Acta Univ. Sapientiae, Mathematica, **16**, 2 (2024) 219–231

DOI: 10.47745/ausm-2024-0017

On stammering p-adic Ruban continued fractions

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Abstract. We establish a new transcendence criterion of Ruban p-adic continued fractions and we prove that a p-adic number whose sequence of partial quotients is bounded in $\mathbb{Q}_p$ and has a stammering continued fraction expansion is either quadratic or transcendental where p is a prime number.

1 Introduction

Continued fractions have a crucial role in number theory. In fact, the continued fraction expansion of algebraic numbers is considered an open and difficult problem, as mentioned by Khintchine [5] in his conjecture, which is classified among the complicated questions in number theory. It remains difficult to give explicit and total answers, however, many authors have been able to establish several continued fraction transcendence criteria. As an example, the first author to give examples of transcendental continued fractions having bounded partial quotients was Maillet [8]. After that, Baker [2] improved Maillet's results using conditions that are simpler and more explicit.

2010 Mathematics Subject Classification: 11A55, 11D88, 11J81, 11J87

Key words and phrases: continued fractions, p-adic numbers, transcendence, subspace theorem

Throughout this paper, $\mathcal{A}$ denotes a countable set, called the *alphabet*. A sequence $\mathbf{a} = (a_n)_{n\geq 1}$ whose elements are from $\mathcal{A}$ is identified by the infinite word $a_1 \dots a_n \dots$. The number of letters composing a finite word W on the alphabet $\mathcal{A}$ is called the *length* of W which is denoted by $|W|$.

In 2013, Bugeaud [3] studied the case of stammering continued fractions given by the following theorem:

Theorem 1 *Let $\boldsymbol{a} = (a_n)_{n\geq 1}$ be a sequence of positive integers not ultimately periodic, $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ three sequences of finite words such that:*

i) For every $n \geq 1$, the word $W_n U_n V_n U_n$ is a prefix of the word $\boldsymbol{a}$;

ii) The sequence $\left(\frac{|V_n|}{|U_n|}\right)_{n\geq 1}$ is bounded;

iii) The sequence $\left(\frac{|W_n|}{|U_n|}\right)_{n\geq 1}$ is bounded;

iv) The sequence $(|U_n|)_{n\geq 1}$ is increasing.

Let $\left(\dfrac{p_n}{q_n}\right)_{n\geq 1}$ denote the sequence of convergents to the real number $\alpha = [0, a_1, \dots, a_n, \dots]$. Assume that the sequence $(q_n^{\frac{1}{n}})_{n\geq 1}$ is bounded. Then, α is transcendental.

There exists a similar theory of continued fractions in the p-adic number field $\mathbb{Q}_p$. In 1968, Schneider [12] gave an algorithm of p-adic continued fraction expansion. After two years, Ruban [10] defined another definition which is more alike the real case. Ever since, a lot of authors studied properties of Ruban's continued fractions. For instance, Ubolsri, Laohakosol, Deze and Wang [7, 4, 13, 14] established multiple Ruban continued fractions transcendence criteria. Add to that, Ooto [9] showed that, for the Ruban continued fractions, the analogue of Lagrange's theorem is not true.

The aim of this paper is to study Bugeaud result's analogue, previously stated, for the p-adic continued fractions. The structure of this paper is as follows. In Section 2, we introduce the field of p-adic numbers $\mathbb{Q}_p$, the p-adic absolute value, the Ruban continued fractions and describe some of their fundamental properties. In Section 3, we give our transcendance criterion in $\mathbb{Q}_p$ as well as its proof. Finally, we close by giving an example to highlight the significance and influence of our finding.

2 Continued fractions of p-adic numbers

Let p be a prime number. We denote by $\mathbb{Q}_p$ the *field* of p-adic numbers with

$$\mathbb{Q}_p = \left\{ \sum_{i \geq j} b_i p^i / b_i \in \{0, \dots, p-1\}, j \in \mathbb{Z} \right\}.$$

We also denote by $\mathbb{Z}_p$ the *ring* of the p-adic integers of $\mathbb{Q}_p$ where

$$\mathbb{Z}_p = \left\{ \sum_{i \geq 0} b_i p^i / b_i \in \{0, \dots, p-1\} \right\}.$$

The p-*adic valution* v_p is defined as follows

$$\begin{aligned} v_p : \mathbb{Q} &\longrightarrow \mathbb{Z} \cup \{+\infty\} \\ \alpha &\mapsto \begin{cases} +\infty & \text{if } \alpha = 0, \\ \inf\{i / b_i \neq 0\} & \text{otherwise.} \end{cases} \end{aligned}$$

The field of p-adic numbers $\mathbb{Q}_p$ is equipped with the p-adic absolute value, called *ultrametric absolute value*, normalized to satisfy $|p|_p = \frac{1}{p}$ and defined by $|\alpha|_p = \dfrac{1}{p^{v_p(\alpha)}}$ and $|0|_p = 0$.

Let $\alpha \in \mathbb{Q}_p$. Then, α can be written in the form

$$\alpha = b_{-m} \frac{1}{p^m} + b_{-m+1} \frac{1}{p^{m-1}} + \dots + b_0 + b_1 p + \dots$$

with $m \in \mathbb{Z}$, $b_{-m} \neq 0$ and $b_i \in \{0, \dots, p-1\}$.

Define

$$[\alpha]_p = b_{-m} \frac{1}{p^m} + b_{-m+1} \frac{1}{p^{m-1}} + \dots + b_0,$$

as the p-*adic floor part* of α.

Set $a_0 = [\alpha]_p$. If $\alpha \neq a_0$, then α becomes

$$\alpha = \alpha_0 = a_0 + \frac{1}{\alpha_1},$$

where $\alpha_1 \in \mathbb{Q}_p$, $|\alpha_1|_p \geq p$ and $[\alpha_1]_p \neq 0$. In the same way, if $\alpha_1 \neq a_1$, with $a_1 = [\alpha_1]_p$, then

$$\alpha_1 = a_1 + \frac{1}{\alpha_2},$$

where $\alpha_2 \in \mathbb{Q}_p$. We continue the above process provided $\alpha_n \neq a_n := [\alpha_n]_p$. Finally, it follows that α can be written as

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{\alpha_n}}}},$$

where $a_k = [\alpha_k]_p$ is called a *partial quotient* of α and α_n is the n^{th} *complete quotient* of α.
We note $\alpha = [a_0, \dots, a_n]_p$ which is defined as the *finite Ruban continued fraction.*
Otherwise, if we have $\alpha = [a_0, \dots, a_n, \dots]_p$ then it is called an *infinite Ruban continued fraction*, where $a_0 \in \mathbb{Z}\left[\frac{1}{p}\right] \cap [0, p)$ and $a_n \in \mathbb{Z}\left[\frac{1}{p}\right] \cap (0, p), \forall\, n \geq 1$.
For an infinite Ruban continued fraction $\alpha = [a_0, \dots, a_n, \dots]_p$, we define non-negative rational numbers p_n and q_n by using recurrence equations as follows

$$p_{-1} = 1, \quad q_{-1} = 0, \quad p_0 = a_0, \quad q_0 = 1$$

and for any $n \geq 1$, we have

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2}, \\ q_n = a_n q_{n-1} + q_{n-2}. \end{cases}$$

In $\mathbb{Q}_p$, p_n and q_n are not integers. Thus, we introduce the following notations:

Notation 1 *For any $n \geq 1$, we set*

$$\begin{cases} p'_n = |p_n|_p p_n, \\ q'_n = |q_n|_p q_n. \end{cases}$$

It is clear that p'_n and q'_n are both integers.

The Ruban continued fraction has the following properties, for all $n \geq 0$, we have

- $\dfrac{p_n}{q_n} = [a_0, \dots, a_n]_p$,
- $\alpha = [a_0, \dots, a_{n-1}, \alpha_n]_p = \dfrac{\alpha_n p_{n-1} + p_{n-2}}{\alpha_n q_{n-1} + q_{n-2}}$,
- $p_{n-1} q_n - p_n q_{n-1} = (-1)^n$.

$\frac{p_n}{q_n}$ is called the n^{th} *convergent* of α and we have in $\mathbb{Q}_p$, $\lim_{n\to+\infty} \frac{p_n}{q_n} = \alpha$.

Lemma 1 *[13] Let $\alpha = [a_0, \ldots, a_n, \ldots]_p$ be a p-adic number. Let $\left(\frac{p_n}{q_n}\right)_{n\geq 0}$ denote the sequence of convergents of α. Then, we have*

- $|q_n|_p = |a_1 \ldots a_n|_p, \ \forall\ n \geq 1,$
- $\begin{cases} |p_n|_p = |a_0 \ldots a_n|_p \ \forall\ n \geq 1, & \textit{if } a_0 \neq 0, \\ |p_1|_p = 1, |p_n|_p = |a_2 \ldots a_n|_p \ \forall\ n \geq 2, & \textit{otherwise} \end{cases}$
- $|q_n|_p \geq p^n, \ \forall n \geq 1,$
- $\begin{cases} |p_n|_p \geq p^n, & \textit{if } a_0 \neq 0, \\ |p_n|_p \geq p^{n-1}, & \textit{otherwise} \end{cases} \quad \forall n \geq 1$
- $\begin{cases} |q_n|_p < |q_{n+1}|_p, \\ |p_n|_p < |p_{n+1}|_p, \end{cases} \quad \forall n \geq 0$
- $\left|\alpha - \frac{p_n}{q_n}\right|_p < |q_n|_p^{-2}, \ \forall\ n \geq 0.$

Lemma 2 *[9] If $\beta = [b_0, \ldots, b_n, \ldots]_p$ is a Ruban continued fraction having the same first $(n+1)$ partial quotients as $\alpha = [a_0, \ldots, a_n, \ldots]_p$, then*

$$|\alpha - \beta|_p \leq |q_n|_p^{-2}.$$

Wang [13] and Laohakosol [6] gave a characterization of rational numbers having Ruban continued fractions as follows.

Proposition 1 *Let α be a p-adic number. Then α is rational if and only if its Ruban continued fraction expansion is finite or ultimately periodic with a period equal to $p - p^{-1}$.*

3 Results

The purpose of our main result is to deal with stammering p-adic continued fractions with bounded partial quotients in $\mathbb{Q}_p$.

Let $\mathbf{a} = (a_i)_{i\geq 1}$ be a sequence of elements from an alphabet $\mathcal{A}$. We say that $\mathbf{a}$ satisfies Condition $(\star)$ if $\mathbf{a}$ is not ultimately periodic and if there exist two sequences of finite words $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ such that:

i) For every $n \geq 1$, the word $U_n V_n U_n$ is a prefix of the word $\mathbf{a}$;

ii) The sequence $\left(\frac{|V_n|}{|U_n|}\right)_{n\geq 1}$ is bounded;

iii) The sequence $(|U_n|)_{n\geq 1}$ is increasing.

We denote by $A = \max\{a_i / i \geq 1\}$ and $B = \frac{A+\sqrt{A^2+4}}{2}$.

We begin now by our main result given by the following theorem:

Theorem 2 *Let p be a prime number. Let $\boldsymbol{a} = (a_i)_{i\geq 1}$ be a sequence of rational numbers in $\mathbb{Z}\left[\frac{1}{p}\right] \cap (0, p)$ not ultimately periodic satisfying Condition $(\star)$ and $\alpha = [0, a_1, \ldots, a_i, \ldots]_p$ be a p-adic number. Assume that $-\nu_p(a_i)$ is bounded. If*

$$\frac{\log B}{\log p} < \frac{1}{4}$$

then α is either quadratic or transcendental.

We show an immediate consequence of Theorem 2.

Corollary 1 *Let $p \geq 7$ be a prime number. Let $\boldsymbol{a} = (a_i)_{i\geq 1}$ be a sequence of rational numbers in $\mathbb{Z}\left[\frac{1}{p}\right] \cap (0, p)$ not ultimately periodic such that $-\nu_p(a_i)$ is bounded. If $A \leq 1$ then the p-adic number $\alpha = [0, a_1, \ldots, a_i, \ldots]_p$ is either quadratic or transcendental.*

The primary tool used for the proof of Theorem 2 is the following version of the Schmidt Subspace Theorem, established by Schlickewei [11], which is recalled below:

Theorem 3 *[11] Let p be a prime number, $L_{1,\infty}, \ldots, L_{m,\infty}$ be m linearly independent forms in the variable $\boldsymbol{x} = (x_1, \ldots, x_m)$ with real algebraic coefficients. Let $L_{1,p}, \ldots, L_{m,p}$ be m linearly independent forms with algebraic p-adic coefficients and in the same variable $\boldsymbol{x} = (x_1, \ldots, x_m)$ and let $\varepsilon > 0$ be a real number. Then, the set of solutions $\boldsymbol{x} \in \mathbb{Z}^m$ of the inequality:*

$$\prod_{i=1}^{m} (|L_{i,\infty}(\boldsymbol{x})||L_{i,p}(\boldsymbol{x})|_p) \leq (\max\{|x_1|, \ldots, |x_m|\})^{-\varepsilon}$$

lies in finitely many proper subspaces of $\mathbb{Q}^m$.

The following lemma is also required for the proof of Theorem 2.

Lemma 3 *[1] Suppose that $a_i \in \mathbb{Q}^*{}_+$ and $\{a_i / i \in \mathbb{N}\}$ is bounded. Set $A = \max\{a_i / i \in \mathbb{N}\}$ and $B = \frac{A+\sqrt{A^2+4}}{2}$. Then, for all $n \geq 0$, we have*

$$p_n \leq B^{n+1} \quad \text{and} \quad q_n \leq B^n.$$

Proof. By induction on n. □

Proof of Theorem 2. Assume that the p-adic number $\alpha = [0, a_1, \ldots, a_i, \ldots]_p$ is algebraic of degree at least three. Set $u_n = |U_n|$ and $v_n = |V_n|$, for $n \geq 1$.
α admits infinitely many good quadratic approximants, set then the quadratic number $\alpha_n = [0, \overline{U_n, V_n}]_p = [0, U_n, V_n, U_n, V_n, \ldots]_p$.
Since α and α_n have the same first $(2u_n + v_n + 1)$ partial quotients, then we get from Lemma 2 that

$$|\alpha - \alpha_n|_p \leq |q_{2u_n+v_n}|_p^{-2}. \tag{1}$$

Furthermore, α_n is a root of the polynomial

$$P_n(X) = q_{u_n+v_n} X^2 - (p_{u_n+v_n} - q_{u_n+v_n-1})X - p_{u_n+v_n-1}.$$

Since $|\alpha|_p \leq 1$ and $|\alpha_n|_p \leq 1$ then $|p_i|_p \leq |q_i|_p$, for $i \geq 1$. We have

$$|q_{u_n+v_n}\alpha - p_{u_n+v_n}|_p < |q_{u_n+v_n}|_p^{-1} \tag{2}$$

likewise,

$$|q_{u_n+v_n-1}\alpha - p_{u_n+v_n-1}|_p < |q_{u_n+v_n-1}|_p^{-1}. \tag{3}$$

Because α_n is a root of the polynomial $P_n(X)$ then $P_n(\alpha_n) = 0$. Using (1), (2) and (3), we obtain

$$\begin{aligned} |P_n(\alpha)|_p &= |P_n(\alpha) - P_n(\alpha_n)|_p \\ &= |(q_{u_n+v_n}(\alpha - \alpha_n)(\alpha + \alpha_n) - (p_{u_n+v_n} - q_{u_n+v_n-1})(\alpha - \alpha_n)|_p \\ &= |\alpha - \alpha_n|_p |(q_{u_n+v_n}(\alpha + \alpha_n) - (p_{u_n+v_n} - q_{u_n+v_n-1})|_p \\ &\leq |q_{u_n+v_n}|_p |q_{2u_n+v_n}|_p^{-2}. \end{aligned} \tag{4}$$

We consider the four following independent linear forms with algebraic p-adic coefficients

$$\begin{cases} L_{1,p}(X_1, X_2, X_3, X_4) = \alpha^2 X_1 - \alpha(X_2 - X_3) - X_4, \\ L_{2,p}(X_1, X_2, X_3, X_4) = \alpha X_1 - X_2, \\ L_{3,p}(X_1, X_2, X_3, X_4) = \alpha X_3 - X_4, \\ L_{4,p}(X_1, X_2, X_3, X_4) = X_3, \end{cases}$$

and the following independent linear forms with algebraic real coefficients

$$L_{i,\infty}(X_1, X_2, X_3, X_4) = X_i, \text{ for } 1 \le i \le 4.$$

Keeping Notations 1, we evaluate the product of these linear forms on the quadruple $X_n = (X_1, X_2, X_3, X_4)$ with $X_1 = \mathfrak{q}'_{u_n+v_n}$, $X_2 = \mathfrak{p}'_{u_n+v_n}$, $X_3 = \mathfrak{q}'_{u_n+v_n-1}$ and $X_4 = \mathfrak{p}'_{u_n+v_n-1}$, we get from (2), (3) and (4)

$$\prod_{i=1}^{4} |L_{i,p}(X_n)|_p \le \frac{1}{|q_{u_n+v_n}|_p^4 |q_{2u_n+v_n}|_p^2}.$$

Hence, from Lemma 1 we get

$$\prod_{i=1}^{4} |L_{i,p}(X_n)|_p \le \frac{1}{|q_{u_n+v_n}|_p^4 p^{2(2u_n+v_n)}} \le \frac{1}{|q_{u_n+v_n}|_p^4 p^{u_n+v_n}}.$$

In addition, we have

$$\begin{aligned}\prod_{i=1}^{4} |L_{i,\infty}(X_n)|_\infty &= |\mathfrak{q}'_{u_n+v_n}|_\infty |\mathfrak{p}'_{u_n+v_n}|_\infty |\mathfrak{q}'_{u_n+v_n-1}|_\infty |\mathfrak{p}'_{u_n+v_n-1}|_\infty \\ &\le |q_{u_n+v_n}|_p^4 q_{u_n+v_n}^4.\end{aligned}$$

By Lemma 3, we obtain

$$\prod_{i=1}^{4} |L_{i,\infty}(X_n)|_\infty \le |q_{u_n+v_n}|_p^4 B^{4(u_n+v_n)}.$$

This easily implies that

$$\prod_{i=1}^{4} (|L_{i,\infty}(X_n)|_\infty \, |L_{i,p}(X_n)|_p) \le \frac{B^{4(u_n+v_n)}}{p^{u_n+v_n}}.$$

Since $-v_p(a_i) \le k, \forall\ i \ge 1$, then we have

$$|X_n|_\infty^\varepsilon = |q_{u_n+v_n}|_p^\varepsilon \; q_{u_n+v_n}^\varepsilon \le p^{k\varepsilon(u_n+v_n)} B^{\varepsilon(u_n+v_n)}.$$

It follows then that

$$|X_n|_\infty^\varepsilon \prod_{i=1}^{4} (|L_{i,\infty}(X_n)|_\infty \, |L_{i,p}(X_n)|_p) \le \left(\frac{B^{4+\varepsilon}}{p^{1-k\varepsilon}}\right)^{u_n+v_n} \le \left[\left(\frac{B^{4+\varepsilon}}{p^{1-k\varepsilon}}\right)^{1+\frac{v_n}{u_n}}\right]^{u_n}.$$

From the hypothesis of Theorem 2, by choosing $\varepsilon = \frac{1}{k^2}$ and the fact that $\frac{4k^2+1}{k(k-1)}$ decreases to 4 as k grows, we can choose k large enough in such a way that $-\nu_p(a_i) \leq k, \forall\ i \geq 1$ and $\frac{\log p}{\log B} > \frac{4k^2+1}{k(k-1)}$. Therefore, we obtain

$$\prod_{i=1}^{4}(|L_{i,\infty}(X_n)|_\infty\, |L_{i,p}(X_n)|_p) \leq |X_n|_\infty^{-\varepsilon}.$$

It follows then from Theorem 3 that the points $X_n = (X_1, X_2, X_3, X_4)$ lie in a finite number of proper subspaces of $\mathbb{Q}^4$. Hence, there exist a non-zero integer quadruple (x_1, x_2, x_3, x_4) and an infinite set of distinct positive integers $\mathcal{N}_1$ such that

$$x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4 = 0.$$

By this equation, we get

$$x_1 q_{u_n+v_n} + x_2 p_{u_n+v_n} + x_3 q_{u_n+v_n-1} + x_4 p_{u_n+v_n-1} = 0. \tag{5}$$

It is clear that $(x_1, x_2) \neq (0, 0)$ since otherwise, by letting n tend to infinity along $\mathcal{N}_1$, we would get that α is rational.
Dividing (5) by $q_{u_n+v_n}$, we get

$$x_1 + x_2\frac{p_{u_n+v_n}}{q_{u_n+v_n}} + x_3\frac{q_{u_n+v_n-1}}{q_{u_n+v_n}} + x_4\frac{p_{u_n+v_n-1}}{q_{u_n+v_n-1}}.\frac{q_{u_n+v_n-1}}{q_{u_n+v_n}} = 0. \tag{6}$$

By letting n tend to infinity along $\mathcal{N}_1$, we obtain from (6) that

$$x_1 + x_2\alpha + x_3\beta + x_4\alpha\beta = 0,$$

with $\beta := \lim\limits_{n\to+\infty} \frac{q_{u_n+v_n-1}}{q_{u_n+v_n}}$. We can observe that β is irrational since otherwise, α would be rational.
For every sufficiently large integer n in $\mathcal{N}_1$, we obtain

$$|q_{u_n+v_n}\beta - q_{u_n+v_n-1}|_p \leq |q_{u_n+v_n-1}|_p^{-1}. \tag{7}$$

Let us consider now the six linearly independent forms with algebraic real and p-adic coefficients

$$\begin{cases} L'_{1,p}(Y_1, Y_2, Y_3) = \alpha Y_1 - Y_3, \\ L'_{2,p}(Y_1, Y_2, Y_3) = \beta Y_1 - Y_2, \\ L'_{3,p}(Y_1, Y_2, Y_3) = Y_2. \\ L'_{i,\infty}(Y_1, Y_2, Y_3) = Y_i, \text{ for } 1 \leq i \leq 3, \end{cases}$$

Keeping Notations 1, we evaluate the product of these linear forms on the triple $Y_n = (Y_1, Y_2, Y_3)$ with $Y_1 = q'_{u_n+v_n}$, $Y_2 = q'_{u_n+v_n-1}$ and $Y_3 = p'_{u_n+v_n}$. We get from (7) and Lemma 3 that

$$|Y_n|_\infty^\varepsilon \prod_{i=1}^{3}(\left|L'_{i,\infty}(Y_n)\right|_\infty \left|L'_{i,p}(Y_n)\right|_p) \leq \left(\frac{B^{3+\varepsilon}}{p^{1-k\varepsilon}}\right)^{u_n+v_n} \leq \left[\left(\frac{B^{4+\varepsilon}}{p^{1-k\varepsilon}}\right)^{1+\frac{v_n}{u_n}}\right]^{u_n}.$$

From the hypothesis of Theorem 2, we can choose $\varepsilon = \frac{1}{k^2}$ such that for n large enough, we get

$$\prod_{i=1}^{3}(\left|L'_{i,\infty}(Y_n)\right|_\infty \left|L'_{i,p}(Y_n)\right|_p) \leq |Y_n|_\infty^{-\varepsilon}.$$

It follows then from Theorem 3 that the points $Y_n = (Y_1, Y_2, Y_3)$ lie in a finite number of proper subspaces of $\mathbb{Q}^3$. Hence, there exist a non-zero integer triple (y_1, y_2, y_3) and an infinite set of distinct positive integers $\mathcal{N}_2 \subset \mathcal{N}_1$ such that

$$y_1 Y_1 + y_2 Y_2 + y_3 Y_3 = 0.$$

From this equation, we obtain

$$y_1 q_{u_n+v_n} + y_2 q_{u_n+v_n-1} + y_3 p_{u_n+v_n} = 0. \tag{8}$$

Dividing (8) by $q_{u_n+v_n}$ and letting n tend to infinity along $\mathcal{N}_2$, we obtain

$$y_1 + y_2\beta + y_3\alpha = 0. \tag{9}$$

To get another equation connecting α and β, let us consider the six linearly independent forms with algebraic real and p-adic coefficients

$$\begin{cases} L''_{1,p}(Z_1, Z_2, Z_3) = \alpha Z_2 - Z_3, \\ L''_{2,p}(Z_1, Z_2, Z_3) = \beta Z_1 - Z_2, \\ L''_{3,p}(Z_1, Z_2, Z_3) = Z_2. \\ L''_{i,\infty}(Z_1, Z_2, Z_3) = Z_i, \text{ for } 1 \leq i \leq 3, \end{cases}$$

Keeping Notations 1, we evaluate the product of these linear forms on the triple $Z_n = (Z_1, Z_2, Z_3)$ with $Z_1 = q'_{u_n+v_n}$, $Z_2 = q'_{u_n+v_n-1}$ and $Z_3 = p'_{u_n+v_n-1}$. We get from (7) and Lemma 3 that

$$|Z_n|_\infty^\varepsilon \prod_{i=1}^{3}(\left|L''_{i,\infty}(Z_n)\right|_\infty \left|L''_{i,p}(Z_n)\right|_p) \leq \left(\frac{B^{3+\varepsilon}}{p^{1-k\varepsilon}}\right)^{u_n+v_n} \leq \left[\left(\frac{B^{4+\varepsilon}}{p^{1-k\varepsilon}}\right)^{1+\frac{v_n}{u_n}}\right]^{u_n}.$$

From the hypothesis of Theorem 2, we can choose $\varepsilon = \frac{1}{k^2}$ such that for n large enough, we obtain

$$\prod_{i=1}^{3}(\left|L''_{i,\infty}(Z_n)\right|_\infty \left|L''_{i,p}(Z_n)\right|_p) \leq |Z_n|_\infty^{-\varepsilon}.$$

It follows then from Theorem 3 that the points $Z_n = (Z_1, Z_2, Z_3)$ lie in a finite number of proper subspaces of $\mathbb{Q}^3$. Hence, there exist a non-zero integer triple (z_1, z_2, z_3) and an infinite set of distinct positive integers $\mathcal{N}_3 \subset \mathcal{N}_2$ such that

$$z_1 Z_1 + z_2 Z_2 + z_3 Z_3 = 0.$$

By this equation, we get

$$z_1 q_{u_n+v_n} + z_2 q_{u_n+v_n-1} + z_3 p_{u_n+v_n-1} = 0. \tag{10}$$

Dividing (10) by $q_{u_n+v_n-1}$ and letting n tend to infinity along $\mathcal{N}_3$, we obtain

$$\frac{z_1}{\beta} + z_2 + z_3\alpha = 0. \tag{11}$$

We deduce from (9) and (11) that

$$(y_3\alpha + y_1)(z_3\alpha + z_2) = y_2 z_1.$$

As we have β is irrational, we obtain from (9) and (11) that $y_3 z_3 \neq 0$. Thus, α is an algebraic number of degree at most two, which is a contradiction with the assumption that α has a degree at least three. Consequently, α is transcendental and the proof of Theorem 2 is reached.

Proof of Corollary 1.

We have $p \geq 7$, therefore $p > \phi^4 \simeq 6,85$, with ϕ is the golden ratio.
Besides, we have $A \leq 1$, then $B \leq \phi$. Thus we obtain from Theorem 2 that $4 < \dfrac{\log p}{\log B}$. This brings us to the end of the proof.

Example 1 *Let* $p = 7$. *Let* $(A_n)_{n\geq 0}$ *be a sequence of blocks defined as follows:*

$$\begin{cases} A_0 = 1\ \frac{1}{p}, \\ A_n = A_{n-1}\ A_{n-1}\ \underbrace{1\ldots1}_{n \text{ times}}\ A_{n-1}. \end{cases}$$

A_{n-1} *is a prefix of* A_n*, then set* $\mathbf{A} = \lim_{n\to+\infty} A_n$*. As stated in Corollary 1,* $(A_n)_{n\geq 0}$ *satisfies Condition* ($\star$)*. Therefore,* $\alpha = [0, \mathbf{A}]_7$ *is either quadratic or transcendental in* $\mathbb{Q}_7$*.*

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Received: August 12, 2023