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# On Chern classes of the tensor product of vector bundles

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**Abstract.** We present two formulas for Chern classes (polynomial) of the tensor product of two vector bundles. In the first formula the Chern polynomial of the product is expressed as determinant of a polynomial in a matrix variable involving the Chern classes of the first bundle with Chern classes of the second bundle as coefficients. In the second formula the total Chern class of the tensor product is expressed as resultant of two explicit polynomials. Finally, formulas for the total Chern class of the second symmetric and the second alternating products are deduced.

## 1 Introduction

One associates a series of cohomological (characteristic) classes $c_i(\mathcal{E}) \in H^{2i}(M)$ called the $i^{\text{th}}$ *Chern class* of $\mathcal{E}$, for any $i = 1, \dots, r$, with a complex vector bundle $\mathcal{E}$ of rank $r$ over a manifold $M$ (cf. [9, Ch. IV] or [3, Ch. I, §4]). One can arrange these classes into a polynomial $c(\mathcal{E}; t) = 1 + c_1(\mathcal{E})t + \cdots + c_r(\mathcal{E})t^r$, called the *Chern polynomial*. Its value $c(\mathcal{E}) = c(\mathcal{E}; 1) = 1 + c_1(\mathcal{E}) + \cdots + c_r(\mathcal{E})$ at $t = 1$ is the *total Chern class* of $\mathcal{E}$.

We recall some basic properties of the Chern classes. If $\mathcal{E}$ and $\mathcal{F}$ are two complex vector bundles over the same manifold then $c(\mathcal{E} \oplus \mathcal{F}; t) = c(\mathcal{E}; t) \cdot c(\mathcal{F}; t)$ by

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the *Whitney product formula* (cf. [9, (20.10.3)]). Computing with Chern classes one can pretend using the *Splitting Principle* (cf. [9, Ch. IV, §21]) that the bundle $\mathcal{E}$ of rank $r$ splits into direct sum of $r$ complex line bundles and the first Chern classes $\alpha_1, \dots, \alpha_r$ of these hypothetical line bundles are the so-called *Chern roots* of $\mathcal{E}$. Hence, by the Whitney product formula we have $c(\mathcal{E}; t) = \prod_{i=1}^{r}(1+\alpha_i t)$, thus $c_k(\mathcal{E}) = e_k(\alpha) = e_k(\alpha_1, \dots, \alpha_r) = \sum_{1 \le i_1 < \cdots < i_k \le r} \alpha_{i_1} \cdots \alpha_{i_k}$ for any $k = 1, \dots, r$, i.e. the Chern classes are *elementary symmetric polynomials* of the Chern roots. The dual bundle $\mathcal{E}^*$ has opposite Chern roots to $\mathcal{E}$, hence its Chern polynomial equals $c(\mathcal{E}^*, t) = \prod_{i=1}^{r}(1 - \alpha_i t) = c(\mathcal{E}, -t)$.

The Chern polynomial does not behave so well for the tensor product like for the direct sum. Nevertheless, for complex line bundles $\mathcal{L}$ and $\mathcal{L}'$ we have $c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}')$ (cf. [9, (20.1)]). Hence, if $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_q$ are Chern roots of $\mathcal{E}$ and $\mathcal{F}$, respectively, then $\alpha_i + \beta_j$, $i = 1, \dots, r$, $j = 1, \dots, q$ are the Chern roots of the tensor product $\mathcal{E} \otimes \mathcal{F}$ and the Chern polynomial of the tensor product equals

$$c(\mathcal{E} \otimes \mathcal{F}; t) = \prod_{i=1}^{r} \prod_{j=1}^{q} (1 + \alpha_i t + \beta_j t). \tag{1}$$

Our goal is to express (1) in terms of Chern classes of $\mathcal{E}$ and $\mathcal{F}$, or equivalently in terms of elementary symmetric polynomials of the Chern roots $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_q$, respectively.

There are several approaches to compute the Chern classes of the tensor product. We mention the four approaches compared in [4]. The first method computes the Chern classes of the tensor product by eliminating the Chern roots $\alpha_1, \dots, \alpha_r$, $\beta_1, \dots, \beta_q$ from $c(\mathcal{E} \otimes \mathcal{F}) = \prod_{i=1}^{r} \prod_{j=1}^{q} (1 + \alpha_i + \beta_j)$ using relations $c_i(\mathcal{E}) = e_i(\alpha_1, \dots, \alpha_r)$ and $c_j(\mathcal{F}) = e_j(\beta_1, \dots, \beta_q)$ for $i = 1, \dots, r$ and $j = 1, \dots, q$. The second approach uses the multiplicativity of the Chern character (cf. [3, Ch. III, §10.1]) and Newton's identities (cf. [7, (2.11′)]). The third uses Lascoux's formula [6] which expresses the Chern classes of the tensor product as linear combination of products of Schur polynomials of Chern classes of $\mathcal{E}$ and $\mathcal{F}$. The last approach is Manivel's formula [8], which has the same form as Lascoux's formula, but computes the coefficients differently. These methods have been implemented in the library CHERN.LIB [5] for the computer algebra system SINGULAR [2].

## 2 Chern polynomial of the tensor product: first approach

**Lemma 1** *Let* $u_1,\dots,u_r$, $v_1,\dots,v_q$ *be formal variables. We consider elementary symmetric polynomials* $e_k(u) = \sum_{1\le i_1<\dots<i_k\le r} u_{i_1}\cdots u_{i_k}$ *for any* $k = 1,\dots,r$ *and we set* $e_0(u) = 1$. *We associate with the list* $(e_1(v),\dots,e_q(v))$ *the following matrix*

$$\Lambda(e(v)) = \begin{pmatrix} e_1(v) & -1 & & \\ \vdots & & \ddots & \\ e_{q-1}(v) & & & -1 \\ e_q(v) & & & \end{pmatrix} \tag{2}$$

(*it has non-zero entries only in the first column and above the diagonal*). *Then we have*

$$\prod_{i=1}^{r}\prod_{j=1}^{q}(1+u_i+v_j) = \det\left(\sum_{k=0}^{r} e_k(u)[I+\Lambda(e(v))]^{r-k}\right),$$

*where* $I = I_q$ *is the* $q$*-by-*$q$ *identity matrix.*

**Proof.** First, we diagonalize the matrix $\Lambda(e(v))$. Therefore, we consider the $q$-by-$q$ matrix $E = E(v_1,\dots,v_q) = \left[e_{i-1}(v_1,\dots,\widehat{v}_j,\dots,v_q)\right]_{i,j=1}^{q}$, where $\widehat{v}_j$ means that the term $v_j$ is omitted. We show that $E$ is non-singular by computing its determinant as follows. We subtract the first column from the other columns, then we raise a $(v_1 - v_j)$-factor from columns $j = 2,\dots,q$, respectively. Expanding the resulting determinant by the first row we get the recurrent relation $\det E(v_1,\dots,v_q) = \prod_{j=2}^{q}(v_1 - v_j)\det E(v_2,\dots,v_q)$, hence $\det E = \prod_{1\le i<j\le q}(v_i - v_j) \neq 0$. Moreover, $\Lambda(e(v))E = E\,\mathrm{diag}(v_1,\dots,v_q)$ by relations $e_i(v_1,\dots,v_q) = e_i(v_1,\dots,\widehat{v}_j,\dots,v_q) + v_j e_{i-1}(v_1,\dots,\widehat{v}_j,\dots,v_q)$, hence $\Lambda(e(v)) = E\mathrm{diag}(v_1,\dots,v_q)E^{-1}$. Furthermore,

$$I+\Lambda(e(v)) = I + E\mathrm{diag}(v_1,\dots,v_q)E^{-1} = E\mathrm{diag}(1+v_1,\dots,1+v_q)E^{-1}$$

is also diagonalizable with eigenvalues $1+v_1,\dots,1+v_q$. Finally,

$$\prod_{j=1}^{q}\prod_{i=1}^{r}(1+u_i+v_j) = \prod_{j=1}^{q}\sum_{k=0}^{r} e_k(u)(1+v_j)^{r-k} =$$

$$= \det\left(E\mathrm{diag}\left(\sum_{k=0}^{r} e_k(u)(1+v_1)^{r-k},\dots,\sum_{k=0}^{r} e_k(u)(1+v_q)^{r-k}\right)E^{-1}\right) =$$

$$= \det\left(\sum_{k=0}^{r} e_k(u) E \operatorname{diag}\left(1+v_1,\ldots,1+v_q\right)^{r-k} E^{-1}\right)$$
$$= \det\left(\sum_{k=0}^{r} e_k(u)[I+\Lambda(e(v))]^{r-k}\right).$$

□

**Theorem 1** *Let $\mathcal{E}$ and $\mathcal{F}$ be two complex vector bundles of rank $r$ and $q$, respectively over the same manifold. Then the Chern polynomial of the tensor product $\mathcal{E}\otimes\mathcal{F}$ equals*

$$c(\mathcal{E}\otimes\mathcal{F};t) = \det\left(\sum_{k=0}^{r} c_k(\mathcal{E}) t^k [I+\Lambda(c(\mathcal{F});t)]^{r-k}\right),$$

*where $c_0(\mathcal{E}) = 1$ and $\Lambda(c(\mathcal{F});t)$ is the matrix (2) with $c_1(\mathcal{F})t,\ldots,c_q(\mathcal{F})t^q$ in the first column.*

**Proof.** Let $\alpha_1,\ldots,\alpha_r$ and $\beta_1,\ldots,\beta_q$ be the Chern roots of $\mathcal{E}$ and $\mathcal{F}$, respectively. Then it is enough to show that

$$\prod_{i=1}^{r}\prod_{j=1}^{q}(1+\alpha_i t+\beta_j t) = \det\left(\sum_{k=0}^{r} e_k(\alpha) t^k [I+\Lambda(e(\beta t))]^{r-k}\right),$$

where $\Lambda(e(\beta t))$ equals the matrix $\Lambda(c(\mathcal{F});t)$ only replacing Chern classes $c_j(\mathcal{F})$ by elementary symmetric polynomials $e_j(\beta) = e_j(\beta_1,\ldots,\beta_q)$ of Chern roots for all $j = 1,\ldots,q$. Finally, substituting $u_1 = \alpha_1 t,\ldots,u_r = \alpha_r t$, $v_1 = \beta_1 t,\ldots,v_q = \beta_q t$ in Lemma 1 we get the desired relation. □

# 3 Resultant and Chern classes of the tensor product: second approach

The second approach uses the resultant of two polynomials. This will lead us to a determinantal formula for Chern classes of the second alternating and the second symmetric products of a vector bundle.

Let $A(t) = a_r + a_{r-1}t + \cdots + a_0 t^r = a_0\prod_{i=1}^{r}(t-\alpha_i)$ and $B(t) = b_q + b_{q-1}t + \cdots + b_0 t^q = b_0\prod_{j=1}^{q}(t-\beta_j)$ be two polynomials in variable $t$ with roots $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_q$, respectively. The *resultant* of polynomials $A$ and $B$ with respect to $t$ is given by

$$\operatorname{res}(A(t), B(t), t) = a_0^q b_0^r \prod_{i=1}^{r} \prod_{j=1}^{q} (\alpha_i - \beta_j) = \begin{vmatrix} a_0 & & & b_0 & & \\ a_1 & \ddots & & b_1 & \ddots & \\ \vdots & & a_0 & \vdots & & b_0 \\ a_r & & \vdots & b_q & & \vdots \\ & \ddots & a_{r-1} & & \ddots & b_{q-1} \\ & & a_r & & & b_q \end{vmatrix},$$

where the first $q$ columns contain the coefficients of $A$, while the last $r$ columns contain the coefficients of $B$ and empty spaces contain zeroes (cf. [1, Ch. III]).

Instead of the Chern polynomial $c(\mathcal{F}; t) = 1 + c_1(\mathcal{F})t + \dots + c_q(\mathcal{F})t^q$ of the rank $q$ vector bundle $\mathcal{F}$ we consider the polynomial with coefficients in reverse order

$$C(\mathcal{F}; t) = \sum_{k=0}^{q} c_k(\mathcal{F})t^{q-k} = c_q(\mathcal{F}) + c_{q-1}(\mathcal{F})t + \dots + c_1(\mathcal{F})t^{q-1} + t^q. \tag{3}$$

They are related by $C(\mathcal{F}; t) = t^q c(\mathcal{F}; t^{-1})$ and moreover, we can recover the total Chern class by substituting $t = 1$, i.e. $c(\mathcal{F}) = C(\mathcal{F}; 1)$. Furthermore, if $\beta_1, \dots, \beta_q$ are Chern roots of $\mathcal{F}$ then $C(\mathcal{F}; t) = \prod_{j=1}^{q}(t + \beta_j)$, i.e. the opposite of Chern roots of $\mathcal{F}$ are roots of the polynomial $C(\mathcal{F}; t)$. We note that for the dual bundle $\mathcal{F}^*$ we have $C(\mathcal{F}^*; t) = (-1)^q C(\mathcal{F}; -t)$.

**Lemma 2** *If $\alpha_1, \dots, \alpha_r$ are the Chern roots of the complex vector bundle $\mathcal{E}$ of rank $r$ then $\prod_{i=1}^{r}(t-s-\alpha_i) = (-1)^r C(\mathcal{E}; s-t) = \sum_{k=0}^{r}(-1)^k d_k(\mathcal{E}; s)t^{r-k}$ with coefficients $d_k(\mathcal{E}; s) = \binom{r}{k}s^k + \binom{r-1}{k-1}c_1(\mathcal{E})s^{k-1} + \dots + c_k(\mathcal{E}) = \sum_{i=0}^{k}\binom{r-i}{k-i}c_i(\mathcal{E})s^{k-i}$.*

**Proof.** Indeed, $\prod_{i=1}^{r}(t - s - \alpha_i) = (-1)^r \prod_{i=1}^{r}(s - t + \alpha_i) = (-1)^r C(\mathcal{E}; s - t)$ and moreover, $\prod_{i=1}^{r}(t - s - \alpha_i) = \sum_{k=0}^{r}(-1)^k e_k(s + \alpha_1, \dots, s + \alpha_r)t^{r-k}$, where

$$e_k(s + \alpha_1, \dots, s + \alpha_r) = \sum_{1 \le i_1 < \dots < i_k \le r} (s + \alpha_{i_1}) \cdots (s + \alpha_{i_k}) =$$

$$= \sum_{1 \le i_1 < \dots < i_k \le r} \left[ s^k + e_1(\alpha_{i_1}, \dots, \alpha_{i_k})s^{k-1} + \dots + e_k(\alpha_{i_1}, \dots, \alpha_{i_k}) \right] =$$

$$= \binom{r}{k}s^k + \binom{r-1}{k-1}e_1(\alpha_1, \dots, \alpha_r)s^{k-1} + \dots + \binom{r-k}{0}e_k(\alpha_1, \dots, \alpha_r) =$$

$$= \sum_{i=0}^{r} \binom{r-i}{k-i} e_i(\alpha_1, \ldots, \alpha_r) s^{k-i} = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(\mathcal{E}) s^{k-i} = d_k(\mathcal{E}; s).$$

□

In the next theorem we express $C(\mathcal{E} \otimes \mathcal{F}; s)$ as resultant of polynomials $(-1)^r C(\mathcal{E}; s-t)$ and $C(\mathcal{F}; t)$. We can also get a formula for the total Chern class of the tensor product by substituting $s = 1$.

**Theorem 2** *If $\mathcal{E}$ and $\mathcal{F}$ are two complex vector bundles of rank $r$ and $q$, respectively over the same manifold then*

$$C(\mathcal{E} \otimes \mathcal{F}; s) = \operatorname{res}((-1)^r C(\mathcal{E}; s-t), C(\mathcal{F}; t), t), \tag{4}$$

*where the polynomial $C$ is defined by (3). Substituting $s = 1$ yields the total Chern class of the tensor product $c(\mathcal{E} \otimes \mathcal{F}) = \operatorname{res}((-1)^r C(\mathcal{E}; 1-t), C(\mathcal{F}; t), t)$. Moreover, the top Chern classes of the tensor product equals*

$$c_{rq}(\mathcal{E} \otimes \mathcal{F}) = (-1)^{rq} \operatorname{res}(c(\mathcal{E}; -t), c(\mathcal{F}; t), t),$$

*while the top Chern classes of the $\mathsf{Hom}(\mathcal{E}, \mathcal{F})$ bundle equals*

$$c_{rq}(\mathsf{Hom}(\mathcal{E}, \mathcal{F})) = (-1)^{rq} \operatorname{res}(c(\mathcal{E}; t), c(\mathcal{F}; t), t).$$

**Proof.** Denote $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_q$ the Chern roots of $\mathcal{E}$ and $\mathcal{F}$, respectively. Then

$$C(\mathcal{E} \otimes \mathcal{F}; s) = \prod_{i=1}^{r} \prod_{j=1}^{q} (s + \alpha_i + \beta_j) = \prod_{i=1}^{r} \prod_{j=1}^{q} (s + \alpha_i - (-\beta_j)),$$

hence $C(\mathcal{E} \otimes \mathcal{F}; s)$ is the resultant of polynomials $\prod_{j=1}^{q}(t + \beta_j) = C(\mathcal{F}; t)$ and $\prod_{i=1}^{r}(t - s - \alpha_i) = (-1)^r C(\mathcal{E}; s-t)$ with respect to the variable $t$, i.e. $C(\mathcal{E} \otimes \mathcal{F}; s) = \operatorname{res}((-1)^r C(\mathcal{E}; s-t), C(\mathcal{F}; t), t)$.

To obtain the top Chern class of the tensor product we substitute $s = 0$ into (4), thus $c_{rq}(\mathcal{E} \otimes \mathcal{F}; t) = \operatorname{res}((-1)^r C(\mathcal{E}; -t), C(\mathcal{F}; t), t)$. The coefficients of polynomials $C(\mathcal{F}; t)$ and $c(\mathcal{F}; t)$ are in reverse order, and similarly the coefficients of polynomials $(-1)^r C(\mathcal{E}; -t)$ and $c(\mathcal{E}; -t)$ are also in reverse order. Hence we get $\operatorname{res}((-1)^r C(\mathcal{E}; -t), C(\mathcal{F}; t), t) = (-1)^{rq} \operatorname{res}(c(\mathcal{E}; -t), c(\mathcal{F}; t), t)$ by reversing the order of rows, the order of the first $q$ columns and last $r$ columns in the defining determinant (3) of the resultant.

Finally, the top Chern class of the $\mathsf{Hom}(\mathcal{E}, \mathcal{F})$-bundle $c_{rq}(\mathsf{Hom}(\mathcal{E}, \mathcal{F}); t) = c_{rq}(\mathcal{E}^* \otimes \mathcal{F}; t) = (-1)^{rq} \operatorname{res}(c(\mathcal{E}^*; -t), c(\mathcal{F}; t), t) = (-1)^{rq} \operatorname{res}(c(\mathcal{E}; t), c(\mathcal{F}; t), t)$. □

# 4 Chern classes the second alternating product $\wedge^2\mathcal{E}$ and the second symmetric product $S^2\mathcal{E}$

We give a different version of Theorem 2, which leads to determinantal formulas for total Chern classes of the second alternating and symmetric products.

**Theorem 3** *If $\mathcal{E}$ and $\mathcal{F}$ are two complex vector bundles of rank $r$ and $q$, respectively over the same manifold, then*

$$C(\mathcal{E}\otimes\mathcal{F};s) = \mathrm{res}((-1)^r C\left(\mathcal{E};\frac{s}{2}-t\right), C\left(\mathcal{F};\frac{s}{2}+t\right), t),$$

*By substituting $s=1$ we get* $c(\mathcal{E}\otimes\mathcal{F}) = \mathrm{res}((-1)^r C\left(\mathcal{E};\frac{1}{2}-t\right), C\left(\mathcal{F};\frac{1}{2}+t\right), t)$.

**Proof.** If $\alpha_1,\dots,\alpha_r$ and $\beta_1,\dots,\beta_q$ are the Chern roots of $\mathcal{E}$ and $\mathcal{F}$, respectively, then

$$C(\mathcal{E}\otimes\mathcal{F};s) = \prod_{i=1}^{r}\prod_{j=1}^{q}(s+\alpha_i+\beta_j) = \prod_{i=1}^{r}\prod_{j=1}^{q}\left(\frac{s}{2}+\alpha_i-\left(-\frac{s}{2}-\beta_j\right)\right),$$

hence $C(\mathcal{E}\otimes\mathcal{F};s)$ is the resultant of $\prod_{i=1}^{q}\left(t+\frac{s}{2}+\beta_j\right) = C\left(\mathcal{F};\frac{s}{2}+t\right)$ and $\prod_{i=1}^{r}\left(t-\frac{s}{2}-\alpha_i\right) = (-1)^r C(\mathcal{E};\frac{s}{2}-t)$. □

If $\alpha_1,\dots,\alpha_r$ are the Chern roots of the vector bundle $\mathcal{E}$ then the total Chern classes of the second alternating and the second symmetric bundles

$$c(\wedge^2\mathcal{E}) = \prod_{1\le i<j\le r}(1+\alpha_i+\alpha_j),$$

$$c(S^2\mathcal{E}) = \prod_{1\le i\le j\le r}(1+\alpha_i+\alpha_j) = c(\mathcal{E};2)c(\wedge^2\mathcal{E}),$$

hence their corresponding $C$ polynomials

$$C(\wedge^2\mathcal{E};s) = \prod_{1\le i<j\le r}(s+\alpha_i+\alpha_j),$$

$$C(S^2\mathcal{E};s) = \prod_{1\le i\le j\le r}(s+\alpha_i+\alpha_j) = 2^r C\left(\mathcal{E};\frac{s}{2}\right) C(\wedge^2\mathcal{E};s).$$

**Theorem 4** *Let* $\bar{d}_k = d_k(\mathcal{E}; \frac{s}{2}) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(\mathcal{E}) (\frac{s}{2})^{k-i}$ *for* $k = 0, 1, \dots, r$ *and* $\bar{d}_k = 0$ *otherwise. With these notations we have*

$$C(\wedge^2 \mathcal{E}; s) = \det \left( \left[ d_{2i-j} \left( \mathcal{E}; \frac{s}{2} \right) \right]_{i,j=1}^{r-1} \right) = \begin{vmatrix} \bar{d}_1 & 1 & & & & \\ \bar{d}_3 & \bar{d}_2 & \bar{d}_1 & 1 & & \\ \bar{d}_5 & \bar{d}_4 & \bar{d}_3 & \bar{d}_2 & \bar{d}_1 & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & \bar{d}_r & \bar{d}_{r-1} & \bar{d}_{r-2} & \bar{d}_{r-3} \\ & & & & \bar{d}_r & \bar{d}_{r-1} \end{vmatrix}. \tag{5}$$

*By substituting* $s = 1$ *we get* $c(\wedge^2 \mathcal{E}) = \det \left( \left[ d_{2i-j}(\mathcal{E}; \frac{1}{2}) \right]_{i,j=1}^{r-1} \right)$.

**Proof.** By Theorem 3 we have $C(\mathcal{E} \otimes \mathcal{E}; s) = \mathrm{res}((-1)^r C(\mathcal{E}; \frac{s}{2} - t), C(\mathcal{E}; \frac{s}{2} + t), t)$. Note that $(-1)^r C(\mathcal{E}; \frac{s}{2} - t) = \sum_{k=0}^{r} (-1)^k d_k(\mathcal{E}; \frac{s}{2}) t^{r-k} = \sum_{k=0}^{r} (-1)^k \bar{d}_k t^{r-k}$ and $C(\mathcal{E}; \frac{s}{2} + t) = \sum_{k=0}^{r} d_k(\mathcal{E}; \frac{s}{2}) t^{r-k} = \sum_{k=0}^{r} \bar{d}_k t^{r-k}$, hence

$$C(\mathcal{E} \otimes \mathcal{E}; s) = \begin{vmatrix} 1 & & & 1 & & \\ -\bar{d}_1 & \ddots & & \bar{d}_1 & \ddots & \\ \bar{d}_2 & \ddots & 1 & \bar{d}_2 & \ddots & 1 \\ \vdots & & -\bar{d}_1 & \vdots & & \bar{d}_1 \\ (-1)^r \bar{d}_r & & \vdots & \bar{d}_r & & \vdots \\ & \ddots & (-1)^{r-1} \bar{d}_{r-1} & & \ddots & \bar{d}_{r-1} \\ & & (-1)^r \bar{d}_r & & & \bar{d}_r \end{vmatrix} \tag{6}$$

We add the $(r+i)^{\text{th}}$ column to the $i^{\text{th}}$ column, then we subtract the $\frac{1}{2}$ of the $i^{\text{th}}$ column from the $(r+i)^{\text{th}}$ column for all $i = 1, \dots, r$. This results the determinant on the left hand side of (7). From the first $r$ columns we raise a $2^r$ factor. Then we switch the $(2i)^{\text{th}}$ and $(r+2i)^{\text{th}}$ columns for all $1 \le i \le \lfloor \frac{r}{2} \rfloor$. This yields the determinant on the right hand side of (7), which has zeroes in the even and odd rows of the first and last $r$ columns, respectively.

$$\begin{vmatrix} 2 & & & 0 & & \\ 0 & 2 & & \bar{d}_1 & 0 & \\ 2\bar{d}_2 & 0 & \ddots & 0 & \bar{d}_1 & \ddots \\ 0 & 2\bar{d}_2 & \ddots & \bar{d}_3 & 0 & \ddots \\ \vdots & 0 & \ddots & \vdots & \bar{d}_3 & \ddots \\ & \vdots & & & \vdots & \\ & & \ddots & & & \ddots \end{vmatrix} = (-1)^{\lfloor \frac{r}{2} \rfloor} 2^r \begin{vmatrix} 1 & & & & 0 & & & \\ 0 & 0 & & & \bar{d}_1 & 1 & & \\ \bar{d}_2 & \bar{d}_1 & 1 & & 0 & 0 & 0 & \\ 0 & 0 & 0 & \ddots & \bar{d}_3 & \bar{d}_2 & \bar{d}_1 & \ddots \\ \vdots & \bar{d}_3 & \bar{d}_2 & \ddots & \vdots & 0 & 0 & \ddots \\ & \vdots & 0 & \ddots & & \vdots & \bar{d}_3 & \ddots \\ & & \vdots & \ddots & & & \vdots & \ddots \end{vmatrix}. \tag{7}$$

Moving the odd rows up and the even rows down yields a 2-by-2 block determinant with zeroes in the off-diagonal blocks and a $(-1)^{r(r-1)/2}$-sign, which cancels the existing $(-1)^{\lfloor r/2 \rfloor}$-sign. We expand this determinant with respect to the first and last rows. These rows contain only zeroes, except the first row has $1$ in the first column and the last row has $\bar{d}_r$ in the last column. After expansion the two diagonal blocks become identical, hence

$$2^r \begin{vmatrix} 1 & & & & & 0 & 0 & 0 & \cdots & 0 \\ \bar{d}_2 & \bar{d}_1 & 1 & & & 0 & & & & 0 \\ \bar{d}_4 & \bar{d}_3 & \bar{d}_2 & \ddots & \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \bar{d}_{r-3} & 0 & & & & 0 \\ & & & & \bar{d}_{r-1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \bar{d}_1 & 1 & & & \\ 0 & & & & 0 & \bar{d}_3 & \bar{d}_2 & \bar{d}_1 & & \\ \vdots & & & & \vdots & \vdots & \bar{d}_4 & \bar{d}_3 & \ddots & \vdots \\ 0 & & & & 0 & & \vdots & \vdots & \ddots & \bar{d}_{r-2} \\ 0 & 0 & 0 & \cdots & 0 & & & & & \bar{d}_r \end{vmatrix} =$$

$$= 2^r \bar{d}_r \begin{vmatrix} \bar{d}_1 & 1 & & & & \\ \bar{d}_3 & \bar{d}_2 & \bar{d}_1 & 1 & & \\ \bar{d}_5 & \bar{d}_4 & \bar{d}_3 & \bar{d}_2 & \bar{d}_1 & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ & & \bar{d}_r & \bar{d}_{r-1} & \bar{d}_{r-2} & \bar{d}_{r-3} \\ & & & & \bar{d}_r & \bar{d}_{r-1} \end{vmatrix}^2 .$$

Note that $C(\mathcal{E};\frac{s}{2}) = \bar{d}_r = d_r(\mathcal{E};\frac{s}{2})$. Finally, by the relation

$$C(\mathcal{E}\otimes\mathcal{E};s) = C(\wedge^2\mathcal{E}\oplus S^2\mathcal{E};s) = C(\wedge^2\mathcal{E};s)C(S^2\mathcal{E};s) = 2^r C\left(\mathcal{E};\frac{s}{2}\right)C(\wedge^2\mathcal{E};s)^2$$

we are able to identify the $C(\wedge^2\mathcal{E};s)$-part in $C(\mathcal{E}\otimes\mathcal{E};s)$ to be (5). □

**Remark 1** *We can also compute* $C(\wedge^{r-2}\mathcal{E};s)$ *from* $C(\wedge^2\mathcal{E};s)$ *by the duality*

$$C(\wedge^{r-2}\mathcal{E};s) = \prod_{1\le i_1<\cdots<i_{r-2}\le r}(s+\alpha_{i_1}+\cdots+\alpha_{i_{r-2}})$$

$$= \prod_{1\le j_1<j_2\le r}(s+c_1(\mathcal{E})-\alpha_{j_1}-\alpha_{j_2}) = (-1)^{\frac{r(r-1)}{2}}C(\wedge^2\mathcal{E};-(s+c_1(\mathcal{E}))).$$

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