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An anticipating stochastic integral with respect to mixed fractional Brownian motion

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Abstract. In this paper, we define a stochastic integral of an anticipating integrand based on Ayed and Kuo's approach [1]. This provides a new concept of stochastic integration of non-adapted processes. In addition, under some conditions, we prove that our anticipating integral is a near-martingale. Furthermore, we deal with some particular cases when the Hurst parameter $H > \frac{3}{4}$.

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1 Introduction

Let $B(t)$ be a Brownian motion and let $\{\mathcal{F}_t; 0 \le t \le T\}$ denote a filtration such that:

1. $f(t)$ is an $\mathcal{F}_t$-adapted stochastic process, i.e. $f(t)$ is $\mathcal{F}_t$-measurable for each $0 \le t \le T$.
2. $g(t)$ is instantly independent with respect to $\mathcal{F}_t$, i.e. $g(t)$ and $\{\mathcal{F}_t\}$ are independent for each $0 \le t \le T$.

Ayed and Kuo [1] defined the anticipating stochastic integral of the product $f(t)g(t)$ as:

$$\int_0^T f(t)g(t)dB(t) = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} f(t_{i-1})g(t_i)(B(t_i) - B(t_{i-1})) \tag{1}$$

provided that the convergence in probability exists, where $\Delta_n = \{0 = t_0 < t_1 < .... < t_n = T\}$ is the partition of interval $[0, T]$.
Notice that the evaluation points are the left endpoints of subintervals for the first process and the right endpoints for the second one.

This new approach has attracted the attention of many researchers. The study of a class of stochastic differential equations with anticipating initial conditions was treated in Khalifa et al. [7]. After that, the concept of near-martingale property of anticipating stochastic integral was introduced in Kuo et al. [8]. It has been proved that both $\int_0^t f(B(s))g(B(T) - B(s))dB(t)$ and $\int_t^T f(B(s))g(B(T) - B(s))dB(t)$ are near-martingales with respect to the forward filtration $\mathcal{F}_t = \sigma\{B(s); 0 \le s \le t\}$ and the backward filtration $\mathcal{F}^{(t)} = \sigma\{B(T) - B(s); 0 \le s \le t\}$, respectively. Interesting literature on the near martingale property can be found in Hwang et al. [6] and Hibino et al. [5]. Recently, Belhadj et al. [2] introduced the anticipating stochastic integral with respect to sub-fractional Brownian motion and discussed the conditions under which this integral satisfies the near-martingale property.

Next, we consider the process

$$M^H(t) = M_t^H(a, b) = aB(t) + bB^H(t), t \in \mathbb{R}_+, \qquad a, b \in \mathbb{R}^*, \tag{2}$$

where B and B^H are independent standard and fractional Brownian motions, respectively. The latter is the centered Gaussian process with a Hurst parameter $H \in (0, 1)$ and covariance function:

$$R^H(s,t) = \frac{1}{2}[t^{2H} + s^{2H} + |t-s|^{2H}], \qquad s,t \geq 0. \tag{3}$$

The linear combination M^H is the so-called mixed fractional Brownian motion (mfBm). This process has been firstly introduced by Chridito [3] to present an interesting stochastic model in financial markets (by taking $b = 1$). The stochastic properties of mfBm have been studied by Zili [13].

It is worth pointing out that, for $H > \frac{3}{4}$, the process M^H is a semimartingale which is equivalent (in distribution) to aB (Chredito [3]), and for $H < \frac{1}{4}$, M^H is equivalent (in distribution) to a bB^H (Van Zanten [10]). Furthermore, we mention that for $H < \frac{3}{4}$, the mixed fBm is not a semi-martingale. Therefore, the techniques of stochastic calculus with respect to fBm should be employed while dealing with a mixed fBm. In the case where $H > \frac{1}{2}$, we can use the pathwise approach that allows us to write the integral as a limit of Riemann sum (Young [11], Zähle[12], and Feyel and Pradelle [4] and the references therein). In our study, we use this approach in order to give a definition of the anticipating integral with respect to a mixed fractional Brownian motion M^H and study the near-martingale property.

1.1 Practical application of our research work

Our study has a notable application in finance and economy. For instance, we consider a financial stock market where the process $f(t)$ is a quantity of the stock at time t, adapted to $\mathcal{F}_t$, the σ-field represents information available by time t, and $B(t)$ (the standard Brownian motion) characterizes the stock price at time t. The integral $\int_0^T f(t)dB(t)$ describes the change of the stock market wealth over the trading period $[0, T]$. By dividing the time integral into the subintervals $[t_{i-1}, t_i]$, $\int_0^T f(t)dB(t)$ can be computed as a limit of Riemann-like sums of $f(t_{i-1})(B(t_i) - B(t_{i-1}))$. The use of the left endpoint of subintervals comes from the fact that $f(t)$ depends on the past and present but not the future. If one comes across the case where the quantity of stock $f(t)$ is independent of past and present, i.e for each $t \in [0, T])$, $f(t)$ is $\mathcal{F}_t$-independent then the future change in stocks can be known and one can use the right endpoint t_i as an evaluation point for the above stochastic integral. On the other hand, it has been interesting, in recent years, to divide the noise of stock price into two parts: the first describes the stochastic behavior of stock markets which is considered as a white noise, the other one represents the random state of the stock price which has a long memory, this motivates researchers to take such a situation into consideration and to provide a mixture of processes in accordance with the requirements of the phenomena.

Furthermore, over the past, there has been an extensive studies on option pricing. It has been shown that the distributions of logarithmic returns of financial assets generally exhibit properties of self-similarity and long-term dependence, and since the fractional Brownian motion has these two important properties, it has the ability to capture the behavior of the underlying asset price. The Black-Scholes model supposed that the volatility of the underlying security is constant, while stochastic volatility models classified the price of the underlying security as a random variable or, more generally, a stochastic process. In turn, the dynamics of this stochastic process can be driven by another process (usually by Brownian motion), see Thao et al. [9]. In a stochastic volatility model, the volatility randomly changes according to stochastic processes. In our paper, the process used is the mixture between fBm (fractional Brownian motion) and Bm(Brownian motion). The current study helps to solve the stochastic differential equations (SDEs) driven by a mixed fractional Brownian motion in the case of no adapted integrands which contributes to the resolution of the phenomena linked to volatility in the above situations.

This paper is arranged as follows. In Section 2, we present some preliminaries on mixed fractional integral as well as pathwise integral with respect to mfBm. In Section 3, we introduce a definition of stochastic integral of a product of instantly independent process and adapted process with respect to M^{H}, $\mathsf{H} > \frac{1}{2}$ as a Riemann sum. Then, we discuss the near-martingale property of our anticipating integral. Section 4 is devoted to some particular cases when $\mathsf{H} > \frac{3}{4}$. We conclude the paper in Section 5.

2 Preliminaries on mixed fractional Brownian motion

The fBm $(\mathsf{B}^{\mathsf{H}}(t); t \geq 0)$ with a Hurst parameter $\mathsf{H} \in (0, 1)$ is a centered Gaussian process with covariance function given by Equation (3). The main properties of B^{H} are self-similarity and the stationary of its increments, it presents a long-range dependence when $\mathsf{H} > \frac{1}{2}$. For $\mathsf{H} = \frac{1}{2}$, B^{H} coincides with the standard Brownian motion.

Note that the mixture M^{H} reserves several properties of the fBm. We recall in this section some basic facts on mixed fractional Brownian motion, the proofs are detailed in Zili [13].

Lemma 1 *(Zili [13]). The mfBm satisfies the following properties:*

- M^H *is a centered gaussian process;*
- *Second moment: for all* $t \in \mathbb{R}_+$; $\mathbb{E}((M_t^H(a,b))^2) = a^2 t + b^2 t^{2H}$.
- *Covariance function: for all* $t, s \geq 0$;

$$\mathrm{Cov}(M_t^H(a,b), M_s^H(a,b)) = a^2 \min(t,s) + \frac{b^2}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

- *The increments of the mfBm are stationary.*
- *Mixed self similarity:* $\left(M_{\alpha t}^H(a,b)\right)_{t\geq 0}$ *and* $\left(M_t^H(a\alpha^{\frac{1}{2}}, b\alpha)\right)_{t\geq 0}$ *have the same distribution.*
- *Hölder continuity: for all* $T > 0$ *and* $\beta < \frac{1}{2} \wedge H$, *the mfBm has a modification which sample paths having a Hölder continuity, with order* β *on the interval* $[0;T]$ *such that, for every* $\alpha > 0$ *:*

$$\mathbb{E}\left(\left|M^H(t) - M^H(s)\right|^{\alpha}\right) \leq C_\alpha |t-s|^{\alpha(\frac{1}{2}\wedge H)}, \qquad t, s \in [0;T],$$

  where C_α *is a positive constant.*

Feyel and Pradelle [4] showed that if f is α-Hölder, g is β-Hölder with $\alpha + \beta > 1$, then the Riemann-Stieltjes integral $\int_0^T f(s)dg(s)$ exists and is β-Hölder. Moreover, for every $0 < \varepsilon < \alpha + \beta - 1$, we have

$$\left|\int_0^T f(s)dg(s)\right| \leq C(\alpha,\beta) \parallel f \parallel_{[0,T],\alpha} \parallel g \parallel_{[0,T],\beta} T^{1+\varepsilon}. \tag{4}$$

Since mfBm has Hölder paths, then it is possible to define the stochastic integral for processes with respect to it in pathwise sense. Particularly, if a process $(u_t)_{t\in[0,T]}$ has α-Hölder paths for some $\alpha > 1 - H$, then the Riemann-Stieltjes integral $\int_0^t u_r dM_r^H$ is well defined and has β-Hölder paths, for every $\beta < H$ (see Young [11] and Zähle [12]).

3 New anticipating integral

Based on the concept presented above, we give a definition of the stochastic integral of the product $f(t)g(t)$, following Definition 2.2 given in Kuo and Ayed [1], by taking the mfBm M^H as an integrator. Formally, we have

Definition 1 *Let $M^H(t)$, $H > \frac{1}{2}$ be a mixed fractional Brownian motion and let $\mathcal{F}_t$ denote the σ-field generated by $\{M^H(t), t \geq 0\}$. For an adapted stochastic process $f(t)$ with respect to the filtration $\mathcal{F}_t$, and an instantly independent stochastic process $g(t)$ with respect to the same filtration. We define the stochastic integral of $f(t)g(t)$ as:*

$$\int_0^T f(t)g(t)dM^H(t) = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} f(t_{i-1})g(t_i)(M^H(t_i) - M^H(t_{i-1})) \qquad (5)$$

provided that the convergence in probability exists.

It is quite clear that the anticipating integral (5) is not a $\mathcal{F}_t$-martingale. Thus, we have to check if this latter satisfies the near-martingale property presented in Kuo et al. [8].

Definition 2 *(Kuo et al. [8]). Let $E|X_t| < \infty$ for all t. We will say that X_t is a near-martingale with respect to a forward filtration $\{\mathcal{F}_t\}$ if*

$$E[X_t - X_s/\mathcal{F}_s] = 0, \qquad \forall s < t. \qquad (6)$$

On the other hand, we say that X_t is a near-martingale with respect to a backward filtration $\{\mathcal{F}^{(t)}\}$ if

$$E[X_t - X_s/\mathcal{F}^{(t)}] = 0, \qquad \forall s < t. \qquad (7)$$

Next, we have to prove that the processes X_t and Y_t defined by (8) and (13) respectively, are near-martingales for an adapted process $f(t)$ and centered instantly independent process $g(t)$ with respect to the forward filtration

$$\mathcal{F}_t = \sigma\{B(s), M^H(s); 0 \leq s \leq t\},$$

Theorem 1 *Let $\mathcal{F}_t$ be a forward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

1. $E\Big[\int_0^T f(B(t))g(B(T) - B(t))dM^H(t)\Big] < +\infty,$

2. $E\big[g(B(T)-B(t))\big]=0.$

Then,

$$X_t=\int_0^t f(B(s))g(B(T)-B(s))dM^H(s); \qquad 0\le t\le T \tag{8}$$

exists and is a near-martingale with respect to the forward filtration $\mathcal{F}_t$.

Proof. We need to verify that $E[X_t-X_s/\mathcal{F}_s]=0$, for $0\le s\le t$. Notice that

$$X_t-X_s=\int_s^t f(B(u))g(B(T)-B(u))dM_u^H.$$

Let $\Delta_n=\{s=t_0<t_1<...<t_{n-1}<t_n=t\}$ be a partition of the interval $[s,t]$ and let $\Delta M_i^H=M^H(t_i)-M^H(t_{i-1})$. Then, we have:

$$E[X_t-X_s/\mathcal{F}_s]=E\left[\int_s^t f(B(u))g(B(T)-B(u))dM^H(u)/\mathcal{F}_s\right]. \tag{9}$$

Making use of Definition 1, we get

$$\begin{aligned} E[X_t-X_s/\mathcal{F}_s] &= E\left[\lim_{\|\Delta_n\|\to 0}\sum_{i=1}^n f(B(t_{i-1}))g(B(T)-B(t_i))\Delta M_i^H/\mathcal{F}_s\right] \\ &= \lim_{\|\Delta_n\|\to 0}\sum_{i=1}^n E\left[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta M_i^H/\mathcal{F}_s\right]. \end{aligned} \tag{10}$$

It is sufficient to show that every component of the last sum is zero. Recall that $f(B(t_{i-1}))$ is $\mathcal{F}_{t_{i-1}}$-measurable and $g(B(T)-B(t_i))$ is independent of $\mathcal{F}_{t_{i-1}}$. Using the properties of conditional expectation, we obtain

$$\begin{aligned} &E\left[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta M_i^H/\mathcal{F}_s\right] \\ &\qquad = E\left[E\big[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta M_i^H/\mathcal{F}_{t_i}\big]/\mathcal{F}_s\right] \\ &\qquad = E\left[f(B(t_{i-1}))\Delta M_i^H E\big[g(B(T)-B(t_i))/\mathcal{F}_{t_i}\big]/\mathcal{F}_s\right]. \end{aligned} \tag{11}$$

Making use of the independence of Brownian increments and the zero expectation of $g(B(T)-B(t_i))$, we get

$$\begin{aligned} E\Big[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta M_i^H/\mathcal{F}_s\Big] \\ &= E\Big[f(B(t_{i-1}))\Delta M_i^H E\big[g(B(T)-B(t_i))\big]/\mathcal{F}_s\Big] \\ &= E\big[g(B(T)-B(t_i))\big]E\Big[f(B(t_{i-1}))\Delta M_i^H/\mathcal{F}_s\Big] \\ &= 0. \end{aligned} \tag{12}$$

Thus, X_t is a near-martingale with respect to $\mathcal{F}_t$. □

Theorem 2 *Let $\mathcal{F}_t$ be a forward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

1. $E\big[\int_0^T f(B(t))g(B(T)-B(t))dM^H(t)\big] < +\infty$
2. $E\big[g(B(T)-B(t))\big] = 0.$

Then,

$$Y_t = \int_t^T f(B(s))g(B(T)-B(s))dM^H(s), \;\; 0 \le t \le T \tag{13}$$

exists and is a near-martingale with respect to the forward filtration $\mathcal{F}_t$.

Proof. For $0 \le s < t \le T$, we have

$$Y_t - Y_s = -\int_s^t f(B(u))g(B(T)-B(u))dM^H(u) = -(X_t - X_s),$$

where X_t is given in Equation (8). Thus, Y_t is a near-martingale with respect to $\mathcal{F}_t$. □

Next, we prove that X_t and Y_t given in Equations (14) and (17) respectively, are near-martingales for a centered adapted process $f(t)$, and instantly independent process $g(t)$ with respect to the backward filtration

$$\mathcal{F}^{(t)} = \sigma\{B(T)-B(s), M^H(T)-M^H(s), \;\; 0 \le s \le t\}.$$

Theorem 3 *Let $\mathcal{F}^{(t)}$ be a backward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

1. $E\big[\int_0^T f(B(t))g(B(T)-B(t))dM^H(t)\big] < +\infty,$

2. $E\left[f(B(t))\right] = 0.$

Then,

$$X_t = \int_0^t f(B(s))g(B(T) - B(s))dM^H(s), \;\; 0 \le t \le T \tag{14}$$

exists and is a near-martingale with respect to the backward filtration $\mathcal{F}^{(t)}$.

Proof. According to the proof of Theorem 1, we have to show that

$$E\left[f(B(t_{i-1}))g(B(T) - B(t_i))\Delta M_i^H / \mathcal{F}^{(t)}\right] = 0,$$

where $0 \le s < t \le T$ and $s = t_0 < t_1 < .... < t_{n-1} < t_n = t$.

It is well known that the increments $M_T^H - M_{t_{i-1}}^H$ and $M_T^H - M_{t_i}^H$ are $\mathcal{F}^{(t_{i-1})}$-measurable, then we have

$$\Delta M_i^H = (M_T^H - M_{t_{i-1}}^H) - (M_T^H - M_{t_i}^H) \in \mathcal{F}^{(t_{i-1})}.$$

By the $\mathcal{F}^{(t_{i-1})}$- measurability of ΔM_i^H and the conditional expectation properties, we obtain

$$\begin{aligned} &E\left[f(B(t_{i-1}))g(B(T) - B(t_i))\Delta M_i^H / \mathcal{F}^{(t)}\right] \\ &\quad = E\left[E\left[f(B(t_{i-1}))g(B(T) - B(t_i))\Delta M_i^H / \mathcal{F}^{(t_{i-1})}\right] / \mathcal{F}^{(t)}\right] \\ &\quad = E\left[g(B(T) - B(t_i))\Delta M_i^H E\left[f(B(t_{i-1})) / \mathcal{F}^{(t_{i-1})}\right] / \mathcal{F}^{(t)}\right]. \end{aligned} \tag{15}$$

Furthermore, $B(T) - B(s)$ is independent of $\mathcal{F}_{t_{i-1}}$ and measurable with respect to $\mathcal{F}^{(t_{i-1})}$ for each $s > t_{i-1}$. This involves the independence of $\mathcal{F}^{(t_{i-1})}$ and $\mathcal{F}_{t_{i-1}}$. Consequently, $f(B(t_{i-1}))$ is independent of $\mathcal{F}^{(t_{i-1})}$ since it is $\mathcal{F}_{t_{i-1}}$ measurable. Hence,

$$\begin{aligned} &E\left[f(B(t_{i-1}))g(B(T) - B(t_i))\Delta M_i^H / \mathcal{F}^{(t)}\right] \\ &\quad = E\left[g(B(T) - B(t_i))\Delta M_i^H E\left[f(B(t_{i-1}))\right] / \mathcal{F}^{(t)}\right] \\ &\quad = E\left[f(B(t_{i-1}))\right] E\left[g(B(T) - B(t_i))\Delta M_i^H / \mathcal{F}^{(t)}\right] \\ &\quad = 0. \end{aligned} \tag{16}$$

□

Theorem 4 *Let $\mathcal{F}^{(t)}$ be a backward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

1. $E\Big[\int_0^T f(B(t))g(B(T)-B(t))dM^H(t)\Big] < +\infty$,

2. $E\Big[f(B(t))\Big] = 0$.

Then,

$$Y_t = \int_t^T f(B(s))g(B(T)-B(s))dM^H(s), \;\; 0 \leq t \leq T \tag{17}$$

exists and is a near-martingale with respect to the backward filtration $\mathcal{F}^{(t)}$.

Proof. From Theorem 3, we have $Y_t - Y_s = -(X_t - X_s)$. This completes the proof of the Theorem. □

4 Some results in the case where $H \in \left(\frac{3}{4}, 1\right)$

This section presents some results establishing the relationship between standard Bm and mixed-fBm in the case where $H > \frac{3}{4}$. We show that our anticipating integral with respect to M^H can be written as a Riemann sum depending on standard Bm satisfying the near martingale property.

Proposition 1 *Let $M^H(t); H > \frac{3}{4}$ be a mixed fractional Brownian motion and $\mathcal{F}_t = \sigma\{M^H(t), t \geq 0\}$. For an $\mathcal{F}_t$-adapted stochastic process $f(t)$ and an $\mathcal{F}_t$-instantly independent stochastic process $g(t)$, we have*

$$\int_0^T f(t)g(t)dM^H(t) = a \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} f(t_{i-1})g(t_i)(B^H(t_i) - B^H(t_{i-1})) \tag{18}$$

provided that the convergence in probability exists.

Proof. The proof is a direct result of Theorem 1.7 of Cheridito [3]. □

Proposition 2 *Let $\mathcal{F}_t$ be a forward filtration, $\mathcal{F}^{(t)}$ denotes the backward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

$$E\Big[\int_0^T f(B(t))g(B(T)-B(t))dM^H(t)\Big] < +\infty.$$

Then,

$$X_t = \int_0^t f(B(s))g(B(T) - B(s))dM^H(s); \qquad 0 \leq t \leq T, \tag{19}$$

and

$$Y_t = \int_t^T f(B(s))g(B(T) - B(s))dM^H(s); \qquad 0 \leq t \leq T \tag{20}$$

exist and are near-martingales with respect to $\mathcal{F}_t$ *and* $\mathcal{F}^{(t)}$ *respectively.*

Proof. The proof of this proposition is based on Theorem 1.7 in Chridito [3] and Theorems 3.5-3.8 given in Kuo et al. [8]. □

In what follows, we give some examples at which we evaluate some anticipating stochastic integrals with respect to mixed fractional Brownian motion when $H > \frac{3}{4}$, using the result obtained in the Proposition 1.

Example 1 *Consider the following integral*

$$\int_0^t B(T)^2 dM^H(s), \qquad 0 \leq t \leq T. \tag{21}$$

The integrand $B(T)^2$ *is decomposed as*

$$B(1)^2 = [(B(T) - B(s))]^2 + 2B(s)[B(T) - B(s)] + B(s)^2. \tag{22}$$

In addition, the integral converges in probability to

$$\sum_{i=1}^{n} \big([(B(T)-B(s_i))]^2+2B(s_{i-1})[B(T)-B(s_i)]+B(s_{i-1})^2\big)(M^H(s_i)-M^H(s_{i-1})).$$

As M^H *and* aB *are equivalent (in law), then the above sum can be expressed as*

$$a\sum_{i=1}^{n} \big([(B(T) - B(s_i))]^2 + 2B(s_{i-1})[B(T) - B(s_i)] + B(s_{i-1})^2\big)(B(s_i) - B(s_{i-1})).$$

Therefore, we have

$$\int_0^t B(T)^2 dM^H(s) = aB(T)^2 B(t) - 2aB(T)t, \qquad 0 \leq t \leq T.$$

In general, for any $n \in \mathbb{N}^*$, *it is easy to check that*

$$\int_0^t B(T)^n dM^H(s) = aB(T)^n B(t) - anB(T)^{n-1}t, \qquad 0 \leq t \leq T.$$

Example 2 *Consider the integrand* $B(s)B(T)$, *equivalently,*

$$B(s)(B(T)-B(s))+B(s)^2.$$

Then,

$$\begin{aligned}\int_0^t & B(s)B(T)dM^H(s)\\ &= a\lim_{\|\Delta_n\|\to 0}\sum_{i=1}^{n}\big(B(s_{i-1})(B(T)-B(s_i))+B(s_{i-1})^2\big)(B(s_i)-B(s_{i-1}))\\ &= \frac{a}{2}B(T)(B(t)^2-t)-a\int_0^t B(s)ds, \qquad 0\le t\le T.\end{aligned} \tag{23}$$

In the same manner, an integrand of the form $\phi(B(s))B(T)$ *can be decomposed as*

$$\phi(B(s))(B(T)-B(s))+\phi(B(s))B(s),$$

for any continuous function $\phi(x)$. *Therefore, the integral*

$$\int_0^t \phi(B(s))B(T)dM^H(s), \qquad 0\le t\le T$$

converges in probability to

$$aB(T)\sum_{i=1}^{n}(\phi(B(s_{i-1}))(B(s_i)-B(s_{i-1}))-a\sum_{i=1}^{n}\phi(B(s_{i-1}))(B(s_i)-B(s_{i-1}))^2,$$

which is equivalent to

$$aB(T)\int_0^t\phi(B(s))dB(s)-a\int_0^t\phi(B(s))ds.$$

Example 3 *The integral*

$$\int_0^t e^{B(T)}dM^H(s), \qquad 0\le t\le T \tag{24}$$

is the limit of the sum

$$e^{B(T)}\sum_{i=1}^{n}e^{(B(s_i)-B(s_{i-1}))}(M(s_i)-M(s_{i-1})).$$

Using Taylor series expansions of exponential function, Equation (24) converges in probability to

$$a e^{B(T)} \sum_{i=1}^{n} \big(1 - (B(s_i) - B(s_{i-1})) - \frac{1}{2}(B(s_i) - B(s_{i-1}))^2 \\ + o((B(s_i) - B(s_{i-1}))^2)(B(s_i) - B(s_{i-1})).$$

Consequently,

$$\int_0^t e^{B(T)} dM^H(s) = a e^{B(T)}(B(t) - t), \qquad 0 \le t \le T.$$

5 Conclusion

In this paper, we introduced an anticipating stochastic integral with respect to a mixed fractional Brownian motion (mfBm) in the case where $H > \frac{1}{2}$, based on Ayed and Kuo's approach [1]. This gives a new concept of stochastic integration of non-adapted processes. Under some conditions, we showed that our anticipating integral turns out to be a near-martingale. In addition, few specific cases when $H > \frac{3}{4}$ have been treated. The present study has a useful application in many areas including finance and economy. For further works, it will be interesting to deal with anticipating stochastic integrals with respect to a weighted fractional Brownian motion and Lévy fractional Brownian motion.

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