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Some results on Caristi type coupled fixed point theorems

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Abstract. In this work we define the concepts of the coupled orbit and coupled orbitally completeness. After then, using the method of Bollenbacher and Hicks [8], we prove some Caristi type coupled fixed point theorems in coupled orbitally complete metric spaces for a function $P : E \times E \to E$. We also give two examples that support our results.

1 Introduction and preliminaries

In the litareture concerning the fixed point theory, one of the most interesting and useful results is the Caristi's fixed point theorem [9], which is equivalent to Ekeland's variational principle [12] and is also a generalization of the famous Banach contraction principle.

In 1976, Caristi proved in [9] that "if S is a self mapping of a complete metric space (E, ρ) such that there is a lower semi-continuous function ψ from E into $[0, \infty)$ satisfying

$$\rho(u, Su) \leq \psi(u) - \psi(Su)$$

for all $u \in E$, then S has a fixed point".

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In this theorem, saying that "ψ is lower semi-continuous at u if for any sequence $\{u_n\} \subset E$, we have $\lim u_n = u$ implies $\psi(u) \leq \liminf \psi(u_n)$".

Several authors have obtained various extensions and generalizations of Caristi's theorem by considering Caristi type mappings on many different spaces. For example [1, 2, 3, 4, 8, 10, 14, 15, 16, 17, 18, 19, 20, 28, 29], and others.

In this paper, by using the method in [8], we give some Caristi type coupled fixed point theorems for a function P from a product space $E \times E$ to E.

The idea of the coupled fixed point was given first by Opoitsev [22, 23] and Opoitsev and Khurodze [24] and then by Guo and Lakhsmikantham in [13]. The first coupled fixed point theorems under the contractive conditions were studied by Bhaskar and Lakhsmikantham, see [7]. Since then various authors have obtained several important, useful and interesting results for the coupled fixed points under different condition [5, 6, 11, 21, 25, 26, 27].

We now give some basic definitions and notions.

Definition 1 ([7]) *Let E be a nonempty set and $P : E \times E \to E$ be a mapping. An element $(u, v) \in E \times E$ is said to be a coupled fixed point of mapping P if $u = P(u, v)$ and $v = P(v, u)$.*

Definition 2 *Let E be a nonempty set and $P : E \times E \to E$ be a mapping. Let u_0 and v_0 are arbitrary two points in E. Consider the sequences $\{u_n\}$ and $\{v_n\}$ by*

$$u_n = P(u_{n-1}, v_{n-1}),\ v_n = P(v_{n-1}, u_{n-1}) \tag{1}$$

for $n = 1, 2, 3, \dots$.

Then the sets

$$O_P(u_0, \infty) = \{u_0, u_1, u_2, \dots\} \quad \text{and} \quad O_P(v_0, \infty) = \{v_0, v_1, v_2, \dots\}$$

are called the coupled orbit of $(u_0, v_0) \in E \times E$.

Now let (E, ρ) be a metric space. If every Cauchy sequence in $O_P(u_0, \infty)$ and $O_P(v_0, \infty)$ converges to a point in E, for some $(u_0, v_0) \in E \times E$, then the (E, ρ) metric space is said to be coupled orbitally complete.

Note that a complete metric space (E, ρ) clearly coupled orbitally complete, but a coupled orbitally complete metric space (E, ρ) does not necessarily complete as in shown by Example 1.

Definition 3 *Let* (E, ρ) *be a metric space,* $P : E \times E \to E$ *a mapping and* $u_0, v_0 \in E$*. A real-valued function* $B : E \times E \to [0, \infty)$ *is said to be* $((u_0, v_0), P)$*–coupled orbitally weak lower semi-continuous (c.o.w.l.s.c.) at* $(u, v) \in E \times E$ *iff* $\{u_n\}$ *and* $\{v_n\}$ *are sequences in* $O_P(u_0, \infty)$ *and* $O_P(v_0, \infty)$ *respectively and*

$$u_n \to u,\ v_n \to v \quad \textit{implies} \quad B(u, v) \le \lim_{n\to\infty} \sup B(u_n, v_n)$$

(See [10]).

2 Main results

The following theorem is a version of Caristi's theorem, which was proved by Bollenbacher and Hicks (See [8]).

Theorem 1 *Let* (E, ρ) *be a metric space. Suppose* $S : E \to E$ *and* $\psi : E \to [0, \infty)$*. Suppose there exists an* u *such that*

$$\rho(v, Sv) \le \psi(v) - \psi(Sv)$$

for every $v \in O_S(u, \infty)$*, and any Cauchy sequence in* $O_S(u, \infty)$ *converges to a point in* E*. Then*:

(a) $\lim S^n u = u'$ *exists*,

(b) $\rho(S^n u, u') \le \psi(S^n u)$,

(c) $Su' = u'$ *iff* $B(u) = \rho(u, Su)$ *is* S*-orbitally lower semi-continuous at* u*,*

(d) $\rho(S^n u, u) \le \psi(u)$ *and* $\rho(u', u) \le \psi(u)$*.*

Now we prove the following coupled fixed point theorem for a function P on the product space $E \times E$.

Theorem 2 *Let* (E, ρ) *be a metric space,* $P : E \times E \to E$ *and* $\psi : E \to [0, \infty)$*. Suppose there exist* $u_0, v_0 \in E$ *such that* (E, ρ) *is coupled orbitally complete and*

$$\max\{\rho(u, P(u, v)), \rho(v, P(v, u))\} \le \psi(u) + \psi(v) - \psi(P(u, v)) - \psi(P(v, u)) \quad (2)$$

for all $u \in O_P(u_0, \infty)$ *and* $v \in O_P(v_0, \infty)$*. Then*:

(a) $\lim u_n = \lim P(u_{n-1}, v_{n-1}) = u'$ *and* $\lim v_n = \lim P(v_{n-1}, u_{n-1}) = v'$ *exist, where the sequences* $\{u_n\}$ *and* $\{v_n\}$ *are defined as in* (1),

(b) $\max\{\rho(u_n, u'), \rho(v_n, v')\} \leq \psi(u_n) + \psi(v_n)$,

(c) (u', v') *is a coupled fixed point of* P *if and only if* $B(u, v) = \rho(P(u, v), u)$ *is* $((u_0, v_0), P)$*– c.o.w.l.s.c. at* (u', v') *and* (v', u'),

(d) $\max\{\rho(u_n, u_0), \rho(v_n, v_0)\} \leq \psi(u_0) + \psi(v_0)$ *and* $\max\{\rho(u', u_0), \rho(v', v_0)\} \leq \psi(u_0) + \psi(v_0)$.

Proof. (a) Using inequality (2) we have

$$\begin{aligned}
S_n &= \sum_{k=0}^{n} \max\{\rho(u_k, u_{k+1}), \rho(v_k, v_{k+1})\} \\
&= \sum_{k=0}^{n} \max\{\rho(u_k, P(u_k, v_k)), \rho(v_k, P(v_k, u_k))\} \\
&\leq \sum_{k=0}^{n} [\psi(u_k) + \psi(v_k) - \psi(P(u_k, v_k)) - \psi(P(v_k, u_k))] \\
&= \sum_{k=0}^{n} [\psi(u_k) - \psi(u_{k+1}) + \psi(v_k) - \psi(v_{k+1})] \\
&= \psi(u_0) - \psi(u_{n+1}) + \psi(v_0) - \psi(v_{n+1}) \\
&\leq \psi(u_0) + \psi(v_0).
\end{aligned}$$

Hence $\{S_n\}$ is bounded above and also non-decreasing, and so convergent.

Now let m, n be any positive integers with $m > n$. Then from triangle inequality of ρ, we have

$$\begin{aligned}
\max\{\rho(u_n, u_m), \rho(v_n, v_m)\} &\leq \max\left\{\sum_{k=n}^{m-1} \rho(u_k, u_{k+1}), \sum_{k=n}^{m-1} \rho(v_k, v_{k+1})\right\} \\
&\leq \sum_{k=n}^{m-1} \max\{\rho(u_k, u_{k+1}), \rho(v_k, v_{k+1})\}. \qquad (3)
\end{aligned}$$

Since $\{S_n\}$ is convergent, for every $\varepsilon > 0$, we can find a sufficiently large positive integer N such that

$$\sum_{k=n}^{\infty} \max\{\rho(u_k, u_{k+1}), \rho(v_k, v_{k+1})\} < \varepsilon$$

for all $n \geq N$. Thus, we get from (3) that

$$\max\{\rho(u_n, u_m), \rho(v_n, v_m)\} < \varepsilon$$

for all $m, n \geq N$, and so $\{u_n\}$ and $\{v_n\}$ are two Cauchy sequences in $O_P(u_0, \infty)$, and $O_P(v_0, \infty)$ respectively. Since (E, ρ) is coupled orbitally complete,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} P(u_{n-1}, v_{n-1}) = u' \text{ and } \lim_{n\to\infty} v_n = \lim_{n\to\infty} P(v_{n-1}, u_{n-1}) = v'$$

exist.

(b) Let m, n be any positive integers with $m > n$. Using inequalities (2) and (3) we have

$$\begin{aligned}
\max\{\rho(u_n, u_m), \rho(v_n, v_m)\} &\leq \sum_{k=n}^{m-1} \max\{\rho(u_k, u_{k+1}), \rho(v_k, v_{k+1})\} \\
&= \sum_{k=n}^{m-1} \max\{\rho(u_k, P(u_k, v_k)), \rho(v_k, P(v_k, u_k))\} \\
&\leq \sum_{k=n}^{m-1} [\psi(u_k) + \psi(v_k) - \psi(u_{k+1}) - \psi(v_{k+1})] \\
&= \psi(u_n) - \psi(u_m) + \psi(v_n) - \psi(v_m) \\
&\leq \psi(u_n) + \psi(v_n).
\end{aligned}$$

Letting m tend to infinity, we have from (a)

$$\max\{\rho(u_n, u'), \rho(v_n, v')\} \leq \psi(u_n) + \psi(v_n).$$

(c) Assume that $u' = P(u', v')$, $v' = P(v', u')$ and $\{u_n\}$, $\{v_n\}$ are sequences in $O_P(u_0, \infty)$ and $O_P(v_0, \infty)$ respectively with $u_n \to u'$, $v_n \to v'$. Then we have,

$$\begin{aligned}
B(u', v') = \rho(P(u', v'), u') = 0 &\leq \limsup \rho(P(u_n, v_n), u_n) \\
&= \limsup B(u_n, v_n)
\end{aligned}$$

and

$$\begin{aligned}
B(v', u') = \rho(P(v', u'), v') = 0 &\leq \limsup \rho(P(v_n, u_n), v_n) \\
&= \limsup B(v_n, u_n)
\end{aligned}$$

and so B is $((u_0, v_0), P)$–c.o.w.l.s.c. at (u', v') and (v', u').

Now let $u_n = P(u_{n-1}, v_{n-1})$, $v_n = P(v_{n-1}, u_{n-1})$ and B is $((u_0, v_0), P)$–c.o.w.l.s.c. at (u', v') and (v', u'). Then from (a) we have

$$0 \leq \rho(P(u', v'), u') = B(u', v') \leq \limsup B(u_n, v_n)$$

$$= \limsup \rho(P(u_n, v_n), u_n) = 0$$

and

$$\begin{aligned} 0 \le \rho(P(v', u'), v') = B(v', u') &\le \limsup B(v_n, u_n) \\ &= \limsup \rho(P(v_n, u_n), v_n) = 0. \end{aligned}$$

Thus $u' = P(u', v')$ and $v' = P(v', u')$.

(d) Using triangle inequality of ρ and inequaliy (2) we have

$$\begin{aligned} \max\{\rho(u_n, u_0), \rho(v_n, v_0)\} &\le \max\left\{ \sum_{k=1}^{n} \rho(u_k, u_{k-1}), \sum_{k=1}^{n} \rho(v_k, v_{k-1}) \right\} \\ &\le \sum_{k=1}^{n} \max\{\rho(u_k, u_{k-1}), \rho(v_k, v_{k-1})\} \\ &= \sum_{k=1}^{n} \max\{\rho(u_{k-1}, P(u_{k-1}, v_{k-1})), \rho(v_{k-1}, P(v_{k-1}, u_{k-1}))\} \\ &\le \sum_{k=1}^{n} [\psi(u_{k-1}) + \psi(v_{k-1}) - \psi(u_k) - \psi(v_k)] \\ &= \psi(u_0) - \psi(u_n) + \psi(v_0) - \psi(v_n) \\ &\le \psi(u_0) + \psi(v_0). \end{aligned}$$

Letting n tend to infinity, we have from (a)

$$\max\{\rho(u', u_0), \rho(v', v_0)\} \le \psi(u_0) + \psi(v_0).$$

□

We now prove the following theorem.

Theorem 3 *Let* (E, ρ) *be a metric space,* $P : E \times E \to E$ *and* $\psi : E \to [0, \infty)$. *Suppose there exist* $u_0, v_0 \in E$ *such that* (E, ρ) *is coupled orbitally complete and*

$$\rho(u, P(u, v)) + \rho(v, P(v, u)) \le \psi(u) + \psi(v) - \psi(P(u, v)) - \psi(P(v, u)) \quad (4)$$

for all $u \in O_P(u_0, \infty)$ *and* $v \in O_P(v_0, \infty)$. *Then:*

(a) $\lim u_n = \lim P(u_{n-1}, v_{n-1}) = u'$ *and* $\lim v_n = \lim P(v_{n-1}, u_{n-1}) = v'$ *exist, where the sequences* $\{u_n\}$ *and* $\{v_n\}$ *are defined as in* (1),

(b) $\rho(u_n, u') + \rho(v_n, v') \leq \psi(u_n) + \psi(v_n)$,

(c) (u', v') *is a coupled fixed point of* P *if and only if* $B(u,v) = \rho(P(u,v), u)$ *is* $((u_0, v_0), P)$*– c.o.w.l.s.c. at* (u', v') *and* (v', u'),

(d) $\rho(u_n, u_0) + \rho(v_n, v_0) \leq \psi(u_0) + \psi(v_0)$ *and* $\rho(u', u_0) + \rho(v', v_0) \leq \psi(u_0) + \psi(v_0)$.

Proof. We have

$$\begin{aligned} \max\{\rho(u, P(u,v)), \rho(v, P(v,u))\} &\leq \rho(u, P(u,v)) + \rho(v, P(v,u)) \\ &\leq \psi(u) + \psi(v) - \psi(P(u,v)) - \psi(P(v,u)). \end{aligned}$$

The results (a) and (c) of this theorem follow immediately from Theorem 2.

(b) Let m, n be any positive integers with $m > n$. Using triangle inequality of ρ and inequality (4), we have

$$\begin{aligned} \rho(u_n, u_m) + \rho(v_n, v_m) &\leq \sum_{k=n}^{m-1} [\rho(u_k, u_{k+1}), +\rho(v_k, v_{k+1})] \\ &= \sum_{k=n}^{m-1} [\rho(u_k, P(u_k, v_k)) + \rho(v_k, P(v_k, u_k))] \\ &\leq \sum_{k=n}^{m-1} [\psi(u_k) + \psi(v_k) - \psi(u_{k+1}) - \psi(v_{k+1})] \\ &= \psi(u_n) - \psi(u_m) + \psi(v_n) - \psi(v_m) \\ &\leq \psi(u_n) + \psi(v_n). \end{aligned}$$

Letting m tend to infinity, we have from (a)

$$\rho(u_n, u') + \rho(v_n, v') \leq \psi(u_n) + \psi(v_n).$$

(d) Using triangle inequality of ρ and inequaliy (4) we have

$$\begin{aligned} \rho(u_n, u_0) + \rho(v_n, v_0) &\leq \sum_{k=1}^{n} [\rho(u_k, u_{k-1}) + \rho(v_k, v_{k-1})] \\ &= \sum_{k=1}^{n} [\rho(u_{k-1}, P(u_{k-1}, v_{k-1})) + \rho(v_{k-1}, P(v_{k-1}, u_{k-1}))] \end{aligned}$$

$$\begin{aligned}&\leq \sum_{k=1}^{n}[\psi(u_{k-1})+\psi(v_{k-1})-\psi(u_k)-\psi(v_k)]\\&=\psi(u_0)-\psi(u_n)+\psi(v_0)-\psi(v_n)\\&\leq \psi(u_0)+\psi(v_0).\end{aligned}$$

Letting n tend to infinity, we have from (a)

$$\rho(u',u_0)+\rho(v',v_0)\leq \psi(u_0)+\psi(v_0).$$

□

Finally, we prove the following theorem.

Theorem 4 *Let* (E,ρ) *be a metric space,* $P:E\times E\to E$ *and* $\psi:E\to[0,\infty)$. *Suppose there exist* $u_0,v_0\in E$ *such that* (E,ρ) *is coupled orbitally complete and*

$$\rho(u,P(u,v)) \leq \psi(u)-\psi(P(u,v)), \tag{5}$$
$$\rho(v,P(v,u)) \leq \psi(v)-\psi(P(v,u)) \tag{6}$$

for all $u\in O_P(u_0,\infty)$ *and* $v\in O_P(v_0,\infty)$. *Then:*

(a) $\lim u_n=\lim P(u_{n-1},v_{n-1})=u'$ *and* $\lim v_n=\lim P(v_{n-1},u_{n-1})=v'$ *exist, where the sequences* $\{u_n\}$ *and* $\{v_n\}$ *are defined as in* (1),

(b) $\rho(u_n,u')\leq\psi(u_n)$ *and* $\rho(v_n,v')\leq\psi(v_n)$,

(c) (u',v') *is a coupled fixed point of* P *if and only if* $B(u,v)=\rho(P(u,v),u)$ *is* $((u_0,v_0),P)$– *c.o.w.l.s.c. at* (u',v') *and* (v',u'),

(d) $\rho(u_n,u_0)\leq\psi(u_0)$ *and* $\rho(u',u_0)\leq\psi(u_0)$, $\rho(v_n,v_0)\leq\psi(v_0)$ *and* $\rho(v',v_0)\leq\psi(v_0)$.

Proof. From inequalities (5) and (6) we have

$$\rho(u,P(u,v))+\rho(v,P(v,u))\leq\psi(u)+\psi(v)-\psi(P(u,v))-\psi(P(v,u)).$$

The results (a) and (c) of this theorem follow immediately from Theorem 3.

(b) Let m,n be any positive integers with $m>n$. Using triangle inequality of ρ and inequality (5) we get

$$\rho(u_n,u_m) \leq \sum_{k=n}^{m-1}\rho(u_k,u_{k+1})=\sum_{k=n}^{m-1}\rho(u_k,P(u_k,v_k))$$

$$\leq \sum_{k=n}^{m-1} [\psi(u_k) - \psi(u_{k+1})] = \psi(u_n) - \psi(u_m) \leq \psi(u_n).$$

Letting m tend to infinity, we have from (a)

$$\rho(u_n, u') \leq \psi(u_n).$$

Similarly, using triangle inequality of ρ and inequality (6) we get

$$\rho(v_n, v') \leq \psi(v_n).$$

(d) Using triangle inequality of ρ and inequality (5) we have

$$\begin{aligned}\rho(u_n, u_0) &\leq \sum_{k=1}^{n} \rho(u_k, u_{k-1}) = \sum_{k=1}^{n} \rho(u_{k-1}, P(u_{k-1}, v_{k-1}))\\ &\leq \sum_{k=1}^{n} [\psi(u_{k-1}) - \psi(u_k)]\\ &= \psi(u_0) - \psi(u_n) \leq \psi(u_0).\end{aligned}$$

Letting n tend to infinity, we have from (a)

$$\rho(u', u_0) \leq \psi(u_0).$$

Similarly, it can be proved that

$$\rho(v_n, v_0) \leq \psi(v_0) \quad \text{and} \quad \rho(v', v_0) \leq \psi(v_0).$$

□

3 Some Examples

We now give two examples which illustrate our results.

Example 1 *Let* $E = [0, 1)$ *with Euclidean metric* ρ.
Define $P : E \times E \longrightarrow E$ *by* $P(u, v) = u/2$ *for all* (u, v) *in* $E \times E$ *and also define* $\psi : E \longrightarrow [0, \infty)$ *by* $\psi(u) = 2u$ *for all* u *in* E.

Let u_0 *and* v_0 *are arbitrary two points in* E. *Then we have*

$$O_P(u_0, \infty) = \left\{u_0, \frac{u_0}{2}, \frac{u_0}{2^2}, \ldots, \frac{u_0}{2^n}, \ldots\right\} \quad \textit{and}$$

$$O_P(v_0,\infty) \quad = \quad \left\{v_0, \frac{v_0}{2}, \frac{v_0}{2^2}, \ldots, \frac{v_0}{2^n}, \ldots\right\}.$$

Clearly, (E,ρ) *is coupled orbitally complete as it is not complete. Further, for all* u *in* $O_P(u_0,\infty)$ *and* v *in* $O_P(v_0,\infty)$*, we have*

$$\begin{aligned}\max\{\rho(u,P(u,v)),\rho(v,P(v,u))\} &= \max\{|u-u/2|\,,|v-v/2|\} = \max\{u/2,v/2\}\\ &\le u+v = \psi(u)+\psi(v)-\psi(P(u,v))-\psi(P(u,v)).\end{aligned}$$

Thus P *satisfies inequality* (2) *with* $\psi(u)=2u$ *and so the conditions of Theorem 2 are satisfied and* $\lim P(u_{n-1},v_{n-1}) = \lim P(v_{n-1},u_{n-1}) = 0$*. Further,* $(0,0)$ *is a coupled fixed point of* P *and* $B(u,v)=\rho(P(u,v),u)$ *is c.o.w.l.s.c. at* $(0,0)$*.*

Example 2 *Let* $E=[0,\infty)$ *with Euclidean metric* ρ *and define*

$$P : E\times E \longrightarrow E \quad by \quad P(u,v)=\begin{cases} 0 & \text{if } u<v\\ 2 & \text{if } u\ge v\end{cases}.$$

for all (u,v) *in* $E\times E$*. If we take* $u_0=2$ *and* $v_0=2$*, then*

$$O_P(2,\infty)=\{2,2,2,\ldots.\} \quad and \quad O_P(2,\infty)=\{2,2,2,\ldots.\}.$$

Clearly, (E,ρ) *is coupled orbitally complete and* P *satisfies inequality* (2) *for all* u *in* $O_P(2,\infty)$ *and* v *in* $O_P(2,\infty)$ *with* $\psi(u)=u$*. So the conditions of Theorem 2 are satisfied and* $\lim P(u_{n-1},v_{n-1}) = \lim P(v_{n-1},u_{n-1}) = 2$*. Further,* $(2,2)$ *is a coupled fixed point of* P *and* $B(u,v)=\rho(P(u,v),u)$ *is c.o.w.l.s.c. at* $(2,2)$*.*

Similarly, if we take $u_0=0$ *and* $v_0=2$*, then*

$$O_P(0,\infty)=\{0,0,0,\ldots.\} \quad and \quad O_P(2,\infty)=\{2,2,2,\ldots.\}.$$

Clearly, (E,ρ) *is coupled orbitally complete and* P *satisfies inequality* (2) *for all* u *in* $O_P(0,\infty)$ *and* v *in* $O_P(2,\infty)$ *with* $\psi(u)=u$*. So the conditions of Theorem 2 are satisfied and* $\lim P(u_{n-1},v_{n-1})=0, \lim P(v_{n-1},u_{n-1})=2$*. Further,* $(0,2)$ *is a coupled fixed point of* P *and* $B(u,v)=\rho(P(u,v),u)$ *is c.o.w.l.s.c. at* $(0,2)$ *and* $(2,0)$*.*

This shows that the coupled fixed point of P *is not unique.*

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