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On existence of fixed points and applications to a boundary value problem and a matrix equation in C^*-algebra valued partial metric spaces

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Abstract. We utilize Hardy-Rogers contraction and $CJM-$contraction in a C^*-algebra valued partial metric space to create an environment to establish a fixed point.

Next, we present examples to elaborate on the novel space and validate our result. We conclude the paper by solving a boundary value problem and a matrix equation as applications of our main results which demonstrate the significance of our contraction and motivation for such investigations.

1 Introduction and preliminaries

Recently Chandok et al. [3] acquainted with the C^*-algebra valued partial metric combining the notions of partial metric (Matthews [12]) and C^*-algebra valued metric (Ma et al. [10]). Tomar and Joshi [17] pointed out, by giving explanatory examples that functions have different natures in different spaces

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and the consequences in C^*-algebra valued metric space can not be reduced to their metric counterparts unless unital C^*-algebra, $\mathbb{A} = \mathbb{R}$. Further, Tomar et al. [16] familiarised contractiveness and expansiveness in a newly introduced space to establish a fixed point and utilized these to solve an integral equation and an operator equation.

In the present work, we familiarize Hardy-Rogers contraction [6] and CJM−contraction [5]. The basic idea comprises utilizing the non-negative elements of an unital C^*-algebra ($\mathbb{A}$) as an alternative to a set of real numbers. Our outcomes are improvements and extensions of the existing results in metric spaces. Further, we provide illustrative examples to validate our result. Applications to a Boundary Value problem and a matrix equation conclude the paper.

Definition 1 [3] *A C^*-algebra valued partial metric is a function $p : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{A}$ on a non-empty set $\mathcal{M}$ if:*

(i) $\theta \preceq p(\mathfrak{w}, \mathfrak{v})$ *and* $p(\mathfrak{w}, \mathfrak{w}) = p(\mathfrak{v}, \mathfrak{v}) = p(\mathfrak{w}, \mathfrak{v})$ *if and only if* $\mathfrak{w} = \mathfrak{v}$, θ *is zero element of* $\mathbb{A}$*;*

(ii) $p(\mathfrak{w}, \mathfrak{w}) \preceq p(\mathfrak{w}, \mathfrak{v})$;

(iii) $p(\mathfrak{w}, \mathfrak{v}) = p(\mathfrak{v}, \mathfrak{w})$;

(iv) $p(\mathfrak{w}, \mathfrak{v}) \preceq p(\mathfrak{w}, \mathfrak{z}) + p(\mathfrak{z}, \mathfrak{v}) - p(\mathfrak{z}, \mathfrak{z})$, $\mathfrak{w}, \mathfrak{v}, \mathfrak{z} \in \mathcal{M}$.

Here, $(\mathcal{M}, \mathbb{A}, p)$ is a C^*-algebra valued partial metric space.
One may refer to [13] and [19], to study in detail on C^*-algebra.

The following example is given by Tomar et al. [16].

Example 1 *Let $F(\mathcal{M})$ be a collection of balls such that $B(\mathfrak{w}_0, \rho) = \{\mathfrak{v} : |\mathfrak{w}_0 - \mathfrak{v}| \leq \rho,\ \rho > 0\}$ and $\mathbb{A} = M_n(\mathbb{C})$ be the C^*-algebra of complex matrices. If $A = [a_{ij}] \in \mathbb{A}$, then $A^* = [\bar{a_{ji}}]$ is a non-zero element of $\mathbb{A}$. Norm is defined as, $\|A\| = \sup\{\|A\alpha\|_2 : \alpha \in \mathbb{C}^n, \|\alpha\|_2 \leq 1\}$, where $\|.\|_2$ is the usual l^2-norm on $\mathbb{C}^n$. Define $p : F(\mathcal{M}) \times F(\mathcal{M}) \longrightarrow \mathbb{A}$ by $p[B(\mathfrak{w}_0, \rho), B(\mathfrak{v}_0, \sigma)] = |\mathfrak{w}_0 - \mathfrak{v}_0|AA^* + \max\{\rho, \sigma\}I$. Then p is a C^*-algebra valued partial metric however, it is neither a C^*-algebra valued metric nor a standard partial metric, since $p[B(\mathfrak{w}_0, \rho), B(\mathfrak{w}_0, \rho)] = \rho \neq \theta$ and*

$$\begin{aligned} p[B(\mathfrak{w}_0, \rho), B(\mathfrak{v}_0, \tau)] &= |\mathfrak{w}_0 - \mathfrak{v}_0|AA^* + \max\{\rho, \tau\} I \\ &\preceq [|\mathfrak{w}_0 - \mathfrak{z}_0| + |\mathfrak{z}_0 - \mathfrak{v}_0|]AA^* + [\max\{\rho, \sigma\} + \max\{\sigma, \tau\} - \sigma]I \\ &= p[B(\mathfrak{w}_0, \rho), B(\mathfrak{z}_0, \sigma)] + p[B(\mathfrak{z}_0, \sigma), B(\mathfrak{v}_0, \tau)] - p[B(\mathfrak{z}_0, \sigma), B(\mathfrak{z}_0, \sigma)]. \end{aligned}$$

The C^*-*algebra valued partial metric reduces to the standard partial metric on taking* $\mathbb{A} = \mathbb{R}$. *For detailed discussions on* C^*-*algebra-valued metric spaces, one may refer to Tomar and Joshi* [17]. *Tomar et al.* [16] *discussed the convergence of the sequence when it converges to a zero element of* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ *and introduced the following definitions to create an environment to establish a fixed point in* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$.

Definition 2 [16]

(i) *A sequence* $\{\mathfrak{w}_n\}_{n\in\mathbb{N}}$ *is called a Cauchy sequence in* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ *if* $\lim_{n,m\longrightarrow\infty} \mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}_m)$ *exists with respect to* $\mathbb{A}$ *and is finite.*

(ii) $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ *is complete if every Cauchy sequence* $\{\mathfrak{w}_n\}_{n\in\mathbb{N}}$ *converges with respect to* $\mathbb{A}$ *in* $\mathcal{M}$, *to a point* $\mathfrak{w} \in \mathcal{M}$ *and satisfy*

$$\lim_{n,m\longrightarrow\infty} \mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}_m) = \lim_{n\longrightarrow\infty} \mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}_n) = \mathfrak{p}(\mathfrak{w}, \mathfrak{w}).$$

(iii) *The sequence* $\{\mathfrak{w}_n\}_{n\in\mathbb{N}}$ *in* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ $\theta-$*converges to a point* $\mathfrak{w} \in \mathcal{M}$ *if*

$$\lim_{n\longrightarrow\infty} \mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}) = \lim_{n\longrightarrow\infty} \mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}_n) = \mathfrak{p}(\mathfrak{w}, \mathfrak{w}) = \theta.$$

(iv) *A sequence* $\{\mathfrak{w}_n\}_{n\in\mathbb{N}}$ *is* $\theta-$*Cauchy if* $\lim_{n,m\to\infty} \mathfrak{p}(\mathfrak{w}_m, \mathfrak{w}_n) = \theta$, θ *is the zero element of* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$.

(v) $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ *is called* $\theta-$ *complete if every* $\theta-$*Cauchy sequence converges to a point* $\mathfrak{w} \in \mathcal{M}$ *and* $\mathfrak{p}(\mathfrak{w}, \mathfrak{w}) = \theta$.

Example 2 (Example 3.5**–Tomar et al. [16])** *Let*

$$\mathfrak{p}(\mathfrak{w}, \mathfrak{v}) = \begin{cases} I, & \text{if } \mathfrak{w} = \mathfrak{v} \\ \mathfrak{p}(\mathfrak{w}, \mathfrak{v}) = 2I, & \text{otherwise.} \end{cases}$$

If $\mathcal{M}$ *is a Hausdorff space and* $B(\mathcal{M})$ *is the set of all bounded functions, then* $B(\mathcal{M})$ *becomes a* C^*-*algebra with* $\|f(\mathfrak{w})\| = \sup_{\mathfrak{w}\in\mathcal{M}} |f(\mathfrak{w})|$. *Here, the sequence* $\{\mathfrak{w}_n\} = a$, $n \geq 1$ *is not* $\theta-$*Cauchy as it converges to* a. *However,* $\{\mathfrak{w}_n\}$ *is a Cauchy sequence. Implying thereby that every* $\theta-$*Cauchy sequence in* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ *is a Cauchy sequence. However, the reverse implication is not necessarily true.*

Remark 1 [16] *It is worth mentioning here that if a sequence* $\theta-$*converges to some point then its self-distance, as well as the self-distance of that point, is equal to zero element of* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$.

2 Main results

In the following, $\mathbb{A}^+$ denotes a set of self-adjoint (positive) operators of $\mathbb{A}$. Now, following Ma et al. [10], we introduce a Hardy - Rogers contraction and a CJM−contraction, then utilize these to establish a fixed point.

Definition 3 *A self-map $\mathcal{T}$ of $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ is called a C^*−algebra valued Hardy-Roger contractive map if*

$$\mathfrak{p}(\mathcal{T}\mathfrak{w}, \mathcal{T}\nu) \preceq \mathcal{A}\mathfrak{p}(\mathfrak{w}, \mathfrak{v}) + \mathcal{B}\mathfrak{p}(\mathfrak{w}, \mathcal{T}\mathfrak{w}) + \mathcal{C}\mathfrak{p}(\mathfrak{v}, \mathcal{T}\mathfrak{v}) + \mathcal{D}\mathfrak{p}(\mathfrak{v}, \mathcal{T}\mathfrak{w}) + \mathcal{E}\mathfrak{p}(\mathfrak{w}, \mathcal{T}\mathfrak{v}), \quad (1)$$

$$\forall\ \mathfrak{w},\ \mathfrak{v} \in \mathcal{M}, \quad \|\mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D} + \mathcal{E}\| \leq 1 \text{ and } \mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \mathcal{D},\ \mathcal{E} \in \mathbb{A}^+.$$

Example 3 *Let $\mathcal{M} = \mathbb{C}$ and $\mathbb{A}$ =Collection of all scalar matrices on $\mathbb{C}$. Let $\mathfrak{p} : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{A}$ be defined as,*

$$\mathfrak{p}(\mathfrak{w}, \mathfrak{v}) = \begin{bmatrix} \max\{|\mathfrak{w}|, |\mathfrak{v}|\} & 0 \\ 0 & \max\{|\mathfrak{w}|, |\mathfrak{v}|\} \end{bmatrix}.$$

So $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ is a C^*−algebra valued partial metric space and

$$\mathfrak{p}(\mathfrak{w}, \mathfrak{w}) = \begin{bmatrix} |\mathfrak{w}| & 0 \\ 0 & |\mathfrak{w}| \end{bmatrix} \neq \theta.$$

A function $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$ defined as

$$\mathcal{T}\mathfrak{w} = \begin{cases} \frac{\mathfrak{w}}{4}, & \mathfrak{w} \text{ is even} \\ \frac{\mathfrak{w}-1}{5}, & \mathfrak{w} \text{ is odd} \\ 0, & \text{otherwise} \end{cases},$$

is a C^*−algebra valued Hardy-Roger contraction for $\theta \preceq \mathcal{A} = \mathcal{D} = \mathcal{E} \prec \frac{I}{7}$ and $\theta \preceq \mathcal{B} = \mathcal{C} \prec \frac{I}{9}$.

It is fascinating to see here that, $\mathcal{T}$ is not a Hardy-Roger contraction [6] as a space under consideration is not a standard metric space.

Definition 4 *A self map $\mathcal{T}$ in $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ is called a C^*−algebra valued CJM−contraction, if*

(a) *for each $\varepsilon \succ \theta$ there exists a number $\delta \succ \theta$ satisfying*

$$\mathfrak{p}(\mathfrak{w}, \mathfrak{v}) \prec \varepsilon + \delta \Longrightarrow \mathfrak{p}(\mathcal{T}\mathfrak{w}, \mathcal{T}\mathfrak{v}) \prec \varepsilon,$$

(b) $\mathfrak{w} \neq \mathfrak{v} \Longrightarrow \mathfrak{p}(\mathcal{T}\mathfrak{w}, \mathcal{T}\mathfrak{v}) \prec \mathfrak{p}(\mathfrak{w}, \mathfrak{v}), \quad \mathfrak{w}, \mathfrak{v} \in \mathcal{M}.$

Example 4 *Let* $\mathcal{M} = \{0, \ 1\} \cup \{2\mathfrak{n} : \mathfrak{n} \in \mathbb{N}\} \cup \{\frac{2\mathfrak{n}-1}{2} + \frac{1}{2\mathfrak{n}-1} : \mathfrak{n} \in \mathbb{N}\}$ *and* $\mathbb{A}$ *=Collection of complex diagonal matrices defined on* $\mathcal{M}$*. Let* $\mathfrak{p} : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{A}$ *be defined as,*
$\mathfrak{p}(\mathfrak{w}, \mathfrak{v}) = \begin{bmatrix} |\mathfrak{w} - \mathfrak{v}| + \max\{\mathfrak{w}, \mathfrak{v}\}, & 0 \\ 0, & \alpha(|\mathfrak{w} - \mathfrak{v}| + \max\{\mathfrak{w}, \mathfrak{v}\}) \end{bmatrix}$*. So* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ *is a* C^*-*algebra valued partial metric space and* $\mathfrak{p}(\mathfrak{w}, \mathfrak{w}) = \begin{bmatrix} \mathfrak{w}, & 0 \\ 0, & \mathfrak{w} \end{bmatrix}$*. A function* $\mathcal{T} : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}$ *be defined as* $\mathcal{T}\mathfrak{w} = \begin{cases} \frac{2\mathfrak{n}-1}{2} + \frac{1}{2\mathfrak{n}-1}, & \mathfrak{w} = 2\mathfrak{n} \\ 0, & \text{otherwise} \end{cases}$*, is a* C^*-*algebra valued* CJM$-$*contraction for* $\varepsilon, \ \delta > \theta$*.*

It is fascinating to see here that, $\mathcal{T}$ is not a CJM$-$ contraction [5] as a space under consideration is not a standard metric space.

Now, we establish our result for C^*-algebra valued Hardy-Rogers contraction.

Theorem 1 *If a self map* $\mathcal{T}$ *is a continuous* C^*-*algebra valued Hardy-Rogers contractive map* (1) *of a* $\theta-$*complete* C^*-*algebra valued partial metric space* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$*, then* $\mathcal{T}$ *has a unique fixed point* $\mathfrak{z} \in \mathcal{M}$ *and* $\mathfrak{p}(\mathcal{T}\mathfrak{z}, \mathcal{T}\mathfrak{z}) = \theta = \mathfrak{p}(\mathfrak{z}, \mathfrak{z})$*.*

Proof. Starting from the given element $\mathfrak{w}_0 \in \mathcal{M}$, form the sequence $\{\mathfrak{w}_\mathfrak{n}\}$, where $\mathfrak{w}_\mathfrak{n} = \mathcal{T}\mathfrak{w}_{\mathfrak{n}-1}, \mathfrak{n} \in \mathbb{N}$. If $\mathfrak{p}(\mathfrak{w}_\mathfrak{n}, \mathfrak{w}_{\mathfrak{n}+1}) = \theta$, for some $\mathfrak{n} \geq 0$, then $\mathcal{T}\mathfrak{w}_\mathfrak{n} = \mathfrak{w}_{\mathfrak{n}+1} = \mathfrak{w}_\mathfrak{n}$ and $\mathfrak{p}(\mathfrak{w}_\mathfrak{n}, \mathfrak{w}_\mathfrak{n}) = \theta$ and this completes the proof.

Further, take $\mathfrak{p}(\mathfrak{w}_\mathfrak{n}, \mathfrak{w}_{\mathfrak{n}+1}) \succ \theta$, $\mathfrak{n} \geq 0$. For $\mathfrak{w} = \mathfrak{w}_{\mathfrak{n}+1}, \mathfrak{v} = \mathfrak{w}_{\mathfrak{n}+2}$, in condition (1),

$$\begin{aligned} \mathfrak{p}(\mathfrak{w}_{\mathfrak{n}+1}, \mathfrak{w}_{\mathfrak{n}+2}) &= \mathfrak{p}(\mathcal{T}\mathfrak{w}_\mathfrak{n}, \mathcal{T}\mathfrak{w}_{\mathfrak{n}+1}) \\ &\preceq \mathcal{A}\mathfrak{p}(\mathfrak{w}_\mathfrak{n}, \mathfrak{w}_{\mathfrak{n}+1}) + \mathcal{B}\mathfrak{p}(\mathfrak{w}_\mathfrak{n}, \mathcal{T}\mathfrak{w}_\mathfrak{n}) + \mathcal{C}\mathfrak{p}(\mathfrak{w}_{\mathfrak{n}+1}, \mathcal{T}\mathfrak{w}_{\mathfrak{n}+1}) \\ &\quad + \mathcal{D}\mathfrak{p}(\mathfrak{w}_{\mathfrak{n}+1}, \mathcal{T}\mathfrak{w}_\mathfrak{n}) + \mathcal{E}\mathfrak{p}(\mathfrak{w}_\mathfrak{n}, \mathcal{T}\mathfrak{w}_{\mathfrak{n}+1}) \\ &\preceq \mathcal{A}\mathfrak{p}(\mathfrak{w}_\mathfrak{n}, \mathfrak{w}_{\mathfrak{n}+1}) + \mathcal{B}\mathfrak{p}(\mathfrak{w}_\mathfrak{n}, \mathfrak{w}_{\mathfrak{n}+1}) + \mathcal{C}\mathfrak{p}(\mathfrak{w}_{\mathfrak{n}+1}, w_{\mathfrak{n}+2}) \\ &\quad + \mathcal{D}\mathfrak{p}(\mathfrak{w}_{\mathfrak{n}+1}, \mathfrak{w}_{\mathfrak{n}+1}) + \mathcal{E}[\mathfrak{p}(\mathfrak{w}_\mathfrak{n}, \mathfrak{w}_{\mathfrak{n}+1}) + \mathfrak{p}(\mathfrak{w}_{\mathfrak{n}+1}, \mathfrak{w}_{\mathfrak{n}+2}) \\ &\quad - \mathfrak{p}(\mathfrak{w}_{\mathfrak{n}+1}, \mathfrak{w}_{\mathfrak{n}+1})] \\ &= (\mathcal{A} + \mathcal{B} + \mathcal{E})\mathfrak{p}(\mathfrak{w}_\mathfrak{n}, \mathfrak{w}_{\mathfrak{n}+1}) + (\mathcal{C} + \mathcal{E})\mathfrak{p}(\mathfrak{w}_{\mathfrak{n}+1}, \mathfrak{w}_{\mathfrak{n}+2}) \\ &\quad + (\mathcal{D} - \mathcal{E})\mathfrak{p}(\mathfrak{w}_{\mathfrak{n}+1}, \mathfrak{w}_{\mathfrak{n}+1}), \end{aligned} \tag{2}$$

and

$$\begin{aligned}
p(\mathfrak{w}_{n+2},\mathfrak{w}_{n+1}) &= p(\mathcal{T}\mathfrak{w}_{n+1},\mathcal{T}\mathfrak{w}_n)\\
&\preceq \mathcal{A}p(\mathfrak{w}_{n+1},\mathfrak{w}_n)+\mathcal{B}p(\mathfrak{w}_{n+1},\mathcal{T}\mathfrak{w}_{n+1})+\mathcal{C}p(\mathfrak{w}_n,\mathcal{T}\mathfrak{w}_n)\\
&\quad+\mathcal{D}p(\mathfrak{w}_{n,\mathcal{T}\mathfrak{w}_{n+1}})+\mathcal{E}p(\mathfrak{w}_{n+1},\mathcal{T}\mathfrak{w}_n)\\
&\preceq \mathcal{A}p(\mathfrak{w}_{n+1},\mathfrak{w}_n)+\mathcal{B}p(\mathfrak{w}_{n+2},\mathfrak{w}_{n+1})+\mathcal{C}p(\mathfrak{w}_{n+1},\mathfrak{w}_n)\\
&\quad+\mathcal{D}[p(\mathfrak{w}_n,\mathfrak{w}_{n+1})+p(\mathfrak{w}_{n+1},\mathfrak{w}_{n+2})-p(\mathfrak{w}_{n+1},\mathfrak{w}_{n+1})]\\
&\quad+\mathcal{E}p(\mathfrak{w}_{n+1},\mathfrak{w}_{n+1})\\
&=(\mathcal{A}+\mathcal{C}+\mathcal{D})p(\mathfrak{w}_n,\mathfrak{w}_{n+1})+(\mathcal{B}+\mathcal{D})p(\mathfrak{w}_{n+1},\mathfrak{w}_{n+2})\\
&\quad+(\mathcal{E}-\mathcal{D})p(\mathfrak{w}_{n+1},\mathfrak{w}_{n+1}).
\end{aligned}\tag{3}$$

Adding (2) and (3)

$$\begin{aligned}
2p(\mathfrak{w}_{n+1},\mathfrak{w}_{n+2})) &\preceq (2\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}+\mathcal{E})P(\mathfrak{w}_n,\mathfrak{w}_{n+1})\\
&\quad+(\mathcal{B}+\mathcal{C}+\mathcal{D}+\mathcal{E})p(\mathfrak{w}_{n+1},\mathfrak{w}_{n+2}),
\end{aligned}$$

that is,

$$(2-\mathcal{B}-\mathcal{C}-\mathcal{D}-\mathcal{E})p(\mathfrak{w}_{n+1},\mathfrak{w}_{n+2}))\preceq(2\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}+\mathcal{E})p(\mathfrak{w}_n,\mathfrak{w}_{n+1}),$$

that is,

$$p(\mathfrak{w}_{n+1},\mathfrak{w}_{n+2}))\preceq\frac{2\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}+\mathcal{E}}{2-\mathcal{B}-\mathcal{C}-\mathcal{D}-\mathcal{E}}p(\mathfrak{w}_n,\mathfrak{w}_{n+1})\preceq\xi\,p(\mathfrak{w}_n,\mathfrak{w}_{n+1}),$$

where, $\xi=\frac{2\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}+\mathcal{E}}{2-\mathcal{B}-\mathcal{C}-\mathcal{D}-\mathcal{E}}$ and $0\le\|\ \xi\|\ <1$.
Now, for $n>m$,

$$\begin{aligned}
p(\mathfrak{w}_n,\mathfrak{w}_m) &\preceq p(\mathfrak{w}_n,\mathfrak{w}_{n-1})+p(\mathfrak{w}_{n-1},\mathfrak{w}_{n-2})+\ldots+p(\mathfrak{w}_{m+1},\mathfrak{w}_m)\\
&\quad-p(\mathfrak{w}_{n-1},\mathfrak{w}_{n-1})-p(\mathfrak{w}_{n-2},\mathfrak{w}_{n-2})-\ldots-p(\mathfrak{w}_{m+1},\mathfrak{w}_{m+1})p(\mathfrak{w}_n,\mathfrak{w}_m)\\
&\preceq p(\mathfrak{w}_n,\mathfrak{w}_{n-1})+p(\mathfrak{w}_{n-1},\mathfrak{w}_{n-2})+\ldots+p(\mathfrak{w}_{m+1},\mathfrak{w}_m)\\
&\preceq(\xi^{n-1}+\xi^{n-2}+\cdots+\xi^m)p(\mathfrak{w}_0,\mathfrak{w}_2),
\end{aligned}$$

and hence $\lim_{n,m\longrightarrow\infty}p(\mathfrak{w}_n,\mathfrak{w}_m)=\theta$, that is, $\{\mathfrak{w}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{M},\mathbb{A},p)$.
Using $\theta-$completeness of $(\mathcal{M},\mathbb{A},p)$, we have $\mathfrak{z}\in\mathcal{M}$ so that $\mathfrak{w}_n\longrightarrow\mathfrak{z}$ in $(\mathcal{M},\mathbb{A},p)$ and $p(\mathfrak{z},\mathfrak{z})=\theta$.
Now,

$$\begin{aligned}
p(\mathfrak{z},\mathcal{T}\mathfrak{z}) &\preceq p(\mathfrak{z},\mathfrak{w}_{n+1})+p(\mathfrak{w}_{n+1},\mathcal{T}\mathfrak{z})-p(\mathfrak{w}_{n+1},\mathfrak{w}_{n+1})\\
&\preceq p(\mathfrak{z},\mathfrak{w}_{n+1})+p(\mathcal{T}\mathfrak{w}_n,\mathcal{T}\mathfrak{z})-p(\mathfrak{w}_{n+1},w_{n+1}).
\end{aligned}$$

Since $\mathcal{T}$ is continuous, $n \longrightarrow \infty$ implies that,

$$p(\mathfrak{z}, \mathcal{T}\mathfrak{z}) \preceq (B + C + D + E)p(\mathfrak{z}, \mathcal{T}\mathfrak{z}) \prec p(\mathfrak{z}, \mathcal{T}\mathfrak{z}),$$

a contradiction, so $p(\mathfrak{z}, \mathcal{T}\mathfrak{z}) = \theta$.
Thus, $p(\mathcal{T}\mathfrak{z}, \mathcal{T}\mathfrak{z}) = p(\mathfrak{z}, \mathcal{T}\mathfrak{z}) = p(\mathfrak{z}, \mathfrak{z}) = \theta$, that is, $\mathfrak{z}$ is a fixed point of $\mathcal{T}$.
To conclude the theorem, suppose $\mathfrak{z}$ and $\mathfrak{w}$ are two different fixed points of $\mathcal{T}$, so

$$\begin{aligned} p(\mathfrak{z}, \mathfrak{w}) = p(\mathcal{T}\mathfrak{z}, \mathcal{T}w) &\preceq \mathcal{A}p(\mathfrak{z}, \mathfrak{w}) + \mathcal{B}p(\mathfrak{z}, \mathcal{T}\mathfrak{z}) + \mathcal{C}p(\mathfrak{w}, \mathcal{T}\mathfrak{w}) \\ &\quad + \mathcal{D}p(\mathfrak{w}, \mathcal{T}\mathfrak{z}) + \mathcal{E}p(\mathfrak{z}, \mathcal{T}\mathfrak{w}), \\ &\preceq (\mathcal{A} + \mathcal{D} + \mathcal{E})p(\mathfrak{z}, \mathfrak{w}) \\ &\prec (\mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D} + \mathcal{E})p(\mathfrak{z}, \mathfrak{w}) \\ &\prec p(\mathfrak{z}, \mathfrak{w}), \end{aligned}$$

a contradiction. So, $p(\mathfrak{z}, \mathfrak{w}) = \theta$. Hence, $\mathfrak{z} = \mathfrak{w}$. □

Next, an example is provided to validate Theorem 1.

Example 5 *Let $\mathcal{M} = \mathbb{C}$ and $\mathbb{A} = M_3(\mathbb{C})$ be the set of complex matrices. Let, for $a > b > c > 0$, $p : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{A}$ be defined as,*

$$p(\mathfrak{w}, \mathfrak{v}) = \begin{bmatrix} af(\mathfrak{w}, \mathfrak{v}) & 0 & 0 \\ 0 & bf(\mathfrak{w}, \mathfrak{v}) & 0 \\ 0 & 0 & cf(\mathfrak{w}, \mathfrak{v}) \end{bmatrix},$$

where, $f(\mathfrak{w}, \mathfrak{v}) = \max\{\|\mathfrak{w}\|, \|\mathfrak{v}\|\}$. *So $(\mathcal{M}, \mathbb{A}, p)$ is a complete C^*-algebra valued partial metric space and*

$$p(\mathfrak{w}, \mathfrak{w}) = \begin{bmatrix} a\|\mathfrak{w}\| & 0 & 0 \\ 0 & b\|\mathfrak{w}\| & 0 \\ 0 & 0 & c\|\mathfrak{w}\| \end{bmatrix} \neq \theta.$$

A continuous function $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$ defined as $\mathcal{T}\mathfrak{w} = \frac{\mathfrak{w}}{2}$, is a C^*-algebra valued Hardy-Roger contraction for $\theta \preceq \mathcal{A} = \mathcal{B} = \mathcal{C} \prec \frac{I}{6}$, $\theta \preceq \mathcal{D} = \mathcal{E} \prec \frac{I}{8}$. Consequently, postulates of Theorem 1 are verified and $\mathcal{T}$ has a unique fixed point at $\mathfrak{w} = 0$.

Remark 2

(i) *Conclusion of Theorem 1 continues to be true if $\mathcal{B} = \mathcal{C} = \mathcal{D} = \mathcal{E} = 0$ and we get an extension of Banach [2], to C^*-algebra-valued partial metric spaces.*

(ii) *Conclusion of Theorem 1 continues to be true if* $\mathcal{A} = \mathcal{D} = \mathcal{E} = 0$ *and* $\mathsf{B} = \mathsf{C}$ *and we get an extension of Kannan* [8] *to* C^*-*algebra-valued partial metric spaces.*

(iii) *Conclusion of Theorem 1 continues to be true if* $\mathcal{A} = \mathcal{B} = \mathcal{C} = 0$ *and* $\mathcal{D} = \mathcal{E}$, *we get an extension of Chatterjea* [4] *to* C^*-*algebra-valued partial metric spaces.*

(iv) *Conclusion of Theorem 1 continues to be true if* $\mathcal{D} = \mathcal{E} = 0$, *we get an extension of Reich* [14] *to* C^*-*algebra-valued partial metric spaces.*

Now, we establish our next result for C^*-algebra valued $CJM-$contraction.

Theorem 2 *Theorem 1 continues to be true if* (1) *is replaced by* C^*-*algebra valued* CJM*-contractive map.*

Proof. Define a Picard sequence $\{\mathfrak{w}_n\} \subseteq \mathcal{M}$, $\mathfrak{w}_{n+1} = \mathcal{T}\mathfrak{w}_n$, $n \in \mathbb{N}_0$. If $p(\mathfrak{w}_n, \mathfrak{w}_{n+1}) = \theta$ for some $n \geq 0$, then $\mathcal{T}\mathfrak{w}_n = \mathfrak{w}_{n+1} = \mathfrak{w}_n$ and $p(\mathfrak{w}_n, \mathfrak{w}_n) = \theta$ and the proof is complete.
Now, let for all $n \in \mathbb{N}_0$, $p(\mathfrak{w}_n, \mathfrak{w}_{n+1}) \succ \theta$. Using (b), we get
$p(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+2}) = p(\mathcal{T}\mathfrak{w}_n, \mathcal{T}\mathfrak{w}_{n+1}) \prec p(\mathfrak{w}_n, \mathfrak{w}_{n+1})$,
that is, the sequence $\{p(\mathfrak{w}_n, \mathfrak{w}_{n+1})\}$ is bounded below and decreasing. Thus, it is convergent and
$\lim_{n\longrightarrow\infty} p(\mathfrak{w}_n, \mathfrak{w}_{n+1}) = \varepsilon \succeq \theta$. If $\varepsilon \succ \theta$, then $\varepsilon \prec p(\mathfrak{w}_n, \mathfrak{w}_{n+1})$, for $n \geq m$ or

$$\varepsilon \prec p(\mathfrak{w}_n, \mathfrak{w}_{n+1}) \prec \varepsilon + \delta(\varepsilon),\ n \geq m,$$

which contradicts condition (a). Thus, $\lim_{n\longrightarrow\infty} p(\mathfrak{w}_n, \mathfrak{w}_{n+1}) = \theta$.
Now, we demonstrate that $\{p(\mathfrak{w}_n, \mathfrak{w}_{n+1})\}$ is a Cauchy sequence. Fix an $\varepsilon \succ \theta$, we may consider $\delta = \delta(\varepsilon) \prec \varepsilon$. Since $\{p(\mathfrak{w}_n, \mathfrak{w}_{n+1})\}$ is monotonically decreasing to θ, there exists $m \in \mathbb{N}$, $\ n \geq m$ satisfying $p(\mathfrak{w}_n, \mathfrak{w}_{n+1}) \prec \frac{\delta}{s}$.
We shall use the principle of mathematical induction to demonstrate that for $l \in \mathbb{N}$

$$p(\mathfrak{w}_m, \mathfrak{w}_{m+l}) \prec \frac{\varepsilon}{s} + \frac{\delta}{s} \prec \varepsilon + \delta. \tag{4}$$

Clearly, Equation (4) holds for $l = 1$. Suppose Equation (4) holds for some l. We shall prove it for $l + 1$. By the property (iv), we have

$$p(\mathfrak{w}_m, \mathfrak{w}_{m+l+1}) \preceq p(\mathfrak{w}_m, \mathfrak{w}_{m+1}) + p(\mathfrak{w}_{m+1}, \mathfrak{w}_{m+l+1}) - p(\mathfrak{w}_{m+1}, \mathfrak{w}_{m+1}).$$

It is enough to show that $p(\mathfrak{w}_{m+1}, \mathfrak{w}_{m+l+1}) \prec \frac{\varepsilon}{s}$. By the induction hypothesis, $p(\mathfrak{w}_m, \mathfrak{w}_{m+l}) \prec \frac{\varepsilon}{s} + \frac{\delta}{s} \prec \frac{\varepsilon}{s} + \delta$. So using (a), $p(\mathfrak{w}_{m+1}, \mathfrak{w}_{m+l+1}) \prec \frac{\varepsilon}{s}$.
Hence, Equation (4) implies that $\{w_n\}$ is a Cauchy sequence in $\mathcal{M}$.

Using $\theta-$completeness of $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$, there exists $\mathfrak{z} \in \mathcal{M}$ so that $\mathfrak{w}_n \longrightarrow \mathfrak{z}$ in $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ and $\mathfrak{p}(\mathfrak{z}, \mathfrak{z}) = \theta$.

Since $\mathcal{T}$ is continuous, $\mathfrak{w}_{n+1} = \mathcal{T}\mathfrak{w}_n \longrightarrow \mathcal{T}\mathfrak{z}$.

Hence, $\mathcal{T}\mathfrak{z} = \mathfrak{z}$, that is, $\mathfrak{z}$ is a fixed point of $\mathcal{T}$.

To conclude the proof, let $\mathfrak{z}$ and $\mathfrak{w}$ be two different fixed points of $\mathcal{T}$.

$$\mathfrak{p}(\mathfrak{z}, \mathfrak{w}) = \mathfrak{p}(\mathcal{T}\mathfrak{z}, \mathcal{T}\mathfrak{w}) \prec \mathfrak{p}(\mathfrak{z}, \mathfrak{w}),$$

a contradiction. So, $\mathfrak{p}(\mathfrak{z}, \mathfrak{w}) = \theta$.

Hence, $\mathfrak{z} = \mathfrak{w}$. □

Next, an example is provided to validate Theorem 2.

Example 6 *Let $\mathcal{M} = \mathbb{C}$ and $\mathbb{A} = M_2(\mathcal{M})$ be the set of complex matrices. Let, for $\alpha > 0$, $\mathfrak{p} : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{A}$ be,*

$$\mathfrak{p}(\mathfrak{w}, \mathfrak{v}) = \begin{bmatrix} |\mathfrak{w} - \mathfrak{v}| + \max\{|\mathfrak{w}|,\ |\mathfrak{v}|\} & 0 \\ 0 & \alpha(|\mathfrak{w} - \mathfrak{v}| + \max\{|\mathfrak{w}|,\ |\mathfrak{v}|\}) \end{bmatrix}.$$

So, $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ is a complete C^*-algebra valued partial metric space and

$$\mathfrak{p}(\mathfrak{w}, \mathfrak{w}) = \begin{bmatrix} |\mathfrak{w}| & 0 \\ 0 & \alpha|w| \end{bmatrix} \neq \theta.$$

A continuous function $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$ given by $\mathcal{T}\mathfrak{w} = \frac{\mathfrak{w}}{7}$, is a C^*-algebra valued $CJM-$contraction. Hence, all the postulates of Theorem 2 are verified and $\mathcal{T}$ has a unique fixed point at $\mathfrak{w} = 0$.

It is interesting to see that Examples 5 and 6 can not be covered by any function in a standard metric space, a partial metric space, or a C^*-algebra valued metric space in the context of Hardy and Roger [6] and Górnicki [5]. Consequently, C^*-algebra-valued partial metric space is an improved version of existing spaces wherein unital C^*-algebra $(\mathbb{A})$ is exploited as an alternative to a set of real numbers and the results in this space are genuine generalizations / improvements / extensions of the corresponding outcomes in the literature in standard metric spaces. Further, the results of C^*-algebra-valued partial metric spaces do not coincide with / derived from the results in other related spaces.

3 Application

Now, we utilize Theorem 1, to solve a boundary value problem.

Theorem 3 *Consider a boundary value problem*

$$\frac{d^2\mathfrak{w}}{dt^2} = -\phi(t, \mathfrak{w}(t)), \ t \in I = [-1, 1] \text{ and } \phi \in C(I, \mathbb{R}) \tag{5}$$

with two-point boundary condition $\mathfrak{w}(-1) = 0, \mathfrak{w}(1) = 0$.
Assume the following:

(i) $\phi : I \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a Lipschitz continuous relative to* $\mathfrak{w}$ *for Lipschitz constant value* $0 \leq \|\xi\| \leq \frac{1}{3}, \forall\ t \in I, \mathfrak{w}_1, \mathfrak{w}_2 \in \mathbb{R}$ such that $\|\phi(t, \mathfrak{w}_1) - \phi(t, \mathfrak{w}_2)\| \leq \xi(t)\|\mathfrak{w}_1 - \mathfrak{w}_2\|$ *and function* ξ *is continuous on* I.

(ii) $|\phi(t, \mathfrak{w})| \leq \mu(t)\ \ |\mathfrak{w}|$, *where,* $0 \leq \|\mu\| \leq \frac{1}{3}$ *and function* μ *is continuous on* I.

Then, the differential equation has exactly one solution $\mathfrak{w}^* \in C(I, \mathbb{R})$.

Proof. The problem in equation (5) may be rewritten as

$$\mathfrak{w}(t) = \int_{-1}^{1} \mathcal{G}(t, u)\phi(u, \mathfrak{w}(u))du, \ \text{for } t \in I, \tag{6}$$

and the Green function $\mathcal{G}(t, u) = \begin{cases} (1-t)(1+u), -1 \leq u \leq t \leq 1 \\ (1-u)(1+t), -1 \leq t \leq u \leq 1 \end{cases}$.
Now, if $\mathfrak{w} \in C^2(I, \mathbb{R})$, then $\mathfrak{w}$ is the solution of (5) if and only if it is the solution of (6).
$\mathcal{M} = C(I)$, the set of a continuous function on I forms a C^*-algebra with pointwise operation with $\|\mathfrak{w}\|_\infty = \max_{t \in I} |\mathfrak{w}|$, $\mathfrak{w} \in \mathcal{M}$.
Define $p : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ by $p(\mathfrak{w}, \mathfrak{v}) = [\|\mathfrak{w} - \mathfrak{v}\| + \|\mathfrak{w}\| + \|\mathfrak{v}\|]f$ is a C^*-algebra valued partial metric space, where, f is the self-adjoint element of $\mathcal{M}$.
Define a self map $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$ by

$$\mathcal{T}\mathfrak{w}(t) = \int_{-1}^{1} \mathcal{G}(t, u)\phi(u, \mathfrak{w}(u))du, \tag{7}$$

for all $w \in \mathcal{M}$ and $t \in I$. Now, our problem (5) may be expressed as determining a fixed point of $\mathcal{T}$. So

$$\begin{aligned}|\mathcal{T}\mathfrak{w}(t)-\mathcal{T}\mathfrak{v}(t)| &= \left|\int_{-1}^{1}\mathcal{G}(t,u)(\phi(u,\mathfrak{w}(u))-\phi(u,\mathfrak{v}(u)))du\right|,\\ &\preceq \int_{-1}^{1}\mathcal{G}(t,u)\left|\phi(u,\mathfrak{w}(u))-\phi(u,\mathfrak{v}(u))\right|du,\\ &\preceq \int_{-1}^{1}\mathcal{G}(t,u)\xi\left|\mathfrak{w}(u)-\mathfrak{v}(u)\right|du\\ &\preceq \xi\left\|\mathfrak{w}(u)-\mathfrak{v}(u)\right\|_{\infty}\sup_{t\in I}\int_{-1}^{1}\mathcal{G}(t,u)du.\end{aligned}$$

Therefore,

$$\|\mathcal{T}\mathfrak{w}(t)-\mathcal{T}\mathfrak{v}(t)\| \leq \|\xi\|\left\|\mathfrak{w}(u)-\mathfrak{v}(u)\right\|_{\infty}. \tag{8}$$

Since, $\int_{-1}^{1}\mathcal{G}(t,u)du = 1-t^2$ and $\sup_{t\in I}\int_{-1}^{1}\mathcal{G}(t,u)du = 1$.
Now,

$$\begin{aligned}|\mathcal{T}\mathfrak{w}(t)| &= \left|\int_{-1}^{1}\mathcal{G}(t,u)\phi(u,\mathfrak{w}(u))du\right|,\\ &\preceq \int_{-1}^{1}\mathcal{G}(t,u)\left|\phi(u,\mathfrak{w}(u))\right|du,\\ &\preceq \int_{-1}^{1}\mu\left|\mathfrak{w}(u)\right|\mathcal{G}(t,u)du,\\ &\preceq \mu\left\|\mathfrak{w}\right\|_{\infty}\int_{-1}^{1}\mathcal{G}(t,u)du.\end{aligned}$$

Therefore,

$$\|\mathcal{T}\mathfrak{w}(t)\|_{\infty} \leq \|\mu\|\left\|\mathfrak{w}\right\|_{\infty}, \tag{9}$$

and also

$$\|\mathcal{T}\mathfrak{v}(t)\|_{\infty} \leq \|\mu\|\left\|\mathfrak{v}\right\|_{\infty}. \tag{10}$$

Now,

$$\begin{aligned}p(\mathcal{T}\mathfrak{w},\mathcal{T}\mathfrak{v}) &= [\|\mathcal{T}\mathfrak{w}-\mathcal{T}\mathfrak{v}\|_{\infty}+\|\mathcal{T}\mathfrak{w}\|_{\infty}+\|\mathcal{T}\mathfrak{v}\|_{\infty}]f\\ &\preceq [\xi\left\|\mathfrak{w}-\mathfrak{v}\right\|_{\infty}+\mu\left\|\mathfrak{w}\right\|_{\infty}+\mu\left\|\mathfrak{v}\right\|_{\infty}]f\\ &\preceq (\xi+2\mu)(\|\mathfrak{w}-\mathfrak{v}\|_{\infty}+\|\mathfrak{w}\|_{\infty}+\|\mathfrak{v}\|_{\infty}]f)\\ &= (\xi+2\mu)p(\mathfrak{w},\mathfrak{v}) \preceq \mathcal{A}p(\mathfrak{w},\mathfrak{v})+\mathcal{B}p(\mathfrak{w},\mathcal{T}\mathfrak{w})+\mathcal{C}p(\mathfrak{w},\mathcal{T}\mathfrak{v})\\ &\quad +\mathcal{D}p(\mathfrak{v},\mathcal{T}\mathfrak{w})+\mathcal{E}p(\mathfrak{w},\mathcal{T}\mathfrak{v}).\end{aligned}$$

Taking $\mathcal{A} = \xi$, $\mathcal{B} = \mathcal{C} = \mathcal{D} = \mathcal{E} = \frac{\mu}{2}$, we may observe that postulates of Theorem 1 are verified, and so $\mathcal{T}$ has only one fixed point $\mathfrak{w}^* \in \mathcal{M}$, that is, boundary value problem (5) has only one solution $\mathfrak{w}^* \in \mathcal{M}$. □

Now, we make use of Theorem 2, to solve a matrix equation to demonstrate the applicability of C^*-algebra valued $CJM-$contraction map. In the following, the symbol $\|.\|$ is the spectral norm of a matrix $\mathcal{P} = [p_{ij}]_{n\times n}$, that is, $\|\mathcal{P}\| = \sqrt{\lambda^+(\mathcal{P}^*\mathcal{P})}$, $\lambda^+(\mathcal{P}^*\mathcal{P})$ is the largest eigenvalue of $\mathcal{P}^*\mathcal{P}$, where $\mathcal{P}^*$ is the conjugate transpose of $\mathcal{P}$. Further, $\|.\|_{tr}$ denotes the trace norm of $\mathcal{P}$ and $\|\mathcal{P}\|_{tr} = \sqrt{\Sigma_{i=1}^n \Sigma_{j=1}^n |p_{ij}|^2} = \sqrt{tr(\mathcal{P}^*\mathcal{P})} = \sqrt{\Sigma_{i=1}^n \sigma_i^2(\mathcal{P})}$, $\sigma_i(\mathcal{P})$, $i = 1, 2, \ldots, n$, denotes largest singular values of $\mathcal{P} \in M_n(\mathbb{C})$. The set of all Hermitian matrices of order n, $H_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$, induced by this trace norm, is a Banach space.

Theorem 4 *Let a non-linear matrix equation be*

$$\mathcal{W} = \Sigma_{i=1}^n \mathcal{P}_i^* f(\mathcal{W})\mathcal{P}_i, \tag{11}$$

where, the C^*-algebra of complex matrices of order n, $\mathcal{M} = M_n(\mathbb{C})$, $\mathcal{P}_i \in M_n(\mathbb{C})$ is an arbitrary matrix of order n. Let $f : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ be a continuous self map satisfying $f(\theta) = \theta$ and

(i) $\max\{|tr(f\mathcal{W})|, |tr(f\mathcal{V})|\}I \preceq \frac{\eta}{2} \max\{|tr(\mathcal{W})|, |tr(\mathcal{V})|\}I_n$,

(ii) $|tr(\mathcal{T}\mathcal{W} - \mathcal{T}\mathcal{V})|I_n \preceq \frac{\eta}{2}|tr(\mathcal{W} - \mathcal{V})|I_n$,

(iii) $tr(\mathcal{W}\mathcal{V}) \le \|\mathcal{W}\| tr(\mathcal{V})$, $\mathcal{W} \in M_n(\mathbb{C})$,

(iv) $\Sigma_{i=1}^n \mathcal{P}_i^* \mathcal{P} \preceq \xi I_n$, *where identity matrix of order n, $I_n \in M_n(\mathbb{C})$ and* $\eta \neq 0$.

Then the matrix equation (11) has exactly one solution $\mathcal{W}^* \in \mathcal{M}$. Further, the iteration $\mathcal{W}_n = \Sigma_{i=1}^n \mathcal{P}_i^* f(\mathcal{W})\mathcal{P}_i$, $\mathcal{W}_0 \in M_n(\mathbb{C})$ such that $\mathcal{W}_0 \preceq \Sigma_{i=1}^n \mathcal{P}_i^* f(\mathcal{W})\mathcal{P}_i$, converges to $\mathcal{W}^* \in \mathcal{M}$ satisfying the nonlinear matrix equation (11).

Proof. Let a map $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$ be defined as

$$\mathcal{T}(\mathcal{W}) = \Sigma_{i=1}^n \mathcal{P}_i^* f(\mathcal{W})\mathcal{P}_i \tag{12}$$

and a C^*-algebra valued partial metric $p : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ be

$$p(\mathcal{W}, \mathcal{V}) = \left[\max\{|tr\mathcal{W}|, |tr\mathcal{V}|\} + |tr(\mathcal{W} - \mathcal{V})|\right] I_n.$$

Noticeably, a fixed point of $\mathcal{T}$ is a solution of a matrix equation (11).

$$
\begin{aligned}
p(\mathcal{T}\mathcal{W},\mathcal{T}\mathcal{V}) &= [\max\{tr|\mathcal{T}\mathcal{W}|, tr|\mathcal{T}\mathcal{V}|\} + |tr(\mathcal{T}\mathcal{W}-\mathcal{T}\mathcal{V})|]I_n \\
&= \Big[\max\{|tr(\Sigma_{i=1}^{n}\mathcal{P}_i^{*}f(\mathcal{W})\mathcal{P}_i)|,\ |tr(\Sigma_{i=1}^{n}\mathcal{P}_i^{*}f(\mathcal{V})\mathcal{P}_i)|\} \\
&\quad + |tr(\Sigma_{i=1}^{n}\mathcal{P}_i^{*}(f(\mathcal{W})-f(\mathcal{V})\mathcal{P}_i))|\Big]I_n \\
&= \Big[\max\{|tr(\Sigma_{i=1}^{n}\mathcal{P}_i^{*}\mathcal{P}_i f(\mathcal{W}))|,\ |tr(\Sigma_{i=1}^{n}\mathcal{P}_i^{*}\mathcal{P}_i f(\mathcal{V}))|\} \\
&\quad + |tr(\Sigma_{i=1}^{n}\mathcal{P}_i^{*}\mathcal{P}_i f(\mathcal{W})-f(\mathcal{V}))|\Big]I_n \\
&\preceq \|\Sigma_{i=1}^{n}\mathcal{P}_i^{*}\mathcal{P}_i\|\Big[\max\{|tr(f\mathcal{W})|,\ |tr(f\mathcal{V})|\} + |f\mathcal{W}-f\mathcal{V}|\Big]I_n \\
&\preceq \|\eta I\|\frac{1}{2\eta}[\max\{|tr(\mathcal{W})|,\ |tr(\mathcal{V})|\} + |tr(f\mathcal{W}-f\mathcal{V})|]I_n \\
&= \frac{1}{2}\Big[\max\{|tr(\mathcal{W})|,\ |tr(\mathcal{V})|\} + |tr(f\mathcal{W}-f\mathcal{V})|\Big]I_n \\
&\prec p(\mathcal{W},\mathcal{V}).
\end{aligned}
$$

Taking $\varepsilon = \frac{1}{2}\Big[\max\{|tr(\mathcal{W})|,\ |tr(\mathcal{V})|\} + |tr(f\mathcal{W}-f\mathcal{V})|\Big]I_n$ and $\delta = \frac{3}{2}\varepsilon$, $p(\mathcal{W},\mathcal{V}) \prec \varepsilon+\delta \implies p(\mathcal{T}\mathcal{W},\mathcal{T}\mathcal{V}) \prec \varepsilon$ and $\mathcal{W} \neq \mathcal{V} \implies p(\mathcal{T}\mathcal{W},\mathcal{T}\mathcal{V}) \prec p(\mathcal{W},\mathcal{V})$.

We may observe that postulates of Theorem 2 are verified, and $\mathcal{T}$ has only one fixed point $\mathcal{W}^* \in \mathcal{M}$, that is, matrix equation (11) has only one solution $\mathcal{W}^* \in \mathcal{M}$. □

4 Conclusion

Acknowledging the C^*-algebra valued partial metric space, we have familiarized Hardy-Roger contraction [6] and CJM−contraction [5] in it to elicit the fixed point theorems in the most generalized environment. From our results, we have deduced results for a C^*-algebra valued variants of Kannan contraction[8], Chatterjee contraction [4], Reich contraction [14] and Banach contraction [2]. Further, we have solved a boundary value problem using C^*-algebra valued Hardy-Roger contraction and a matrix equation using C^*-algebra valued CJM− contraction. The motivation behind using this space is its application in quantum field theory and statistical mechanics. It is worth to mention that there may be some circumstances when it is possible to apply C^*-algebra valued partial metric results, however it is not possible to apply standard metric results . These novel ideas promote further examinations and applications.

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