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On auto-nilpotent polygroups

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Abstract. The purpose of this paper is to introduce the concept of autonilpotent polygroups and investigate their properties concerning the automorphism of polygroups. To realize the article's goals, we present the notation of $\mathfrak{m}$-very thin polygroups and construct the (non) commutative very thin polygroups on every (infinite) finite non–empty set, where $\mathfrak{m} \in \mathbb{N}$. As a result of the research, is to show that the set of automorphism of some very thin polygroups is equal to the set of automorphism of special groups. The paper includes implications for the development of automorphism of polygroups, and shows that under some conditions very thin polygroups are autonilpotent polygroups and investigates the connection between of autonilpotent polygroups and nilpotent polygroups. The new conception of autonilpotent polygroups was broached for the in this paper the first time.

1 Introduction

The hyper compositional structure theory as an extension of classic structures, was firstly introduced, by F. Marty in 1934 [16]. In algebraic hyper compositional system, output from the hyperoperation on elements is a set and so any

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algebraic system is an algebraic hypercompositional system. Marty extended the concept of groups to hypergroups and other researchers presented the algebra hypercompositional structures concepts such as hyperring, hypermodule, hyperfield, hypergraph, polygroup, multiring, etc in this similar way [18]. Algebraic hypercompositional structures are applied in several branches of sciences such as artificial intelligence and (hyper) complex networks [7]. Polygroup, as an important subclass of hypergroups is introduced by Bonansinga and Corsini citebc and is discussed by many scholars [1, 3, 20]. Comer used playgroups to study color algebra [3, 4] and considered some algebraic and combinatorial properties of playgroups [5, 6]. Further materials regarding polygroups and hypergroup such as permutation polygroups [9], isomorphism in polygroups [10], weak polygroups [11], rough subpolygroups in factor polygroup [12], automorphism group of very thin H_v–groups [13], divisible groups derived from divisible hypergroups [14] etc are investigated. An important class of groups as the concept of autonilpotent groups introduced by Moghaddam et. al [17, 19]. Recently Hamidi et al. introduced the concept of auto-Engel polygroups via the heart of hypergroups and investigated the relation between auto–Engel polygroups and auto-nilpotent polygroups. They showed that the concept of the heart of hypergroups plays an important role in the construction of auto-engel polygroups and proved the heart of hypergroups is a characteristic set in hypergroups [15].

This paper introduces the concept of $\mathfrak{m}$-very thin polygroups and constructs finite and infinite very thin polygroups, where $\mathfrak{m} \in \mathbb{N}$. We compute the number of very thin polygroups up to isomorphic. The motivation of our work is the generalization of nilpotent playgroups to autonilpotent polygroups. So we introduce the concept of autonilpotent polygroups and investigate the automorphism of $\mathfrak{m}$–very thin polygroups. It considered some properties of autonilpotent polygroups and connected the autonilpotent polygroups and the quotient of autonilpotent polygroups via the fundamental relations. This study considers the relation between autonilpotent polygroups and nilpotent polygroups and extends the autonilpotent polygroups by the quotient of autonilpotent ploygroup and the direct product of autonilpotent polygroups.

2 Preliminaries

In this section, we review some definitions and results from [8, 20], which we need in what follows. Assume that $H \neq \emptyset$ be an arbitrary set and $P^*(H) = \{G \mid \emptyset \neq G \subseteq H\}$. Each map $\rho : H^2 \longrightarrow P^*(H)$ is said to be a *hyperop-*

eration, hyperstructure (H, ρ) is called a *hypergroupoid* and for every $\emptyset \neq A, B \subseteq H, \rho(A, B) = \bigcup\limits_{a \in A, b \in B} \rho(a, b)$. A *hypergroupoid* (H, ρ) together with an associative binary hyperoperation is said a *semihypergroup* and a semihypergroup (H, ρ) is called a *hypergroup* if for any $x \in H, \rho(x, H) = \rho(H, x) = H$(*reproduction axiom*).

Definition 1 [8] *A semihypergroup* (H, ρ) *is said to be a polygroup, if* (i) *there exists* $e \in H$ *such that for all* $x \in H, \quad \rho(e, x) = \rho(x, e) = \{x\}$, (ii) $x \in \rho(y, z)$ *concludes that* $y \in \rho(x, \vartheta(z))$ *and* $z \in \rho(\vartheta(y), x)$*, where* ϑ *is an unitary operation on* $H\big($*it follows that for all* $x \in H$ *there exists a uiniqe* $\vartheta(x) \in H$ *i.e* $e \in (\rho(x, \vartheta(x)) \cap (\rho(\vartheta(x), x)), \vartheta(e) = e, \vartheta(\vartheta(x)) = x\big)$ *and is denoted by* (H, ρ, e, ϑ) *or* $(H, \cdot, e,^{-1})$*, for simplify. A set* $\emptyset \neq K \subseteq H$ *is said to be a subpolygroup of* H*, if for all* $x, y \in K, \rho(x, \vartheta(y)) \subseteq K$ *and it is denoted by* $K \leq H$.

Definition 2 [20] *Suppose that* (H, ρ) *is a hypergroup. For any given an equivalence relation* ω *on* H*, a hyperoperation* σ *on* $\frac{H}{\omega}$ *is defined by* $\sigma(\omega(a), \omega(b)) = \{\omega(c) \mid c \in \rho(\omega(a), \omega(b))\}$.

Theorem 1 [2] *Let* (H, ρ) *be a hypergroup. Then* $\big(\frac{H}{\omega}, \sigma\big)$ *is a hypergroup if and only if* ω *is a regular equivalence relation and* $\big(\frac{H}{\omega}, \sigma\big)$ *is a group if and if only* ω *is a strongly regular equivalence relation.*

One of famous algebraic relation on any given hypergroup is β which is defined by $a\beta b$ if and only if there exists $u \in \mathcal{U}(H)$ s.t $\{a, b\} \subseteq u$, where $\mathcal{U}(H)$ is denoted by the set of all finite product of elements of H. The smallest transitive relation in a way contains β is denoted by β^* and it means the *transitive closure* of β and $\big(\frac{H}{\beta^*}, \sigma\big)$ is said the *fundamental group* of (H, ρ) [20].

Definition 3 [20] *A map* $f : H_1 \to H_2$ *is called a homomorphism of hypergroups if* $\forall\, x, y \in H_1$*, we have* $f(\rho_1(x, y)) = \rho_2(f(x), f(y))$ *and it is said to be an isomorphism if it is a one to one and onto homomorphism. In similar to algebraic system,* $Aut(H) = \{f : H \to H \mid f$ *is an isomorphism on hypergroup* $H\}$ *is defined. Assume that* $\varphi : H \longrightarrow H/\beta^*$ *by* $\varphi(x) = \beta^*(x)$ *is the canonical homomorphism, then* $w_H = \{x \in H \mid \varphi(x) = 1\}$ *means heart of* H.

Definition 4 [8] *For each* $\emptyset \neq X \subseteq H$*, a subpolygroup generated by* X *is the intersection of all subpolygroups of* H *which contain* X *and is denoted by* $< X >$.

(i) *In every hypergroup* H*, a commutator of* $x, y \in H$ *is shown by* $[x, y] = \{h \in H \mid \rho(x, y) \cap \rho(h, y, x) \neq \emptyset\}$ *and* $H = L_0(H) \supseteq L_1(H) \supseteq \cdots$ *is called a lower series of* H*, where for any* $n \in \mathbb{N}^*$, $L_{n+1}(H) = \{h \in [x, y] \mid x \in L_n(H), y \in H\}$.

(ii) *In every hypergroup* H*,* $H = \Gamma_0(H) \supseteq \Gamma_1(H) \supseteq \cdots$ *is called a derived series of* H*, where for each* $n \in \mathbb{N}^*$, $\Gamma_{n+1}(H) = \{h \in [x, y] \mid x, y \in \Gamma_n(H)\}$. *A polygroup* (H, ρ, e, ϑ) *means a nilpotent polygroup, if for some given integer* $n \in \mathbb{N}, \rho(l_n(H), w_H) = w_H$*, where* $l_{n+1}(H) = \langle\{h \in [x, y] \mid x \in l_n(H), y \in H\}\rangle$ *and* $l_0(H) = H$ *(if there exists a smallest integer* c *in a way that* $\rho(l_c(H), w_H) = w_H$*, and* c *is called the nilpotency class for* H*).*

(iii) *In every hypergroup* H*, for each given* $n \in \mathbb{N}$*, define* $H' = H^{(1)} = \langle\Gamma_1(H)\rangle$ *and* $H^{(n+1)} = (H^{(n)})'$.

3 Automorphism of very thin polygroups

In this section, we introduce the concept of very thin polygroups and for given an arbitrary set constructed at least a very thin polygroup. Moreover, we obtain the number of automorphism group of some very thin polygroups.

Proposition 1 *Let* $(G, \cdot, e)$ *be a polygroup. If for all* $x \in G$ *we have* $x \cdot x^{-1} = \{e\}$*, then* G *is a group.*

Proof. Let $x, y \in G$ and $|x \cdot y| \geq 2$. Then there exists $z_1, z_2 \in x \cdot y$. It follows that $y \in x^{-1} \cdot z_2$ and so $z_1 \in x \cdot y \subseteq x \cdot (x^{-1} \cdot z_2) = (x \cdot (x^{-1}) \cdot z_2 = e \cdot z_2 = \{z_2\}$. Hence G is a group. □

Let $(G, \cdot, e,^{-1})$ be a polygroup, where $n, r \in \mathbb{N}$ and $|G| = n$. Consider $\mathcal{A}^{(r)} = \{x \cdot y \mid r = |x \cdot y| \text{ and } x, y \in G\}$ and $\mathcal{A} = \bigcup_{1 \leq r \leq n} \mathcal{A}^{(r)}$.

Definition 5 *A polygroup* $(G, \cdot, e,^{-1})$ *is said to be an* m*–very thin polygroup if* $|\mathcal{A}^{(m)}| = 1$*, where* $m \in \mathbb{N}$.

It is clear that every 1–very thin polygroup is isomorphic to a group and we consider any 2–very thin polygroup as a very thin polygroup.

Example 1 *Let* $(G, \cdot, e)$ *be a (non)commutative group and* $g \notin G$*. Define a hyperoperation "*$\cdot_g$*" on* G' *as follows:*

$$x \cdot_g y = \begin{cases} \{x \cdot y\} & x, y \in G, x \neq y^{-1} \\ \{e, g\} & x = y^{-1} \in G \setminus \{e\}, \\ \{e\} & x = y = g, \\ y & x = g, y \in G \setminus \{e\}, \\ x & y = g, x \in G \setminus \{e\} \end{cases}$$

and $e \cdot_g g = g \cdot_g e = g$*. Some modifications and computations show that* $(G', \cdot_g, e)$ *is a (non)commutative very thin polygroup.*

Definition 6 *Let* $(G, \cdot)$ *be a polygroup. Then* G *is called a cyclic polygoup, if there exists* $g \in G$ *in such a way that* $G = \bigcup_{n \in \mathbb{Z}} g^n$*, where* $g^0 = g \cdot g^{-1}$*,* $g^n = \underbrace{g \cdot g \cdot \ldots \cdot g}_{n-\text{times}}$ *and it is denoted by* $G = \langle g \rangle$*.*

Example 2 (i) *Let* $G = \{0, a\}, r \notin G$*, and* $G' = G \cup \{r\}$*. Then for* $1 \leq i \leq 3$*,* $(G', +_r^{(i)}, 0, -)$ *are cyclic polygroups as follows:*

$+_r^{(1)}$	0	a	r
0	{0}	{a}	{r}
a	{a}	{0, r}	{a}
r	{r}	{a}	{0}

,

$+_r^{(2)}$	0	a	r
0	{0}	{a}	{r}
a	{a}	{0}	{r}
r	{r}	{r}	{0, a}

and

$+_r^{(3)}$	0	a	r
0	{0}	{a}	{r}
a	{a}	{0, r}	{a}
r	{r}	{a}	{0, r}

.

(ii) *Let* $G = \{e, a, b, c\}$*. Then* $(G, \cdot, e, ^{-1})$ *is a cyclic polygroup as follows:*

$\cdot$	e	a	b	c
e	{e}	{a}	{b}	{c}
a	{a}	{a}	G	c
b	{b}	{e, a, b}	{b}	{b, c}
c	{c}	{a, c}	{c}	G

,

where $G = \langle c \rangle$*.*

The above Example, shows that cyclic polygroups necessarily are not commutative polygroup.

Corollary 1 *Let* $n \in \mathbb{N}$*. Then there exists a cyclic polygroup* $(G, \cdot)$ *in such a way that* $|G| = n$*.*

Proof. Let $(H, \cdot)$ be a cyclic group and $g \notin H$ and $G = H \cup \{g\}$. Hence similar to Example 1, we can see that G is a cyclic very thin polygroup, where $G = \bigcup_{n \in \mathbb{Z}} a^n$ and $a^0 = a \cdot a^{-1}$. □

Theorem 2 *Let $G = \langle a \rangle$ be a cyclic polygroup, G' be a polygroup and $f : G \longrightarrow G'$ be an on to homomorphism, where $a \in G$. Then*

(i) *G' is a cyclic polygroup,*

(ii) *G/β^* is a cyclic group,*

(iii) *if $K \leq G$ and K be a complete part of G, then K is a cyclic subpolygroup of G.*

Proof. (i) *Consider $G' = \langle f(a) \rangle$ and so G' is a cyclic polygroup.*

(ii) *Consider $G/\beta^* = \langle \beta^*(a) \rangle$ and so G/β^* is a cyclic group.*

(iii) *Consider $K = \langle a^n \rangle$, where n is the smallest natural number such that $a^n \cap K \neq \emptyset$. Since $x \in K$ implies that there exists $m \in \mathbb{N}$ such that $x \in a^m$, then $a^m \cap K \neq \emptyset$ and so $S = \{l \in \mathbb{N} \mid a^l \cap K \neq \emptyset\} \neq \emptyset$ and so there exist the smallest natural number such that $a^n \cap K \neq \emptyset$. But K is a complete part of G, then $a^n \subseteq K$ and for $m \in \mathbb{N}$ we have $(a^n)^m \subseteq K$. In addition, if $x \in K$, then there exists $m \in \mathbb{N}$ such that $x \in a^m$. If there exists $0 \leq r < n \leq m$, where $m = nq + r$ and $r \neq 0$, then we have*

$$K = x \cdot K \subseteq a^m \cdot K \subseteq (a^{nq} \cdot a^r) \cdot K = a^r \cdot K.$$

It follows that $a^r \cap K \neq \emptyset$, which is a contradiction and so $K = \langle a^n \rangle$. □

The concept of strong homomorphism is defined in [8]. Let $(G_1, \cdot_1, e_1)$ and $(G_2, \cdot_2, e_2)$ be polygroups and $\alpha : G_1 \longrightarrow G_2$ be a map. We define α is a homomorphism, if for all $x, y \in G_1, \alpha(x \cdot_1 y) = \alpha(x) \cdot_2 \alpha(y)$.

Lemma 1 *Let G_1 and G_2 be polygroups and $\alpha : G_1 \longrightarrow G_2$ be a homomorphism. Then*

(i) *$e_2 \in Im(\alpha)$ if and only if $\alpha(e_1) = e_2$,*

(ii) *for all $x \in G_1$, $\alpha(e_1) = e_2$ implies that $\alpha(x^{-1}) = (\alpha(x))^{-1}$.*

Proof. (i) Since $e_2 \in Im(\alpha)$, there exists $x \in G_1$ in such a way that $e_2 = \alpha(x)$. So $e_2 = \alpha(x) = \alpha(e_1 \cdot_1 x) = \alpha(e_1) \cdot_2 \alpha(x) = \alpha(e_1) \cdot_2 e_2 = \alpha(e_1)$.

(ii) By definition and the item (i), is obtained. □

Table 1: polygroup G

$\cdot$	a_0	a_1	a_2	a_3	a_4	a_5	a_6
a_0	a_0	a_1	a_2	a_3	a_4	a_5	a_6
a_1	a_1	T	a_3	a_2	a_1	a_1	a_1
a_2	a_2	a_3	T	a_1	a_2	a_2	a_2
a_3	a_3	a_2	a_1	T	a_3	a_3	a_3
a_4	a_4	a_1	a_2	a_3	T	$T\setminus\{a_0\}$	$T\setminus\{a_0\}$
a_5	a_5	a_1	a_2	a_3	$T\setminus\{a_0\}$	T	$T\setminus\{a_0\}$
a_6	a_6	a_1	a_2	a_3	$T\setminus\{a_0\}$	$T\setminus\{a_0\}$	T

Theorem 3 *Let* $(G,\cdot)$ *be a polygroup,* $g \notin G$ *and* $G' = G \cup \{g\}$*. Then*

(i) *If* $\alpha \in \mathrm{Aut}(G', \cdot_g, ^{-1}, e)$*, then* $\alpha(e) = e$ *and* $\alpha(g) = g$,

(ii) $\mathrm{Aut}(G', \cdot_g, ^{-1}, e) \cong \mathrm{Aut}(G, ., ^{-1}, e)$.

Proof. (i) By Lemma 1, $\alpha(e) = e$. Clearly, $e = \alpha(e) = \alpha(g \cdot g) = \alpha(g) \cdot \alpha(g)$, so $\alpha(g) = e$ or $\alpha(g) = g$. If $\alpha(g) = e$, it follows that $\alpha(g) = \alpha(e)$. Since α is an one to one map, we get $g = e$ that is a contradiction. Thus $\alpha(g) = g$.

(ii) Let $\alpha \in \mathrm{Aut}(G', \cdot_g, ^{-1}, e)$. Then $f : \mathrm{Aut}(G', \cdot_g, ^{-1}, e) \to \mathrm{Aut}(G, ., ^{-1}, e)$ by $f(\alpha) = \alpha|_G$ is an isomorphism and so $\mathrm{Aut}(G', \cdot_g, ^{-1}, e) \cong \mathrm{Aut}(G, ., ^{-1}, e)$. $\square$

Corollary 2 *Let* $n \in \mathbb{N}$*. Then*

(i) $|\mathrm{Aut}(\mathbb{Z}_n, +_g, -, \bar{0})| = \varphi(n)$.

(ii) $|\mathrm{Aut}(\mathbb{Z}, +_g, -, 0)| = 2$.

(iii) $|\mathrm{Aut}(D_{2n}, \cdot_g, -, \bar{0})| = |\mathrm{Aut}(D_{2n})| = n\varphi(n)$.

(iv) $|\mathrm{Aut}(S_n, \cdot_g, -, \bar{0})| = |\mathrm{Aut}(S_n)| = n!$.

Example 3 *Let* $G = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6\}$*. Then* $(G,\cdot)$ *is a polygroup in Table 1, where* $T = \{a_0, a_4, a_5, a_6\}$*. Simple computations show that* $\mathrm{Aut}(G) = \{xy \mid x \in S_3, y \in S_X,$ *where* $X = \{4,5,6\}\}, S_3 \trianglelefteq \mathrm{Aut}(G), S_X \trianglelefteq \mathrm{Aut}(G), S_3 \cap S_X = \{e\}$ *for all* $\alpha \in \mathrm{Aut}(G)$ *we have ,* $\alpha(T) = T$ *and so* $\mathrm{Aut}(G,\cdot) \cong S_3 \times S_3$.

Let G be a non–empty set. We denote the set of all very thin polygroups on G by $VP(G)$, the number of all very thin polygroups on G by $|VP(G)|$ and the number of all very thin polygroups up to isomorphic on G by $\|VP(G)\|$.

Theorem 4 *Let* $m \in \mathbb{N}$, $(G, \cdot, e)$ *be an* m*–very thin polygroup and* $x, y \in G$. *If* $|x \cdot y| \neq 1$, *then* $e \in x \cdot y$.

Proof. Let $e \notin x \cdot y$ and there exists $a \in G$ such that $|a \cdot a^{-1}| \neq 1$. Since $(G, \cdot, e)$ is an m–very thin polygroup $e \in a^{-1} \cdot a = x \cdot y$, which is a contradication. Thus for any $a \in G, |a \cdot a^{-1}| = 1$, so G must be a group, which is a contradiction. □

Corollary 3 *Let* $m \in \mathbb{N}$ *and* (G, e) *be an* m*–very thin polygroup. Then*

(i) *there exist* $a_1, a_2, \ldots a_{m-1}, x \in G$ *such that* $x \cdot x^{-1} = \{e, a_1, a_2, \ldots a_{m-1}\}$,

(ii) *if* $m = 2$ *and* $a_1 \in x \cdot x^{-1}$, *then for all* $\alpha \in Aut(G)$ *we have* $\alpha(a_1) = a_1$.

Proof. (i) By definition is clear. (ii) Let $\alpha \in Aut(G)$. If $x \cdot x^{-1} = \{e, a_1\}$, we have $\alpha(x) \cdot \alpha^{-1}(x) = \alpha(x) \cdot \alpha(x^{-1}) = \alpha(x \cdot x^{-1}) = \alpha(\{e, a_1\}) = \{e, \alpha(a_1)\}$. Hence $\alpha(a_1) \neq e$ and $\{e, a_1\} = \{e, \alpha(a_1)\}$, imply that $\alpha(a_1) = a_1$. □

Theorem 5 *Let* G *be a non–empty set and* $e \in G$.

(i) *If* $|G| = 2$, *then* $|VP(G)| = 2$ *and* $\|VP(G)\| = 1$.

(ii) *If* $|G| = 3$, *then* $|VP((G, e))| = 4$ *and* $\|VP((G, e))\| = 2$,

(iii) *If* $|G| = 3$, *then* $|VP(G)| = 12$ *and* $\|VP(G)\| = 2$.

Proof. (ii), (iii) Let $G = \{e, a, b\}$. For all $x, y \in G, |x \cdot y| \neq 1$ implies that $|x \cdot y| = 2$. If (G, e) is a very thin polygroup, then $a \cdot a = \{e\}, b \cdot b = \{e, a\}$ or $a \cdot a = b \cdot b = \{e, a\}$ or $a \cdot a = b \cdot b = \{e, b\}$, or $b \cdot b = \{e\}, a \cdot a = \{e, b\}$. So $|VP((G, e))| = 4$ and $\|VP((G, e))\| = 3$. □

Let G be a non–empty set. We can compute the set of all very thin polygroups on G and the number of all very thin polygroups up to isomorphic on G, whence $|G| \leq 3$. But can't compute for $|G| \geq 4$.

Open Problem 1 *Let* $m, n \in \mathbb{N}$, $|G| = n$ *and* G *be an* m*–very thin polygroup. Then* $|VP(G)| = ?$ *and* $\|VP(G)\| = ?$

Theorem 6 *Let* $|G| \in \{2, 3\}$. *If* G *is a very thin polygroup, then it is a commutative very thin polygroup and* $|Aut(G)| = 1$.

Proof. It is obtained by Corollary 3 and Theorem 5. □

Example 4 *Consider the very thin polygroup* $G = \mathbb{Z}_3 \cup \{g\}$, *where* $g \notin \mathbb{Z}_3$. *By Corollary 2, we have* $|Aut(G)| = 2$. *It shows that if* G *is a very thin polygroup and* $|G| \geq 4$, *then necessarily* $|Aut(G)| \neq 1$.

Proposition 2 *Let* G *be a hypergroup,* $x \in G$, $\overline{G} = G/\beta$ *and* $\alpha \in Aut(G)$. *Then*

(i) $\overline{\alpha} \in Aut(\overline{G})$, *where* $\overline{\alpha}(\overline{x}) = \overline{\alpha(x)}$ *and* $\overline{x} = \beta^*(x)$,

(ii) $\overline{Aut(G)} \subseteq Aut(\overline{G})$, *where* $\overline{Aut(G)} = \{\overline{\alpha} \mid \alpha \in Aut(G)\}$,

Proof. (i) Let $\overline{x} = \overline{y}$. Then there exists $u \in \mathcal{U}$ in such a way that $\{x, y\} \subseteq u$ and so $\{\alpha(x), \alpha(y)\} \subseteq \alpha(u) \in \mathcal{U}$. Hence, $\overline{\alpha}(\overline{x}) = \overline{\alpha}(\overline{y})$ and then $\overline{\alpha}$ is a well-defined map. In similar a way one can see that $\overline{\alpha}$ is an isomorphism. □

Corollary 4 *Let* G *be a hypergroup,* $\overline{G} = G/\beta^*$ *and* $\alpha, \theta \in Aut(G)$. *Then*

(i) $\overline{\alpha}^{-1} = \overline{\alpha^{-1}}$,

(ii) $\overline{\alpha \circ \theta} = \overline{\alpha} \circ \overline{\theta}$,

(iii) $\overline{Aut(G)} \leq Aut(\overline{G})$.

Proof. Let $x \in G$. Define $\overline{\alpha}(\beta^*(x)) = \beta^*(\alpha(x))$. So the proof is obtained. □

4 Autonilpotent polygroups

In this section, we introduce the concept of autonilpotent polygroup and via the fundamental relations and regular relations consider some conditions to construct autonilpotent groups and autonilpotent polygroups.

Let G be a hypergroup, $x \in G$ and $\alpha \in Aut(G)$. Define $[x, \alpha] = \{g \in G \mid x \in g \cdot \alpha(x)\}$ and will call an autocommutator of x and α. Inductively, for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in Aut(G), [x, \alpha_1, \alpha_2, \ldots, \alpha_n] = \big[[x, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}], \alpha_n\big]$ is an autocommutator of $x, \alpha_1, \alpha_2, \ldots, \alpha_n$ of weight $n+1$, where for all $X \subseteq G$ we have $[X, \alpha] = \bigcup_{x \in X} [x, \alpha]$. Let $K_0(G) = G$ and for every $n \in \mathbb{N}^*$, consider $K_{n+1}(G) = \{g \in [x, \alpha] \mid x \in K_n(G), \alpha \in Aut(G)\}$.

Definition 7 *Let* $n \in \mathbb{N}$, G *be a polygroup. Then* G *is called an autonilpotent polygroup of class at most* n, *if* $K_n(G) \subseteq \omega_G$.

Proposition 3 *Let* (G, e) *be a polygroup,* $x \in G, n \in \mathbb{N}$ *and* $\alpha \in Aut(G)$.

(i) $[x, id] = [e, x] = x \cdot x^{-1}$ *and* $[e, \alpha] = e$,

(ii) $[x, \alpha] = x \cdot \alpha(x^{-1})$,

(iii) $\beta^*([x, \alpha]) = [\beta^*(x), \overline{\alpha}]$,

(iv) $[x, \alpha]^{-1} = [\alpha(x), \alpha^{-1}]$,

(v) $K_n(G) = \{h \in [g, \alpha_1, \alpha_2, \ldots, \alpha_n] \mid g \in G, \alpha_1, \alpha_2, \ldots, \alpha_n \in Aut(G)\}$,

(vi) $K_{n+1}(G) \subseteq K_n(G)$.

Proof. Since $[x, \alpha] = \{g \in G \mid x \in g \cdot \alpha(x)\} = \{g \in G \mid g \in x \cdot \alpha(x^{-1})\}$ and $\beta([x, \alpha]) = \beta(x\alpha(x^{-1})) = [\beta(x), \alpha^{-1}]$, we get the results. □

Example 5 (i) *Consider the very thin polygroup* $G = (\mathbb{Z}, +_g, -, 0)$. *Routine computations show that* $K_1(G) = 2\mathbb{Z} \cup \{g\}, K_2(G) = 2^2\mathbb{Z} \cup \{g\}$ *and for every* $n \in \mathbb{N}$ *we have* $K_n(G) = 2^n\mathbb{Z} \cup \{g\}$, *while* $w_G = \{0, g\}$. *Hence* G *is not an autonilpotent polygroup.*

(ii) *Consider the very thin polygroup* $G = (D_6, +_g, -, 0)$. *Routine computations show that for every* $n \in \mathbb{N}$ *we have* $K_n(G) = \{id, (1, 2, 3), (1, 3, 2), g\}$ *while* $w_G = \{0, g\}$. *Hence* G *is not an autonilpotent polygroup.*

Theorem 7 *Let* $(G, \cdot, e, ^{-1})$ *be a group and* $g \notin G$. *Then*

(i) $(G', \cdot_g, e, ^{-1})/\beta^* \cong (G, \cdot, e, ^{-1})$ *and* $\overline{Aut(G')} = Aut(\overline{G'})$,

(ii) *for all* $n \in \mathbb{N}^*$, *we have* $K_n(G) \cup \{g\} = K_n(G')$,

(iii) $(G', \cdot_g, e, ^{-1})$ *is an autonilpotent polygroup if and only if* $(G, \cdot, e, ^{-1})$ *is an autonilpotent group.*

Proof. (i) It is easy to see that $\overline{e} = \overline{g} = \{e, g\}$ and for all $x \notin \overline{e}$ we have $\overline{x} = x$.

(ii) Obviously we have $K_0(G) \cup \{g\} = K_0(G')$ and by induction one can see that $K_n(G) \cup \{g\} = K_n(G')$.

(iii) Since $w_{G'} = \{e, g\}$ and by the item (ii) the proof is obtained. □

Example 6 *Let* $G = \{e, a, b\}$. *Then* $(G, \cdot, e, ^{-1})$ *is a polygroup as follows:*

$\cdot$	e	a	b
e	$\{e\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{e, b\}$	$\{a, b\}$
b	$\{b\}$	$\{a, b\}$	$\{e, a\}$

.

It is easy to see that $Aut(G) = \{id, \alpha = (a\ b)\}$ *and* $\overline{Aut(G)} = Aut(\overline{G})$*. In addition for every* $n \in \mathbb{N}^*, K_n(G) = w_G = G$*, implies that* G *is an autonilpotent polygroup.*

Theorem 8 *Let* $n \in \mathbb{N}$*,* $k = 2^n$*,* $g \notin G'$ *be a cyclic group and* $|G'| = k$ *. Then* $G = (G' \cup \{g\}, \cdot_g, 0)$ *is an autonilpotent polygroup of class at most* n*.*

Proof. Let $G' = \langle a \rangle$ and $a \in G$. By Corollary 2, $|Aut(G)| = 2^n - 2^{n-1}$. Let $\alpha \in Aut(G)$. Then $\alpha(a) = a^r$, where $r \in S = \{1, 3, 5, 2^n - 1\}$. So

$$K_1(G) = \{a^{r-1}, g \mid r \in S\}, K_2(G) = \{a^{2(r-1)}, g \mid r \in S\},$$
$$K_3(G) = \{a^{4(r-1)}, g \mid r \in S\}, \dots \quad \text{and} \quad K_n(G) = \{a^{2^{n-1}(r-1)}, g \mid r \in S\} = \{0, g\}.$$

Hence $w_G = \{0, g\} = K_n(G)$ and so G is an autonilpotent polygroup. □

Corollary 5 *Let* $k \in \mathbb{N}$*,* $n = 2^k$*. Then* $G = (\mathbb{Z}_n \cup \{\sqrt{2}\}, \cdot_{\sqrt{2}}, \overline{0})$ *is an autonilpotent polygroup.*

Proof. Let $k \in \mathbb{N}$, $n = 2^k$. Since $\mathbb{Z}_{2^k}$ is an autonilpotent group, by definition of G, $G = (\mathbb{Z}_n \cup \{\sqrt{2}\}, \cdot_{\sqrt{2}}, \overline{0})$ is an autonilpotent polygroup. □

Definition 8 *Let* G *be a polygroup. Define* $Z_0(G) = w_G$*, for every* $n \in \mathbb{N}^*, Z_{n+1}(G) = \{x \mid [x, \alpha] \subseteq Z_n(G), \forall \alpha \in Aut(G)\}$ *and we called it by absolute center of* G*.*

Theorem 9 *Let* G *be a polygroup,* $x \in G$ *and* $n \in \mathbb{N}^*$*. Then*

(i) $Z_n \subseteq Z_{n+1}$ *and so* $w_G \subseteq Z_n$*,*

(ii) $Z_n(G)$ *is a complete part of* G*,*

(iii) $[x, id] \subseteq Z_n(G)$*,*

(iv) *if* $|Aut(G)| = 1$*, then* G *is autonilpotent.*

Proof. (i) Let $\alpha \in Aut(G)$. Since $\alpha(w_G) \subseteq w_G$, we get that $Z_0(G) \subseteq Z_1(G)$ and so by induction the proof is obtained.

(ii) By item (i), $w_G \subseteq Z_n$ implies that $C(Z_n(G)) = Z_n(G) \cdot w_G = Z_n(G)$. Thus $Z_n(G)$ is a complete part of G.

(iii) It is obtained by item (i).

(iv) Let $x \in K_n(G)$ and $h \in [x, id]$. Then by definition we have $h \in x \cdot x^{-1} \subseteq w_G$ that it follows $K_{n+1}(G) \subseteq w_G$. Thus G is autonilpotent. □

Corollary 6 *Let* G *be a very thin polygroup. If* $|G| \leq 3$*, then* G *is an autonilpotent polygroup.*

Proof. It is obtained from Theorems 6 and 9. □

Theorem 10 *Let* G *be a polygroup and* $n \in \mathbb{N}$. $K_n(G) \subseteq \omega_G$ *if and only if* $Z_n(G) = G$.

Proof. Let $Z_n(G) = G$. Then by induction on i, we have $K_i(G) \subseteq Z_{n-i}(G)$. Now for $i = n$ we obtain that $K_n(G) \subseteq Z_0(G) = \omega_G$.

Conversely, if $K_n(G) \subseteq \omega_G$, then by induction we conclude $K_{n-i}(G) \subseteq Z_i(G)$. Letting $i = n$ implies that $G = K_0(G) \subseteq Z_n(G) \subseteq G$. □

Example 7 (i) *Let* $G = \mathbb{Z}_4 \cup \{g\}$*. Then for all* $n \geq 2$*, we have* $Z_n(G) = Z_n(\mathbb{Z}_4) \cup \{g\} = G$ *and so it is an autonilpotent polygroup.*

(ii) *Let* $G = S_3 \cup \{g\}$*. Then* $Z_n(G) = Z_n(S_3) \cup \{g\} = \{e, g\}$ *and so it is not an autonilpotent polygroup.*

Corollary 7 *Let* G *be a polygroup.* G *is an autonilpotent polygroup if and only if there exists some* $n \in \mathbb{N}$ *in such a way that* $Z_n(G) = G$.

Theorem 11 *Let* $G \neq \{e\}$ *be an autonilpotent group. Then* $Z_1(G) \neq \{e\}$.

Proof. Since G is an autonilpotent group, by Corollary 7, there exists some $n \in \mathbb{N}$ in such a way that $Z_n(G) = G$. Let $Z_1(G) = \{e\}$. Then for all $n \geq 2$ we obtain that $Z_n(G) = \{e\}$, which is a contradiction. □

Theorem 12 *Let* G *be a polygroup,* $\mathrm{Aut}(G)$ *be a commutative group and* $\alpha \in \mathrm{Aut}(G)$*. Then*

(i) $\alpha(K_n(G)) \subseteq K_n(G)$,

(ii) *if* $x \in K_n(G)$*, then* $x^{-1} \in K_n(G)$,

(iii) *if for all* $x \in G, x \cdot x^{-1} = \{e\}$*, then* $L_n(G) \subseteq K_n(G)$,

(iv) *if* G *is an autonilpotent group, then it is an nilpotent group.*

Proof. (i) Let $h \in K_{n+1}(G)$ and $f \in \mathrm{Aut}(G)$. Then there exist $x \in K_n(G)$ and $\alpha \in \mathrm{Aut}(G)$ such that $h \in [x, \alpha]$ and so $f(h) \in f(x \cdot \alpha(x^{-1}))) = f(x) \cdot f(\alpha(x^{-1})) = f(x)\alpha(f(x^{-1})) = [f(x), \alpha]$. So by induction hypothesis, $x \in K_n(G)$ implies that $f(x) \in K_n(G)$ and so $f(h) \in K_{n+1}(G)$.

(ii) Let $h \in K_{n+1}(G)$. Then there exist $x \in K_n(G)$ and $\alpha \in Aut(G)$ in such a way that $h \in [x, \alpha]$. Using Proposition 3, we have $h^{-1} \in [\alpha(x), \alpha^{-1}]$ and by the item (i), we obtain that $h^{-1} \in K_{n+1}(G)$.

$(iii), (iv)$ It is obtained by induction. Let $h \in [x, y]$ and $x \in L_n(G)$. Then induction assumption, implies that $x \in K_n(G)$. Let $y \in G$ and $\varphi_y \in Inn(G)$, where for all $a \in G$, we have $\varphi_y(a) = y{\cdot}a{\cdot}y^{-1}$. Thus $h = x{\cdot}y{\cdot}x^{-1}{\cdot}y^{-1} = [x, \varphi_y]$. Hence by the item (ii), we conclude so $h \in K_{n+1}(G)$. □

Theorem 13 *Let* G *be a hypergroup,* $n \in \mathbb{N}, \overline{G} = G/\beta^*$ *and* $Aut(\overline{G}) \subseteq \overline{Aut(G)}$. *Then*

(i) $K_n(\overline{G}) = \{\overline{t} \mid t \in K_n(G)\}$,

(ii) G *is an autonilpotent polygroup if and only if* $\overline{G}$ *is an autonilpotent group.*

Proof.

(i) Let $\overline{a} \in K_{n+1}(\overline{G})$. Then there exist $\alpha \in Aut(\overline{G})$ and $\overline{x} \in K_n(\overline{G})$ in such a way that $\overline{a} = [\overline{x}, \alpha]$. Thus there exists $\alpha_0 \in Aut(G)$ such that $\overline{\alpha_0} = \alpha$. Using induction hypotheses implies that there exists $t \in K_n(G)$ such that $\overline{x} = \overline{t}$. If $b \in [t, \alpha_0]$, then $b \in K_{n+1}(G)$ and $\overline{b} = [\overline{x}, \overline{\alpha_0}] = [\overline{x}, \alpha] = \overline{a}$. The converse is clear.

(ii) Let G be an autonilpotent polygroup. Then there exists $n \in \mathbb{N}$ in such a way that $K_n(G) \subseteq \omega_G$. It follows $K_n(\overline{G}) = \{e\}$. The converse is similarly. □

In [8], it is shown that every polygroup G is a nilpotent polygroup if and only if G/β^* is a nilpotent group. So we have the following theorem.

Theorem 14 *Let* G *be an autonilpotent polygroup and* $Aut(G/\beta^*) \subseteq \overline{Aut(G)}$. *Then* G *is a nilpotent polygroup.*

Proof. Applying, Theorem 13, G/β^* is an autonilptent group, so there exists $n \in \mathbb{N}$ in such a way that $K_n(G/\beta^*) = \{\beta^*(e)\}$. By Theorem 12, $L_n(G/\beta^*) = \{\beta^*(e)\}$ and so it is a nilpotent group. Therefore, G is a nilpotent polygroup. □

For any given autonilpotent polygroup G, we can't prove that prove or disprove that it is a nilpotent polygroup, so we give up it as the following open problem.

Open Problem 2 *Let* G *be an autonilpotent polygroup. Then it is a nilpotent polygroup.*

Example 8 *Consider the polygroup* $G = \mathbb{Z}_6 \cup \{g\}$, *where* $g \notin G$. *Thus the converse of Theorem 14 it is not necessarily true.*

Theorem 15 *Let* G_1, G_2 *be hypergroups. Then*

(i) $K_n(G_1) \times K_n(G_2) \subseteq K_n(G_1 \times G_2)$,

(ii) $w_{G_1 \times G_2} = w_{G_1} \times w_{G_2}$,

(iii) *if* $K_n(G_1) \times K_n(G_2) \subseteq w_{G_1} \times w_{G_2}$, *then* $K_n(G_1) \subseteq w_{G_1}$ *and* $K_n(G_2) \subseteq w_{G_2}$.

Proof. $(i),(ii)$ *We prove by induction. Let* $(h_1, h_2) \in K_{n+1}(G_1) \times K_{n+1}(G_2)$. *Then there exist* $x_1 \in K_(G_1), x_2 \in\in K_(G_2), \alpha_1 \in Aut(G_1)$ *and* $\alpha_2 \in Aut(G_2)$ *in such a way that* $h_1 \in [x_1, \alpha_1]$ *and* $h_2 \in [x_2, \alpha_2]$. *Define* $\alpha = (\alpha_1, \alpha_2)$ *by* $\alpha(x, y) = (\alpha_1(x), \alpha_2(y))$. *Clearly* $\alpha \in Aut(G_1 \times G_2)$ *and so by induction assumption,* $(x_1, x_2) \in K_n(G_1) \times K_n(G_2) \subseteq K_n(G_1 \times G_2)$. *So* $(h_1, h_2) \in [(x_1, x_2), (\alpha_1, \alpha_2)] \subseteq K_{n+1}(G_1 \times G_2)$.
(ii) [8]. □

Example 9 *Consider the polygroup* $G_1 = G_2 = \mathbb{Z}_2$. *It is easy to see that* $\{(\overline{0}, \overline{0})\} = K_1(G_1) \times K_1(G_1) \subset K_1(G_1 \times G_2)$. *So necessarily for all* $n \in \mathbb{N}, K_n(G_1) \times K_n(G_2) \neq K_n(G_1 \times G_2)$.

Theorem 16 *Let* G_1 *and* G_2 *be polygroups. If* $G_1 \times G_2$ *is an autonilpotent polygroup, then* G_1 *and* G_2 *are autonilpotent polygroups.*

Proof. Since $G_1 \times G_2$ is an autonilpotent polygroup, then there exists $n \in \mathbb{N}$ such that $K_n(G_1 \times G_2) \subseteq w_{G_1 \times G_2} = w_{G_1} \times w_{G_2}$. Applying Theorem 15, we have

$$K_n(G_1) \times K_n(G_2) \subseteq K_n(G_1 \times G_2) \subseteq w_{G_1} \times w_{G_2}.$$

It follows that $K_n(G_1) \subseteq w_{G_1}$ and $K_n(G_2) \subseteq w_{G_2}$. Hence G_1 and G_2 are autonilpotent polygroups. □

Example 10 *Let* $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. *Then* $Aut(G) \cong S_3$ *and* $K_1(G) = G$. *So* G *is not an autonilpotent polygroup, whlie* $K_1(\mathbb{Z}_2) = \{\overline{0}\}$ *implies that* $\mathbb{Z}_2$ *is an autonilpotent polygroup.*

The Example 10, shows that the convese of Theorem 16, is not necessarily true.

5 Application

We refer to some applications of our work to sample as follows.

Economic Hypernetwork: Let $G = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6\}$ be a set of some peoples that want to share in an economic benefit and follow the instructions of this partnership based on their abilities. We assume that the instructions are based on the axioms of polygroups and so construct a polygroup as Table 1. This polygroup shows that the people a_0, a_4, a_5, a_6 must be only help in the investment of each person so that the result of the work can be balanced as $T = \{a_0, a_4, a_5, a_6\}$ and sometimes the person a_0 must be removed in this regards.

Artificial Hypernetwork: Let $G = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6\}$ be a set of 7 computers that are used in a intelligent hypernetwork. We want to input layers and output layers of our data satisfy in a certain law, so we put this law in the form of axioms of a polygroup as Table 1 and use their automorphisms as information transfer. Thus we have for all $\alpha \in Aut(G)$ we have, $\alpha(T) = T$ and $Aut(G, \cdot) \cong S_3 \times S_3$ and so we can do this work in 36 ways.

6 Conclusion and discussion

The current paper introduced the notion of $m-$ very thin polygroups, the concept of autonilpotent polygroups and investigated some properties of autonilpotent polygroups. Such as:

(i) For any non–empty set, (non)commutative very thin polygroups are constructed.

(ii) Using the concept of homomorphisms, we obtain the set of autonilpotent of very thin polygroups.

(iii) We show that the set of automorphism of very thin polygroups are equal to set of automorphism of some groups.

(iv) With respect to the concept of nilpotent polygroups, we investigated the relation between of autonilpotent polygroups and nilpotent polygroups.

(v) Through the concept of direct product of autonilpotent polygroups, we extend the autonilpotent polygroups.

We hope that these results are helpful for furthers studies in autonilpotent polygroups. In our future studies, we hope to obtain more results regarding autonilpotent polygroups, autosolvable polygroups, nilpotent polygroups, solvable polygroups and their applications.

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