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The conditional quantile function in the single-index

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Abstract. The main contribution of the present paper is to give the conditional quantile estimator and we establish the pointwise and the almost complete convergence of the kernel estimate of this model in the functional single-index model.

1 Introduction

In recent years, nonparametric statistics have undergone a very important development. As well as the single fictional index models which are used in different fields, namely, medical, economic, epidemiology, and others. In the literature, the prediction problem has been widely studied when the two variables are of finite dimensions and in the case of functional variables. When the explanatory variable is functional and the response is still real. Note that the modeling of functional data is becoming more and more popular. In 1985, Härdle et al. The first who are interested in the nonparametric estimations of the regression functions [21] and in 2005, Ferraty and vieu gave a good synthesis on the conditional models using the nucleus method [15]. The approach of single-index is to widely applied in econometrics as a reasonable compromise between nonparametric and parametric models. A number of works dealing

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with index model can be found in the literature when the explanatory variable is multivariate. Without claiming to be exhaustive, we quote for example Härdle et al. [20], Hristache et al. [23]. A first work linking the single-index model and the nonparametric regression models for functional random variables for independent observations can be found in [13]. Their results were extended to dependent case by Ait saidi et al. [2, 3]. Concerning the conditional density estimate, Attaoui et al. [4] studied the estimation of the single functional index and established some asymptotic results. Their work extend, in different way, the works of Delecroix et al.[10]. In 2017, Hamdaoui and al. have studied the asymptotic normality of the conditional distribution function in the single index model [19].

Much has been done on conditional quantile estimates. For example, Berlitet et al. [6, 5], have studied proprits and normality asymptottic od the conditional quantile and Dabo-Niang et al. have also studied the estimation of the quantile regression [8, 9]. We can also cite the work of Ezzahrioui and Ould Said (see [11]) and Honda (see [22]) carried out the study on estimator in the α mixing case. We also have Ferraty and al. in the case of dependent data [14]. we can also cite other works, such as those of Gannoun and al. on the median and the quantile [16], Koenker on the quantile linked to the regression [17, 18], Laksaci and Maref [24] and Wang and Zhao [26].

In this work, we consider the problem of estimating the conditional quantile function of a scalar response variable Y given a Hilbertian random variable X when the explanation of Y given X is done through its projection on one functional direction. Following this study we can build a prediction method based on the conditional quantile estimation with simple functional index. This alternative method is more robust than the conditional method. This result allows us to calculate the prediction expectation which is very sensitive to the errors of the observations when the data of heteroscedasty, or asymmetry and in the case where the distribution is bimodal. Ait saidi et al. ([3]) studied the expectation when we regress a real random variable on a functional random variable (in the case of infinite dimension).

In this article, we are first interested in the estimation of the conditional quantile by the kernel method for the functional single index model. Subsequently, we study the pointwise convergence and almost complete convergence of the estimate of the kernel of this model in the functional single index model.

2 Model and estimator

Let (X, Y) be a couple of random variables taking its values in $\mathcal{H} \times \mathbf{R}$, where $\mathcal{H}$ is a separable real Hilbert space with the inner product $\langle ., . \rangle$. Consider now the sample $(X_i, Y_i)_{i=1,\dots,n}$ of n independent pairs identically distributed as the pair (X, Y). Assume that the conditional cumulative distribution function (c.d.f.) of Y given X has a single-index structure. Such structure supposes that the explanation of Y from X is done through a fixed functional index θ in $\mathcal{H}$. More precisely, we suppose that the conditional c.d.f. of Y given $X = x$, denoted by $F(. \mid x)$, is given by

$$\forall y \in \mathbf{R} \quad F(y \mid x) = F(y \mid \langle \theta,\ x \rangle).$$

The functional index θ appears as a filter allowing the extraction of the part of X explaining the response Y and represents a functional direction which reveals pertinent explanation of the response variable. Concerning the identifiability of this model, we consider the same conditions as those in Ferraty et al. [13] on the regression operator. In other words, we assume that the F is differentiable with respect to x and θ such that $\langle \theta, e_1 \rangle = 1$, where e_1 is the first vector of an orthonormal basis of $\mathcal{H}$. Clearly, under this condition, we have, for all $x \in \mathcal{H}$,

$$F_1(. \mid \langle \theta_1,\ x \rangle) = F_2(. \mid \langle \theta_2,\ x \rangle) \Longrightarrow \theta_1 = \theta_2 \ \text{ and } \ F_1 \equiv F_2.$$

We consider the semi-metric d_θ, associated to the single-index $\theta \in \Theta_{\mathcal{H}} \subset \mathcal{H}$ defined by $\forall x_1, x_2 \in \mathcal{H} : d_\theta\,(x_1, x_2) = |\langle x_1 - x_2,\ \theta \rangle|$. In what follows we denote by $F_\theta(., x)$ the conditional c.d.f. of Y given $\langle \theta, x \rangle$ and we we define the Kernel estimator $\widehat{F}_\theta(y, x)$ of $F_\theta(y, x)$ by

$$\widehat{F}_\theta(y, x) = \frac{\sum_{i=1}^{n} K\left(\frac{d_\theta(X_i, x)}{h}\right) H\left(\frac{y - Y_i}{g}\right)}{\sum_{i=1}^{n} K\left(\frac{d_\theta(X_i, x)}{h}\right)}, \quad \forall y \in \mathbb{R} \tag{1}$$

With the convention $0/0 = 0$, where H is defined by :

$$\forall u \in \mathbb{R} \quad H(u) = \int_{-\infty}^{u} K_0(v) dv.$$

The function K is a kernel of type I or of type II and the function K_0 is a kernel of type 0 and $h = h(n)$ (resp. $g = g(n)$) is a sequence of positive real

numbers which goes to zero as n tends to infinity. This estimate extend, in different way, the works of Samanta [25]) in the real case and Ferraty et al. [12] in the functional case.
Recall that a function K from $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K = 1$ is called kernel of type I if there exist two real constants $0 < C_1 < C_2 < \infty$ such that

$$C_1 1_{[0,1]} \leq K \leq C_2 1_{[0,1]}.$$

It is called kernel of type II if its support is $[0, 1]$ and if its derivative K' exists on $[0, 1]$ and satisfies for two real constants $-\infty < C_4 < C_3 < 0$:

$$C_4 \leq K' \leq C_3.$$

A function K_0 from $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K_0 = 1$ is called kernel of type 0 if its compact support is $[-1, 1]$ and such that $\forall u \in (0, 1)\,,\ K(u) > 0$.

Let $\alpha \in]0, 1[$, the conditional quantile function of Y given $X = x$, denoted by $Q_{\theta,\ \alpha}(x)$, is given by

$$Q_{\theta,\ \alpha}(x) = \inf\{y \in \mathbb{R}, F_\theta(y, x) \geq \alpha\} \tag{2}$$

and we can write That:

$$F_\theta(Q_{\theta,\ \alpha}(x), x) = \alpha$$

The fact that the conditional c.d.f $F_\theta\,(y, x)$ is strictly increasing, insures the existence and unicity of the conditional quantile c.q.f.
The kernel estimate $\widehat{Q}_{\theta,\ \alpha}(x)$, of the conditional quantile $Q_{\theta,\ \alpha}(x)$ is defined by

$$\widehat{F}_\theta(\widehat{Q}_{\theta,\ \alpha}(x), x) = \alpha$$

3 Main results

All along the paper, we will denote by C and C' some strictly positive generic constants.

3.1 Pointwise almost complete convergence

Let x (resp. y) be a fixed element of $\mathcal{H}$ (resp. $\mathbb{R}$), let $\mathcal{N}_x \subset \mathcal{H}$ be a neighborhood of x and $\mathbb{S}$ be a fixed compact subset of $\mathbb{R}$. In order to establish the almost complete (a.co.) convergence of our estimate we will introduce some hypotheses.

(H_1) The probability of the functional variable on a small ball is non null:

$$P\left(d_\theta\left(X, x\right) < h\right) = \varphi_{\theta,x}(h) > 0, \tag{3}$$

(H_2) h is a sequence of positive numbers satisfying

$$\lim_{n\to\infty} \frac{\log n}{n\varphi_{\theta,x}(h)} = 0, \tag{4}$$

(H_3) About the small ball conditional probability $\varphi_{\theta,x}(.)$, we assume that :

$$\exists C > 0,\ \exists \varepsilon_0,\ \forall \varepsilon < \varepsilon_0,\ \int_0^\varepsilon \varphi_{\theta,x}(u)du > C\varepsilon\varphi_{\theta,x}(\varepsilon), \tag{5}$$

(H_4) Now, we suppose that the operator F_θ satisfy the following Hölder-type condition :

$$\begin{cases} \exists C_{\theta,x} > 0 \text{ such that } \forall\left(y_1, y_2\right) \in \mathbb{S}^2,\ \forall\left(x_1, x_2\right) \in \mathcal{N}_x \times \mathcal{N}_x, \\ |F_\theta(y_1, x_1) - F_\theta(y_2, x_2)| \le C_{\theta,x}\left(d_\theta^{\beta_1}\left(x_1, x_2\right) + |y_1 - y_2|^{\beta_2}\right),\ \beta_1 > 0,\ \beta_2 > 0. \end{cases} \tag{6}$$

Theorem 1 *Under the hypotheses (H_1)-(H_4), as n goes to infinity, we have*

$$\left|\widehat{Q}_{\theta,\ \alpha}(y, x) - Q_{\theta,\ \alpha}(y, x)\right| = \bigcirc(h^{\beta_1}) + \bigcirc(g^{\beta_2}) + \bigcirc_{a.co.}\left(\sqrt{\frac{\log n}{n\varphi_{\theta,x}(h)}}\right). \tag{7}$$

Recall that a sequence $\left(X_n\right)_{n\in\mathbb{N}^*}$ of a real-valued random variables is said to converge almost completely (a.co.) to a real-valued varaible X if and only if

$$\forall \epsilon > 0,\ \sum_{n\in\mathbb{N}^*} P\left(|X_n - X| > \epsilon\right) < \infty).$$

This mode of convergence implies both almost sure and in probability convergence (see for instance Bosq and Lecoutre, [7])

Proof. Since, we have $\lim\limits_{n\to\infty} g = 0$ and K_0 is a kernel of type 0 the estimated conditional c.d.f. $\widehat{F}_\theta(., x)$ is continuous and strictly increasing. So, the function $\widehat{F}_\theta^{-1}(., x)$ exists and is continuous. The continuity property of $\widehat{F}_\theta(., x)$ at point $\widehat{F}_\theta(Q_{\theta,\ \alpha}(x), x)$ can be written as: $\forall \epsilon > 0,\ \exists \delta\left(\epsilon\right) > 0, \forall y$,

$$\left|\widehat{F}_\theta(y, x) - \widehat{F}_\theta(Q_{\theta,\ \alpha}(x), x)\right| \le \delta\left(\epsilon\right) \Longrightarrow |y - Q_{\theta,\ \alpha}(x)| \le \epsilon.$$

In the special case when $y = \widehat{Q}_\theta(x)$, we have: $\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0$,

$$\left|\widehat{F}_\theta(\widehat{Q}_{\theta,\ \alpha}(x), x) - \widehat{F}_\theta(Q_{\theta,\ \alpha}(x), x)\right| \leq \delta(\epsilon) \Longrightarrow \left|\widehat{Q}_{\theta,\ \alpha}(x) - Q_{\theta,\ \alpha}(x)\right| \leq \epsilon,$$

in such a way that we arrive at: $\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0$,

$$\begin{aligned} P\left(\left|\widehat{Q}_{\theta,\ \alpha}(x) - Q_{\theta,\ \alpha}(x)\right| > \epsilon\right) &\leq P\left(\left|\widehat{F}_\theta(\widehat{Q}_{\theta,\ \alpha}(x), x) - \widehat{F}_\theta(Q_{\theta,\ \alpha}(x), x)\right| > \delta(\epsilon)\right) \\ &\leq P\left(\left|F_\theta(Q_{\theta,\ \alpha}(x), x) - \widehat{F}_\theta(Q_{\theta,\ \alpha}(x), x)\right| > \delta(\epsilon)\right), \end{aligned}$$

the last inequality following from the simple observation that

$$F_\theta(Q_{\theta,\ \alpha}(x), x) = \widehat{F}_\theta(\widehat{Q}_{\theta,\ \alpha}(x), x) = \alpha. \tag{8}$$

The pointwise almost complete convergence of the kernel conditional c.d.f. estimate $\widehat{F}_\theta(., x)$ given by Ait Aidi and Mecheri (2016) (see Theorem 1 [1]),we get the result directly:

$$\forall \epsilon > 0,\ \sum_{n=1}^{\infty} P\left(\left|\widehat{Q}_{\theta,\ \alpha}(x) - Q_{\theta,\ \alpha}(x)\right| > \epsilon\right) < \infty, \tag{9}$$

Finally, we get the result. □

3.2 Pointwise almost complete rate of convergence

The aim of this section we study the rate of convergence of our conditional quantile estimator $Q_{\theta,\ \alpha}(x)$. As it is usual in conditional quantiles estimation, the rate of convergence can be linked with the flatness of the cond-cdf $F(.|x)$ around the conditional quantile $Q_{\theta,\ \alpha}(x)$.However, the behavior of the conditional quantiles estimation depends on the flatness of F_θ around the point $Q_{\theta,\ \alpha}(x)$.
In order to study the rate of convergence of this conditional estimator, we must introduce other hypotheses.

(H_5) $F(.|x)$ is j-times continuously differentiable in some neighbourhood of $Q_{\theta,\ \alpha}(x)$,

(H_6) $F_\theta(y, x)$ is strictly increasing and if we suppose that exists $l \in \{1, ..., j\}$ such that $F_\theta^{(l)}(., x)$ is Lipschitz continuous of order β_0 :

$$\exists C \in (0, \infty),\ \forall \left(y, y'\right) \in \mathbb{R}^2,\ \left|F_\theta^{(l)}(y, x) - F_\theta^{(l)}\left(y', x\right)\right| \leq C\left|y - y'\right|^{\beta_0}, \tag{10}$$

(H_7) $\exists j > 0, \forall l = 1, ..., j-1,$

$$\begin{cases} F_\theta^{(l)}(Q_{\theta,\ \alpha}(x), x) = 0, \\ F_\theta^{(j)}(Q_{\theta,\ \alpha}(x), x) > 0. \end{cases} \tag{11}$$

(H_8) The coumulative kernel H is j-times continuously differentiable.

$$\lim_{n\longrightarrow\infty} \frac{\log n}{n g^{2j-1}\varphi_{\theta,x}(h)} = 0, \tag{12}$$

Theorem 2 *Under the conditions (H_1)-(H_8),*
we have:

$$\widehat{Q}_{\theta,\ \alpha}(x) - Q_{\theta,\ \alpha}(x) = \bigcirc\left((h^{\beta_1} + g^{\beta_2})^{\frac{1}{j}}\right) + \bigcirc_{a.co.}\left(\left(\frac{\log n}{\varphi_{\theta,x}(h)}\right)^{\frac{1}{2j}}\right). \tag{13}$$

Proof. Taylor expansion of the function $\widehat{F}_\theta$ leads the existence of some $Q^*_{\theta,\ \alpha}$ between $\widehat{Q}_{\theta,\ \alpha}(x)$ and $Q_{\theta,\ \alpha}(x)$ such that :

$$\widehat{F}_\theta(Q_{\theta,\ \alpha}(x), x) - \widehat{F}_\theta(\widehat{Q}_{\theta,\ \alpha}(x), x) = \sum_{l=1}^{j-1} \frac{\left(Q_{\theta,\ \alpha}(x) - \widehat{Q}_{\theta,\ \alpha}(x)\right)^l}{l!}\widehat{F}_\theta^{(l)}(Q_{\theta,\ \alpha}(x), x)$$

$$+\frac{\left(Q_{\theta,\ \alpha}(x) - \widehat{Q}_{\theta,\ \alpha}(x)\right)^j}{j!}\widehat{F}_\theta^{(j)}(Q^*_{\theta,\ \alpha}, x).$$

Because of (9), this can be rewritten as :

$$\widehat{F}_\theta(Q_{\theta,\ \alpha}(x), x) - \widehat{F}_\theta(\widehat{Q}_{\theta,\ \alpha}(x), x) = \sum_{l=1}^{j-1} \frac{(Q_{\theta,\ \alpha}(x) - \widehat{Q}_{\theta,\ \alpha}(x))^l}{l!}\times$$

$$\left(\widehat{F}_\theta^{(l)}(Q_\nu(x), x) - F_\theta^{(l)}(Q_{\theta,\ \alpha}(x), x)\right) + \frac{(Q_{\theta,\ \alpha}(x) - \widehat{Q}_{\theta,\ \alpha}(x))^j}{j!}\widehat{F}_\theta^{(j)}(Q^*_{\theta,\ \alpha}, x).$$

Because $\widehat{F}_\theta(\widehat{Q}_{\theta,\ \alpha}(x), x) = F_\theta(Q_{\theta,\ \alpha}(x), x) = \alpha$, we have

$$(Q_{\theta,\ \alpha}(x) - \widehat{Q}_{\theta,\ \alpha}(x))^j\widehat{F}_\theta^{(j)}(Q^*_{\theta,\ \alpha}, x) = \bigcirc\left(\widehat{F}_\theta(Q_{\theta,\ \alpha}(x), x) - F_\theta(Q_{\theta,\ \alpha}(x), x)\right)$$

$$+\bigcirc\left(\sum_{l=1}^{j-1}(Q_\theta(x) - \widehat{Q}_{\theta,\ \alpha}(x))^l\left(\widehat{F}_\theta^{(l)}(Q_{\theta,\ \alpha}(x), x) - F_\theta^{(l)}(Q_{\theta,\ \alpha}(x), x)\right)\right)$$

By combining the results of following Lemma 1 and Theorem 2, together with the fact that $Q^*_{\theta,\ \alpha}$ is lying between $\widehat{Q}_{\theta,\ \alpha}(x)$ and $Q_{\theta,\ \alpha}(x)$, it follows that

$$\lim_{n\longrightarrow\infty}\widehat{F}^{(j)}_{\theta}(Q^*_{\theta,\ \alpha},x)=F^{(j)}_{\theta}(Q_{\theta,\ \alpha}(x),x),\ \text{a.co.}$$

Lemma 1 (*See Ait Saidi and Mecheri (2016)* [1] *and Ferraty et al. (2005)* [14])

Let be an integer $l\in\{1,...,j\}$. *Under the conditions of theorem 2, we have*

$$\lim_{n\longrightarrow\infty}\widehat{F}^{(l)}_{\theta}(y,x)=F^{(l)}_{\theta}(y,x),\ \text{a.co.} \tag{14}$$

In addition the function $F^{(l)}_{\theta}(.,x)$ *is Lipschitz continuous of order* β_0, *that is if*

$$\exists C\in(0,+\infty),\ \forall\left(y,y'\right)\in\mathbb{R}^2,\ \left|F^{(l)}_{\theta}(y,x)-F^{(l)}_{\theta}(y',x)\right|\leq C\left|y-y'\right|^{\beta_0}, \tag{15}$$

then we have

$$F^{(l)}_{\theta}(y,x)-\widehat{F}^{(l)}_{\theta}(y,x)=\bigcirc(h^{\beta_1})+\bigcirc(g^{\beta_0})+\bigcirc_{a.co.}\left(\sqrt{\frac{\log n}{ng^{2l-1}\varphi_{\theta,x}(h)}}\right). \tag{16}$$

Proof of Lemma 1

The proof is given by Ait Saidi and Mecheri (2016) (See [1]).

Because the second part of assumption (12) is insuring that this limit is not 0, it follows by using proposition A.6-ii Ferraty and Vieu (2006) [15] that :

$$(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x))^j=\bigcirc_{a.co.}\left(\widehat{F}_{\theta}(Q_{\theta,\ \alpha}(x),x)-F_{\theta}(Q_{\theta,\ \alpha}(x),x)\right)$$
$$+\bigcirc_{a.co.}\left(\sum_{l=1}^{j-1}(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x))\right)^l\left(\widehat{F}^{(l)}_{\theta}(Q_{\theta,\ \alpha}(x),x)-F^{(l)}_{\theta}(Q_{\theta,\ \alpha}(x),x)\right).$$

Because (9), for all $l\in\{0,1,...,j\}$ and for all y in a neighborhood of $Q_{\theta,\ \alpha}(x)$, it exists $Q^*_{\theta,\ \alpha}$ between y and $Q_{\theta,\ \alpha}$ such that :

$$F^{(l)}_{\theta}(y,x)-F^{(l)}_{\theta}(Q_{\theta,\ \alpha}(x),x)=\frac{(y-Q_{\theta,\ \alpha}(x))^{j-l}}{(j-l)\,!}F^{(j)}_{\theta}(Q^*_{\theta,\ \alpha},x)$$

wich implies that $F^{(l)}_{\theta}$ is Lipschitz continuous around $Q_{\theta,\ \alpha}(x)$ with order $j-l$. So, by using now Theorem 1 of Ait Saidi and Mecheri (2016) (See [1]) together with the following Lemma with the suitable Lipschitz orders, one get:

$$(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x))^j = \bigcirc(h^{\beta_1}) + \bigcirc(g^{\beta_2}) + \bigcirc_{a.co.}\left(\sqrt{\frac{\log n}{n\varphi_{\theta,x}(h)}}\right) + \bigcirc_{a.co.}\left(\sum_{l=1}^{j-1} A_{n,l}\right) + \bigcirc_{a.co.}\left(\sum_{l=1}^{j-1} B_{n,l}\right), \tag{17}$$

where

$$A_{n,l} = (Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x))^l\left(\sqrt{\frac{\log n}{ng^{2l-1}\varphi_{\theta,x}(h)}}\right)$$

and

$$B_{n,l} = (Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x))^l g^{j-l}.$$

- Now we suppose that it exists $l \in \{1,...,j-1\}$ such that $(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x))^j = \bigcirc\ (A_{n,l})$, we can write that:

$$\left|Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right|^j \leq C\left|Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right|^l\left(\sqrt{\frac{\log n}{ng^{2l-1}\varphi_{\theta,x}(h)}}\right),$$

which implies that

$$\left|Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right|^{j-l} \leq C\left(\sqrt{\frac{\log n}{ng^{2l-1}\varphi_{\theta,x}(h)}}\right)$$

and

$$\left|Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right|^j \leq C\left(\frac{\log n}{ng^{2l-1}\varphi_{\theta,x}(h)}\right)^{\frac{j}{2(j-l)}}.$$

So, because (10), as soon as it exists l such that $\left(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right)^j = \bigcirc$ $(A_{n,l})$, then we have

$$\left(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right)^j = \bigcirc\left(\sqrt{\frac{\log n}{n\varphi_{\theta,x}(h)}}\right). \tag{18}$$

- In the same way, we will deny that if $l \in \{1,...,j-1\}$ such that $(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x))^j = \bigcirc\ (B_{n,l})$, we have :

$$\left|Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right|^j \leq C\left|Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right|^l g^{j-l},$$

which implies that

$$\left|Q_{\theta,\ \alpha}(x) - \widehat{Q}_{\theta,\ \alpha}(x)\right|^j \le Cg^j.$$

So, as soon as it exists l such that $\left(Q_{\theta,\ \alpha}(x) - \widehat{Q}_{\theta,\ \alpha}(x)\right)^j = \bigcirc\ (B_{n,l})$, then we have

$$\left(Q_{\theta,\ \alpha}(x) - \widehat{Q}_{\theta,\ \alpha}(x)\right)^j = \bigcirc\left(g^{\beta_1}\right). \tag{19}$$

Finally, we get the result

$$\left(Q_{\theta,\ \alpha} - \widehat{Q}_{\theta,\ \alpha}(x)\right)^j = \bigcirc(h^{\beta_1}) + \bigcirc(g^{\beta_2}) + \bigcirc_{a.co.}\left(\sqrt{\frac{\log n}{n\varphi_{\theta,x}(h)}}\right).$$

The proof is finished. □

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