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Some equalities and inequalities in 2-inner product spaces

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Abstract. We obtain some new results concerning some equalities and inequalities in a 2-inner product space. These inequalities are a generalization of the Cauchy–Schwarz inequality. Also a reverse of Cauchy–Schwarz's inequality in this space is given.

1 Introduction

In 1964, Gähler [10] introduced the concepts of 2-norm and 2-inner product spaces as a generalization of norm and inner product spaces, respectively, which have been intensively studied by many authors in the last four decades. A presentation of the results related to the theory of 2-inner product spaces can be found in [2]. The Cauchy–Schwarz inequality is one of the many inequalities related to inner product spaces. The theory of such inequalities plays an important role in modern mathematics together with numerous applications

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for the nonlinear analysis, approximation and optimization theory, numerical analysis, probability theory, statistics, and other fields.
The Cauchy–Schwarz inequality has been frequently used for obtaining bounds or estimating the errors in various approximation formulas occurring in the above domains. Thus, any new advantages will have a number of important consequences in the mathematical fields, where inequalities are basic tools. Cho, Matic, and Pecaric [3] proved the Cauchy–Schwarz inequality in 2-inner product spaces. In this paper, we obtain some equalities and inequalities in a 2-inner product space, and then we get the Cauchy–Schwarz inequality. Finally, we state a reverse Cauchy–Schwarz inequality in a 2-inner product space.

2 Notation and preliminary results

In this section, we recall some basic notations, definitions, and some important properties, which will be used. For more detailed information, one can see [2, 3].

2.1 2-inner Product space

Let $\mathcal{X}$ be a linear space of dimension greater than 1 over the field $\mathcal{K}$, where $\mathcal{K}$ is the real or complex numbers field. Suppose that $(\cdot,\cdot|\cdot)$ is a $\mathcal{K}$-valued function defined on $\mathcal{X}\times\mathcal{X}\times\mathcal{X}$ satisfying the following conditions:

(I1) $(x,x|z)\geq 0$ and $(x,x|z)=0$ if and only if x and z are linearly dependent,

(I2) $(x,x|z)=(z,z|x)$,

(I3) $(y,x|z)=\overline{(x,y|z)}$,

(I4) $(\alpha x,y|z)=\alpha(x,y|z)$ for any scalar $\alpha\in\mathcal{K}$,

(I5) $(x+x^{'},y|z)=(x,y|z)+(x^{'},y|z)$,

where x,$x^{'}$,y,$z\in\mathcal{X}$.
Indeed, $(\cdot,\cdot|\cdot)$ is called a 2-inner product on $\mathcal{X}$ and $(\mathcal{X},(\cdot,\cdot|\cdot))$ is called a 2-inner product space (or 2-pre-Hilbert space). Some properties of 2-inner product $(\cdot,\cdot|\cdot)$ can be obtained as follows:

(1) If $\mathcal{K}=\mathcal{R}$, then (I3) reduces to

$$(y,x|z)=(x,y|z).$$

(2) From (I3) and (I4), we have

$$(0, y|z) = 0, \quad (x, 0|z) = 0$$

and also

$$(x, \alpha y|z) = \overline{\alpha}(x, y|z). \tag{1}$$

(3) Using (I3)–(I5), we have

$$(z, z|x \pm y) = (x \pm y, x \pm y|z) = (x, x|z) + (y, y|z) \pm 2\mathrm{Re}(x, y|z)$$

and

$$\mathrm{Re}(x, y|z) = \frac{1}{4}\left[(z, z|x + y) - (z, z|x - y)\right]. \tag{2}$$

In the real case $\mathcal{K} = \mathcal{R}$, equation (2) reduces to

$$(x, y|z) = \frac{1}{4}\left[(z, z|x + y) - (z, z|x - y)\right], \tag{3}$$

and using this formula, it is easy to see that, for any $\alpha \in \mathcal{R}$,

$$(x, y|\alpha z) = \alpha^2 (x, y|z). \tag{4}$$

In the complex case, using (1) and (2), we have

$$\mathrm{Im}(x, y|z) = \mathrm{Re}[-i(x, y|z)] = \frac{1}{4}\left[(z, z|x + iy) - (z, z|x - iy)\right],$$

which, in combination with (2), yields

$$(x, y|z) = \frac{1}{4}\left[(z, z|x + y) - (z, z|x - y)\right] + \frac{i}{4}\left[(z, z|x + iy) - (z, z|x - iy)\right]. \tag{5}$$

Using the above formula and (1), for any $\alpha \in \mathcal{C}$, we have

$$(x, y|\alpha z) = |\alpha|^2 (x, y|z). \tag{6}$$

Moreover, for $\alpha \in \mathcal{R}$, equation (6) reduces to (4). Also, it follows from (6) that

$$(x, y|0) = 0.$$

(4) For any three given vectors $x, y, z \in \mathcal{X}$, consider the vector $u = (y, y \mid z)x - (x, y \mid z)y$. We know that $(u, u \mid z) \geq 0$ with the equality if and only if u and z are linearly dependent. The inequality $(u, u \mid z) \geq 0$ can be rewritten as follows:

$$(y, y \mid z)\left[(x, x \mid z)(y, y \mid z) - \mid (x, y \mid z) \mid^2\right] \geq 0. \tag{7}$$

If $x = z$, then (7) becomes

$$-(y, y \mid z) \mid (z, y \mid z) \mid^2 \geq 0,$$

which implies that

$$(z, y \mid z) = (y, z \mid z) = 0, \tag{8}$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (8) holds too.

Thus (8) is true for any two vectors $y, z \in \mathcal{X}$. Now, if y and z are linearly independent, then $(y, y \mid z) > 0$, and from (7), it follows the Cauchy–Bunyakovsky–Schwarz inequality (CBS-inequality, for short) for 2-inner products:

$$\mid (x, y \mid z) \mid^2 \leq (x, x \mid z)(y, y \mid z). \tag{9}$$

Using (8), it is easy to check that (9) is trivially fulfilled when y and z are linearly dependent. Therefore, inequality (9) holds for any three vectors $x, y, z \in \mathcal{X}$ and is strict unless the vectors $u = (y, y \mid z)x - (x, y \mid z)y$ and z are linearly dependent. In fact, we have the equality in (9) if and only if the three vectors x, y, and z are linearly dependent.
In any given 2-inner product space $(\mathcal{X}, (\cdot, \cdot|\cdot))$, we can define a function $\|\cdot, \cdot\|$ on $\mathcal{X} \times \mathcal{X}$ by

$$\|x, z\| = (x, x|z)^{\frac{1}{2}} \tag{10}$$

for all $x, z \in \mathcal{X}$.

It is easy to see that, this function satisfies the following conditions:

(N1) $\|x, z\| \geq 0$ and $\|x, z\| = 0$ if and only if x and z are linearly dependent,

(N2) $\|x, z\| = \|z, x\|$,

(N3) $\|\alpha x, z\| = |\alpha| \|x, z\|$ for any scalar $\alpha \in \mathcal{K}$,

(N4) $\|x + x', z\| \leq \|x, z\| + \|x', z\|$.

The function $\|\cdot,\cdot\|$ defined on $\mathcal{X}\times\mathcal{X}$ and satisfying the above conditions is called a 2-norm on $\mathcal{X}$, and $(\mathcal{X},\|\cdot,\cdot\|)$ is called a linear 2-normed space; see [8]. Whenever a 2-inner product space $(\mathcal{X},(\cdot,\cdot|\cdot))$ is given, we consider it as a linear 2-normed space $(\mathcal{X},\|\cdot,\cdot\|)$ with 2-norm defined by (11).
In terms of 2-norms, the (CBS)-inequality (9) can be written as

$$|\ (x,y\mid z)\ |^2\leq \|x,z\|^2\|y,z\|^2. \tag{11}$$

The equality in (11) holds if and only if x,y, and z are linearly dependent.
For recent inequalities, see [1, 7, 9, 11].

3 Main results

We present some equalities and inequalities in a 2-inner product space. The first part is devoted to illustrate these equalities and inequalities that the Cauchy–Schwarz inequality is one of them. Thereafter in the second part, we show a reverse of Cauchy–Schwarz inequality in 2-inner product spaces.

3.1 Some equalities and inequalities in 2-inner product space

Let $\mathcal{X}$ be a 2-inner product space over the field $\mathcal{K}$, where $\mathcal{K}$ is the field of real numbers $\mathcal{R}$ or complex numbers $\mathcal{C}$. The 2-inner product space $(\cdot,\cdot|\cdot)$ induces an associated norm, given by $\|x,z\|=(x,x|z)^{\frac{1}{2}}$, for all $x,z\in\mathcal{X}$, thus $\mathcal{X}$ is a linear 2-normed space. In this section, we establish several new results related to the equalities and inequalities in a 2-inner product space that lead to the Cauchy–Schwarz inequality. Then we deduce some relations.

Theorem 1 *In a 2-Inner product space $\mathcal{X}$ over the field of complex numbers $\mathcal{C}$, we have*

$$\begin{aligned}&\frac{1}{\|y,z\|^2}(\alpha y-x,x-\beta y\mid z)\\&=\left[\alpha-\frac{(x,y\mid z)}{\|y,z\|^2}\right]\left[\frac{\overline{(x,y\mid z)}}{\|y,z\|^2}-\overline{\beta}\right]-\frac{1}{\|y,z\|^2}\left\|x-\frac{(x,y\mid z)}{\|y,z\|^2}y,z\right\|^2\end{aligned}$$

for every $\alpha,\beta\in\mathcal{C}$ and for all $x,y,z\in\mathcal{X}$, where y and z are linearly independent.

Proof. The proof is obtained from the following:

$$\frac{1}{\|y,z\|^2}(\alpha y-x,x-\beta y\mid z)$$

$$
\begin{aligned}
&= \frac{1}{\|y,z\|^2}\left[(\alpha y, x \mid z) - (\alpha y, \beta y \mid z) - (x, x \mid z) + \overline{\beta}\,(x, y \mid z)\right] \\
&= \alpha\frac{\overline{(x,y \mid z)}}{\|y,z\|^2} - \alpha\overline{\beta} - \frac{\|x,z\|^2}{\|y,z\|^2} + \overline{\beta}\frac{(x,y \mid z)}{\|y,z\|^2} \\
&= \left[\alpha - \frac{(x,y \mid z)}{\|y,z\|^2}\right]\left[\frac{\overline{(x,y \mid z)}}{\|y,z\|^2} - \overline{\beta}\right] - \frac{1}{\|y,z\|^2}\left[\|x,z\|^2 - \frac{|(x,y \mid z)|^2}{\|y,z\|^2}\right] \\
&= \left[\alpha - \frac{(x,y \mid z)}{\|y,z\|^2}\right]\left[\frac{\overline{(x,y \mid z)}}{\|y,z\|^2} - \overline{\beta}\right] - \frac{1}{\|y,z\|^2}\left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\|^2.
\end{aligned}
$$

□

Corollary 1 *In a 2-Inner product space $\mathcal{X}$ over the field of real numbers $\mathcal{R}$, we have*

$$
\begin{aligned}
\frac{1}{\|y,z\|^2}(\alpha y - x, x - \beta y \mid z) = &\left[\alpha - \frac{(x,y \mid z)}{\|y,z\|^2}\right]\left[\frac{(x,y \mid z)}{\|y,z\|^2} - \beta\right] \\
&- \frac{1}{\|y,z\|^2}\left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\|^2
\end{aligned}
$$

for every $\alpha, \beta \in \mathcal{R}$ and for all $x, y, z \in \mathcal{X}$, where y and z are linearly independent.

Corollary 2 *In a 2-Inner product space X over the field of complex numbers $\mathcal{C}$, we have*

$$
\|x - \alpha y, z\|^2 = \left|\alpha\|y,z\| - \frac{(x,y \mid z)}{\|y,z\|}\right|^2 + \left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\|^2 \tag{12}
$$

for every $\alpha \in \mathcal{C}$ and for all $x, y, z \in \mathcal{X}$, where y and z are linearly independent.

Proof. Putting $\alpha = \beta$ in Theorem 1, we get

$$
\begin{aligned}
\frac{1}{\|y,z\|^2}(\alpha y - x, x - \alpha y \mid z) &= -\frac{1}{\|y,z\|^2}\|x - \alpha y, z\|^2 \\
&= \left[\alpha - \frac{(x,y \mid z)}{\|y,z\|^2}\right]\left[\frac{\overline{(x,y \mid z)}}{\|y,z\|^2} - \overline{\alpha}\right] \\
&\quad - \frac{1}{\|y,z\|^2}\left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\|^2.
\end{aligned}
$$

Now multiplying this equality by $-\|y,z\|^2$ we get the desired result. □

Corollary 3 *In a 2-Inner product space $\mathcal{X}$ over the field of complex numbers $\mathcal{C}$, we have*

$$\|x-\alpha y, z\| \geq \left\| x - \frac{(x, y \mid z)}{\|y, z\|^2} y, z \right\| \tag{13}$$

and

$$\|x-\alpha y, z\| \geq \left| \alpha \|y, z\| - \frac{(x, y \mid z)}{\|y, z\|} \right|, \tag{14}$$

for every $\alpha \in \mathcal{C}$ and for all $x, y, z \in \mathcal{X}$, where y and z are linearly independent.

Proof. In the proof of (12), we see that

$$\frac{1}{\|y, z\|^2} \left\| x - \frac{(x, y \mid z)}{\|y, z\|^2} y, z \right\|^2 \geq 0.$$

Hence, (13) and (14) are obtained. □

Remark 1 *If y and z are linearly independent, then from relation (14), for $\alpha = 0$, we obtain the following inequality of Cauchy–Schwartz:*

$$|(x, y \mid z)| \leq \|x, z\| \; \|y, z\|.$$

Now we are ready to state the following result.

Corollary 4 *In a 2-inner product space $\mathcal{X}$ over the field of complex numbers $\mathcal{C}$, we have*

$$\|x, z\|^2 \|y, z\|^2 = |(x, y \mid z)|^2 + \left\| \; \|y, z\| x - \frac{(x, y \mid z)}{\|y, z\|} y, z \right\|^2, \tag{15}$$

$$\begin{aligned} &\|x, z\|^2 \|y, z\|^2 \left\| \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|}, z \right\|^2 \\ &= \left| \|x, z\| \; \|y, z\| - (x, y \mid z) \right|^2 + \|x, z\|^2 \|y, z\|^2 - |(x, y \mid z)|^2 \end{aligned} \tag{16}$$

for all $x, y, z \in \mathcal{X}$, where y and z are linearly independent and x and z are linearly independent, too.

$$\mathrm{Re}(x, y \mid z) = \frac{1}{2} \left(\|x, z\|^2 + \|y, z\|^2 - \|x - y, z\|^2 \right), \tag{17}$$

$$\mathrm{Im}(x, y \mid z) = \frac{1}{2} \left(\|x, z\|^2 + \|y, z\|^2 - \|x - iy, z\|^2 \right), \tag{18}$$

for all $x, y, z \in \mathcal{X}$.

Proof. From (12), for $\alpha = 0$, we have

$$\|x, z\|^2 = \frac{|(x, y \mid z)|^2}{\|y, z\|^2} + \left\| x - \frac{(x, y \mid z)}{\|y, z\|^2} y, z \right\|^2 .$$

Therefore

$$\|x, z\|^2 \|y, z\|^2 = |(x, y \mid z)|^2 + \left\| \, \|y, z\| x - \frac{(x, y \mid z)}{\|y, z\|} y, z \right\|^2 ,$$

and (15) is obtained.

Also, for $\alpha = \frac{\|x, z\|}{\|y, z\|}$ in (12), it follows that

$$\begin{aligned} &\|x\|y, z\| - \|x, z\|y, z\|^2 \\ &= | \; \|x, z\| \; \|y, z\| - (x, y \mid z)|^2 + \left\| \, \|y, z\| x - \frac{(x, y \mid z)}{\|y, z\|} y, z \right\|^2 , \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\|x, z\|^2 \|y, z\|^2 \| \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|}, z\|^2 \\ &= | \; \|x, z\| \; \|y, z\| - (x, y \mid z)|^2 + \left\| \, \|y, z\| x - \frac{(x, y \mid z)}{\|y, z\|} y, z \right\|^2 . \end{aligned}$$

Indeed, by (15), we have $\|x, z\|^2 \|y, z\|^2 - |(x, y \mid z)|^2 = \left\| \, \|y, z\| x - \frac{(x, y \mid z)}{\|y, z\|} y, z \right\|^2$, therefore (16) holds.

For $\alpha = 1$, (12) implies that

$$\|x - y, z\|^2 = \left| \, \|y, z\| - \frac{(x, y \mid z)}{\|y, z\|} \right|^2 + \left\| x - \frac{(x, y \mid z)}{\|y, z\|^2} y, z \right\|^2 . \tag{19}$$

On the other hand, we have

$$\left| \, \|y, z\| - \frac{(x, y \mid z)}{\|y, z\|} \right|^2 = \left(\|y, z\| - \frac{\mathrm{Re}(x, y \mid z)}{\|y, z\|} \right)^2 + \frac{\mathrm{Im}^2(x, y \mid z)}{\|y, z\|^2} . \tag{20}$$

Indeed

$$\left| \, \|y, z\| - \frac{(x, y \mid z)}{\|y, z\|} \right|^2 = \left(\|y, z\| - \frac{(x, y \mid z)}{\|y, z\|} \right) \left(\|y, z\| - \frac{\overline{(x, y \mid z)}}{\|y, z\|} \right)$$

$$
\begin{aligned}
&= \|y,z\|^2 - (y,x \mid z) - (x,y \mid z) + \frac{|(x,y \mid z)|^2}{\|y,z\|^2}\\
&= \|y,z\|^2 - 2\mathrm{Re}(x,y \mid z) + \frac{|(x,y \mid z)|^2}{\|y,z\|^2}\\
&= \|y,z\|^2 - 2\mathrm{Re}(x,y \mid z) + \frac{\mathrm{Re}^2(x,y \mid z)}{\|y,z\|^2} + \frac{\mathrm{Im}^2(x,y \mid z)}{\|y,z\|^2}\\
&= \left(\|y,z\| - \frac{\mathrm{Re}(x,y \mid z)}{\|y,z\|}\right)^2 + \frac{\mathrm{Im}^2(x,y \mid z)}{\|y,z\|^2}.
\end{aligned}
$$

Also,

$$
\left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\|^2 = \|x,z\|^2 - \frac{|(x,y \mid z)|^2}{\|y,z\|^2}, \tag{21}
$$

since

$$
\begin{aligned}
\left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\|^2 &= \left(x - \frac{(x,y \mid z)}{\|y,z\|^2}y, x - \frac{(x,y \mid z)}{\|y,z\|^2}y \mid z\right)\\
&= (x,x \mid z) - \left(x, \frac{(x,y \mid z)}{\|y,z\|^2}y \mid z\right) - \left(\frac{(x,y \mid z)}{\|y,z\|^2}y, x \mid z\right)\\
&\quad + \left(\frac{(x,y \mid z)}{\|y,z\|^2}y, \frac{(x,y \mid z)}{\|y,z\|^2}y \mid z\right)\\
&= \|x,z\|^2 - \frac{\overline{(x,y \mid z)}}{\|y,z\|^2}(x,y \mid z) - \frac{(x,y \mid z)}{\|y,z\|^2}\overline{(x,y \mid z)}\\
&\quad + \frac{(x,y \mid z)}{\|y,z\|^2}\frac{\overline{(x,y \mid z)}}{\|y,z\|^2}(y,y)\\
&= \|x,z\|^2 - \frac{|(x,y \mid z)|^2}{\|y,z\|^2}.
\end{aligned}
$$

Now by substitutions (20) and (21) in (19), it follows that

$$
\begin{aligned}
\|x-y,z\|^2 &= \left|\ \|y,z\| - \frac{(x,y \mid z)}{\|y,z\|}\right|^2 + \left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\|^2\\
&= \left(\|y,z\| - \frac{\mathrm{Re}(x,y \mid z)}{\|y,z\|}\right)^2 + \frac{\mathrm{Im}^2(x,y \mid z)}{\|y,z\|^2} + \|x,z\|^2 - \frac{|(x,y \mid z)|^2}{\|y,z\|^2}\\
&= \|y,z\|^2 + \frac{\mathrm{Re}^2(x,y \mid z)}{\|y,z\|^2} - 2\mathrm{Re}(x,y \mid z)\\
&\quad + \frac{\mathrm{Im}^2(x,y \mid z)}{\|y,z\|^2} + \|x,z\|^2 - \frac{|(x,y \mid z)|^2}{\|y,z\|^2}
\end{aligned}
$$

$$= \|x,z\|^2 + \|y,z\|^2 - 2\mathrm{Re}(x,y \mid z).$$

Hence, $\mathrm{Re}(x,y \mid z) = \frac{1}{2}\left(\|x,z\|^2 + \|y,z\|^2 - \|x-y,z\|^2\right)$ and (17) is proved. Using (12) for $\alpha = i$, we obtain

$$\|x - iy,z\|^2 = \left| i\|y,z\| - \frac{(x,y \mid z)}{\|y,z\|} \right|^2 + \left\| x - \frac{(x,y \mid z)}{\|y,z\|^2} y, z \right\|^2. \tag{22}$$

On the other hand, we have

$$\left| i\|y,z\| - \frac{(x,y \mid z)}{\|y,z\|} \right|^2 = \frac{\mathrm{Re}^2(x,y \mid z)}{\|y,z\|^2} + \left(\|y,z\| - \frac{\mathrm{Im}(x,y \mid z)}{\|y,z\|} \right)^2, \tag{23}$$

since

$$\begin{aligned}
\left| i\|y,z\| - \frac{(x,y \mid z)}{\|y,z\|} \right|^2 &= \left(i\|y,z\| - \frac{(x,y \mid z)}{\|y,z\|} \right)\left(i\|y,z\| - \frac{\overline{(x,y \mid z)}}{\|y,z\|} \right) \\
&= \|y,z\|^2 - i(y,x \mid z) - i(x,y \mid z) + \frac{|(x,y \mid z)|^2}{\|y,z\|^2} \\
&= \|y,z\|^2 - 2\mathrm{Im}(x,y \mid z) + \frac{|(x,y \mid z)|^2}{\|y,z\|^2} \\
&= \|y,z\|^2 - 2\mathrm{Im}(x,y \mid z) + \frac{\mathrm{Re}^2(x,y \mid z)}{\|y,z\|^2} + \frac{\mathrm{Im}^2(x,y \mid z)}{\|y,z\|^2} \\
&= \frac{\mathrm{Re}^2(x,y \mid z)}{\|y,z\|^2} + \left(\|y,z\| - \frac{\mathrm{Im}(x,y \mid z)}{\|y,z\|} \right)^2.
\end{aligned}$$

From (21), we have $\|x - \frac{(x,y \mid z)}{\|y,z\|^2} y, z\|^2 = \|x,z\|^2 - \frac{|(x,y \mid z)|^2}{\|y,z\|^2}$, so we put (23) and (21) in (22) and it follows that

$$\begin{aligned}
\|x - iy,z\|^2 &= \left| i\|y,z\| - \frac{(x,y \mid z)}{\|y,z\|} \right|^2 + \left\| x - \frac{(x,y \mid z)}{\|y,z\|^2} y, z \right\|^2 \\
&= \frac{\mathrm{Re}^2(x,y \mid z)}{\|y,z\|^2} + \left(\|y,z\| - \frac{\mathrm{Im}(x,y \mid z)}{\|y,z\|} \right)^2 + \|x,z\|^2 - \frac{|(x,y \mid z)|^2}{\|y,z\|^2} \\
&= \frac{\mathrm{Re}^2(x,y \mid z)}{\|y,z\|^2} + \|y,z\|^2 + \frac{\mathrm{Im}^2(x,y \mid z)}{\|y,z\|^2} \\
&\quad - 2\mathrm{Im}(x,y \mid z) + \|x,z\|^2 - \frac{|(x,y \mid z)|^2}{\|y,z\|^2}
\end{aligned}$$

$$= \|x, z\|^2 + \|y, z\|^2 - 2\mathrm{Im}(x, y \mid z).$$

Therefore (18) is obtained. □

3.2 A new reverse of the Cauchy–Schwarz inequality in 2-inner product spaces

Reverses of Cauchy–Schwarz inequality in 2-inner product spaces, usually establish upper bounds for one of the following nonnegative quantities:

$$\|x, z\|\|y, z\| - \mid (x, y \mid z) \mid, \quad \|x, z\|^2\|y, z\|^2 - \mid (x, y \mid z) \mid^2,$$
$$\frac{\|x, z\|\|y, z\|}{\mid (x, y \mid z) \mid}, \quad \frac{\|x, z\|^2\|y, z\|^2}{\mid (x, y \mid z) \mid^2}.$$

Classical examples of such inequalities can be found in [4, 5, 6, 12]. Otachel [13] obtained several reverses of the Cauchy–Schwarz inequality in inner product space. In this section, we present a reverse of the Cauchy–Schwarz inequality in 2-inner product space.
First we fix some notations. Suppose that $(\mathcal{X}, (\cdot, \cdot \mid \cdot))$ is a 2-inner product space over the field $\mathcal{K}$, where $\mathcal{K}$ is the real or complex numbers field, and that $z \in \mathcal{X}$. For given $A, a, B, b \in \mathcal{X}$, we define $M = \|A - a, z\| + \frac{\|A + a, z\|}{\|B + b, z\|} \min\{\|B + b, z\|, \|B - b, z\|\}$ if $B+b$ and z are linearly independent. Now we prove the following Lemma.

Lemma 1 *Let $A, a \in \mathcal{X}$. Then*

$$\mathrm{Re}(A - x, x - a \mid z) \geq 0 \quad \textit{if and only if} \quad \|x - \frac{A + a}{2}, z\| \leq \frac{1}{2}\|A - a, z\| \quad (24)$$

for $x \in \mathcal{X}$.

Proof. Suppose that $\mathrm{Re}(A - x, x - a \mid z) \geq 0$. Using (2), we have $\frac{1}{4}[(z, z \mid A - a) - (z, z \mid A - 2x + a)] \geq 0$, that is, $(z, z \mid A - 2x + a) \leq (z, z \mid A - a)$. Therefore, $\|A - 2x + a, z\| \leq \|A - a, z\|$ or $\left\|-2(x - \frac{A + a}{2}), z\right\| \leq \|A - a, z\|$. It follows that $\left\|x - \frac{A + a}{2}, z\right\| \leq \frac{1}{2}\|A - a, z\|$. □

Theorem 2 *Let $A, a, B, b \in \mathcal{X}$ with $A + a, B + b \in \mathrm{span}\{v\}$ for a certain $0 \neq v \in \mathcal{X}$.*
If

$$\mathrm{Re}(A - x, x - a \mid z) \geq 0 \quad \text{and} \quad \mathrm{Re}(B - y, y - b \mid z) \geq 0 \quad (25)$$

for x,y,$z \in \mathcal{X}$*, then the following inequalities hold:*

$$0 \le \|x,z\|^2\|y,z\|^2 - |\,(x,y\mid z)\,|^2$$
$$\le \begin{cases} \frac{1}{4}M^2\min\{\|x,z\|^2, \|y,z\|^2\}, & B+b \text{ and } z \text{ are linearly independent,} \\ \frac{1}{4}\min\{\|A-a,z\|^2\|y,z\|^2, \|B-b,z\|^2\|x,z\|^2\}, & A+a \text{ and } z \text{ also } B+b \text{ and } z \text{ are linearly dependent.} \end{cases}$$

where M *is a real number, which does not depend from* x *and* y*.*

Proof. If x and z, or y and z are linearly dependent, the inequalities hold. Let y and z be linearly independent. For any $c \in \mathcal{K}$, we have

$$\left\|x - \frac{(x,y\mid z)}{\|y,z\|^2}y, z\right\| \le \|x-cy,z\| = \left\|x - \frac{1}{2}(A+a) + \frac{1}{2}(A+a) - cy, z\right\|$$
$$\le \left\|x - \frac{1}{2}(A+a), z\right\| + \left\|\frac{1}{2}(A+a) - cy, z\right\|.$$

Since $A+a, B+b \in \operatorname{span}\{v\}$ with $v \neq 0$, $B+b$ and z are linearly independent. Put
$A+a = \varepsilon\left(\frac{\|A+a,z\|}{\|B+b,z\|}\right)(B+b)$, for a certain $\varepsilon \in \mathcal{K}$ with $|\,\varepsilon\,| = 1$.
Hence, letting $c = \frac{(A+a,y\mid z)}{2\|y,z\|^2}$, we obtain

$$\left\|x - \frac{(x,y\mid z)}{\|y,z\|^2}y, z\right\| \le \left\|x - \frac{1}{2}(A+a), z\right\| + \left\|\frac{1}{2}(A+a) - \frac{(A+a,y\mid z)}{2\|y,z\|^2}y, z\right\|$$
$$= \left\|x - \frac{1}{2}(A+a), z\right\| + \frac{\|A+a,z\|}{\|B+b,z\|}\left\|\frac{1}{2}(B+b) - \frac{(B+b,y\mid z)}{2\|y,z\|^2}y, z\right\|.$$

Obviously, $\left\|\frac{1}{2}(B+b) - \frac{(B+b,y\mid z)}{2\|y,z\|^2}y, z\right\| \le \left\|\frac{1}{2}(B+b) - \tilde{c}y, z\right\|$ for any $\tilde{c} \in \mathcal{K}$.

Next, substituting consecutively $\tilde{c} = 0$ and $\tilde{c} = 1$, we obtain

$$\left\|\frac{1}{2}(B+b) - \frac{(B+b,y\mid z)}{2\|y,z\|^2}y, z\right\| \le \min\left\{\frac{1}{2}\|B+b,z\|, \left\|\frac{1}{2}(B+b) - y, z\right\|\right\}.$$

Consequently, by using (24) and (25), we have

$$\left\|x - \frac{(x,y\mid z)}{\|y,z\|^2}y, z\right\| \le \left\|x - \frac{1}{2}(A+a), z\right\|$$

$$+\frac{\|A+a,z\|}{\|B+b,z\|}\min\left\{\frac{1}{2}\|B+b,z\|,\left\|\frac{1}{2}(B+b)-y,z\right\|\right\}$$
$$\leq\frac{1}{2}\left(\|A-a,z\|+\frac{\|A+a,z\|}{\|B+b,z\|}\min\{\|B+b,z\|,\|B-b,z\|\}\right).$$

Finally,

$$\|x,z\|^2-\frac{|\,(x,y\mid z)\,|^2}{\|y,z\|^2}=\left\|x-\frac{(x,y\mid z)}{\|y,z\|^2}y,z\right\|^2$$
$$\leq\frac{1}{4}\left(\|A-a,z\|+\frac{\|A+a,z\|}{\|B+b,z\|}\min\{\|B+b,z\|,\|B-b,z\|\}\right)^2.$$

Multiplying the both of sides by $\|y,z\|^2>0$, we have

$$\|x,z\|^2\|y,z\|^2-|\,(x,y\mid z)\,|^2\leq\frac{1}{4}M^2\|y,z\|^2.$$

Similarly, we obtain

$$\|x,z\|^2\|y,z\|^2-|\,(x,y\mid z)\,|^2\leq\frac{1}{4}M^2\|x,z\|^2.$$

Hence

$$\|x,z\|^2\|y,z\|^2-|\,(x,y\mid z)\,|^2\leq\frac{1}{4}M^2\min\{\|x,z\|^2,\|y,z\|^2\}.$$

If $A+a$ and z and also $B+b$ and z are linearly dependent, then the right side inequality (24) takes the form $\|x,z\|\leq\frac{1}{2}\|A-a,z\|$ and $\|y,z\|\leq\frac{1}{2}\|B-b,z\|$. Hence, we have

$$\|x,z\|^2\|y,z\|^2-|\,(x,y\mid z)\,|^2\leq\|x,z\|^2\|y,z\|^2\leq\begin{cases}\frac{1}{4}\|A-a,z\|^2\|y,z\|^2,\\ \frac{1}{4}\|B-b,z\|^2\|x,z\|^2.\end{cases}$$

Therefore,

$$\|x,z\|^2\|y,z\|^2-|\,(x,y\mid z)\,|^2\leq\frac{1}{4}\min\{\|A-a,z\|^2\|y,z\|^2,\|B-b,z\|^2\|x,z\|^2\}.$$

□

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