, 6], see R.C. Singh and Khan [21].*
(iv) *with the* I*-function defined by Saxena [15], see Singh and Khan [22].*

8 Concluding Remarks

The Aleph-function of two variables and the Kampe de Fériet function presented in this paper are fundamentally simple in nature. By specializing the parameters of these functions, we can derive various Fourier series expansions related to other special functions, such as the I-function of two variables, the H-function of two variables, the I-function, Fox's H-function, Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, MacRobert's E-function, generalized hypergeometric functions, the Bessel function of the first kind, the modified Bessel function, the Whittaker function, the exponential function, the binomial function, and more. Consequently, numerous unified integral representations can be obtained as special cases of our results.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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