Acta Univ. Sapientiae, Mathematica, **14**, 2 (2022) 308–316

DOI: 10.2478/ausm-2022-0020

Co-unit graphs associated to ring of integers modulo n

S. Pirzada
Department of Mathematics,
University of Kashmir,
Srinagar, India
email:
pirzadasd@kashmiruniversity.ac.in

Aaqib Altaf
Department of Mathematics,
University of Kashmir,
Srinagar, India
email: aaqibwani777@gmail.com

Abstract. Let R be a finite commutative ring. We define a co-unit graph, associated to a ring R, denoted by $G_{nu}(R)$ with vertex set $V(G_{nu}(R)) = U(R)$, where $U(R)$ is the set of units of R, and two distinct vertices x, y of $U(R)$ being adjacent if and only if $x + y \notin U(R)$. In this paper, we investigate some basic properties of $G_{nu}(R)$, where R is the ring of integers modulo n, for different values of n. We find the domination number, clique number and the girth of $G_{nu}(R)$.

1 Introduction

A graph $G = (V, E)$ consists of the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E(G)$. Further, $|V(G)| = n$ is the *order* and $|E(G)| = m$ is the *size* of G. The *degree* of a vertex v, denoted by $d_G(v)$ (we simply write d_v) is the number of edges incident on the vertex v.

A path of length n is denoted by P_n and a cycle of length n is denoted by C_n. A graph G is connected if their is at least one path between every pair of distinct vertices, otherwise disconnected. As usual, K_n denotes a complete

2010 Mathematics Subject Classification: 05C50, 05C12, 05A18
Key words and phrases: unit graph, co-unit graph, ring of integers modulo n, domination number

graph with n vertices and $K_{a,b}$ denotes a complete bipartite graph with $a+b$ vertices. Also a graph G is said to be $k-regular$ if degree of every vertex of G is k. The $girth$ of a graph G, denoted by $gr(G)$, is the length of the shortest cycle contained G. In G, an independent set is a subset S of the vertex set $V(G)$ if no two vertices of S are adjacent. The $independence\ number$ of G, denoted by $\alpha(G)$, is defined as $\alpha(G) = max\{|S| : S\ is\ an\ independent\ set\ of\ G\}$. Two graphs G_1 and G_2 are said to be isomorphic if there exists a bijection between vertices and edges so that the incidence relationship is preserved and is written as $G_1 \cong G_2$. A subset D of $V(G)$ is called a $dominating\ set$ of G if every vertex in $V \setminus D$ is adjacent to at least one vertex in D. A dominating set of minimum cardinality is called a $\gamma - set$ of G. The $domination\ number$ of G, denoted by $\gamma(G)$, is the cardinality of a $\gamma - set$ of G.

Let R be a finite commutative ring and let $U(R)$ be the set of units of R. Let $R \cong R_1 \times R_2 \times ... \times R_n$ be the direct product of the finite rings R_i. If a_i is a unit in R_i, where $1 \leq i \leq n$, then $(a_1, a_2, a_3, \ldots, a_n)$ is the unit element of $R_1 \times R_2 \times ... \times R_n$.

Let n be a positive integer and let $\mathbb{Z}_n$ be the ring of integers modulo n. Grimaldi [4] defined the unit graph $G(\mathbb{Z}_n)$ whose vertex set is the set of elements of $\mathbb{Z}_n$ and two distinct vertices x and y are adjacent if and only if $x+y$ is a unit of $\mathbb{Z}_n$. Ashrafi et. al [2] extended the concept of $G(\mathbb{Z}_n)$ to $G(R)$, where R is any arbitrary associative ring with nonzero identity. More literature on this can be seen in [1, 5, 6, 13, 15, 14].

We define a *co-unit graph* associated to a ring R, denoted by $G_{nu}(R)$, with vertex set as the set $U(R)$ and two vertices $x, y \in U(R)$ are adjacent if and only if $x + y \notin U(R)$. We observe that $G_{nu}(R)$ is an empty graph when R is the ring of real numbers or the ring of rational numbers. More generally, if $\mathbb{R}$ is a field, then $G_{nu}(\mathbb{R})$ is an empty graph. Also, for the ring of integers $\mathbb{Z}$, $G_{nu}(\mathbb{Z}) \cong K_2$, since $U(\mathbb{Z}) = \{-1, 1\}$ and $-1 + 1 = 0 \notin U(\mathbb{Z})$ which implies that the vertex corresponding to the unit -1 is adjacent to the vertex corresponding to the unit 1 and hence becomes K_2.

In Section 2, we characterize the graphs $G_{nu}(\mathbb{Z}_n)$, for different values of n. Also, we find the domination number, clique number and the girth of $G_{nu}(\mathbb{Z}_n)$.

2 On graphs $G_{nu}(\mathbb{Z}_n)$ associated to the ring $\mathbb{Z}_n$

Definition 1 *The Euler's phi function* $\phi(n)$*, where* n *is positive integer, is defined as the number of non-negative integers less than* n *that are relatively prime to* n*. If* $n \geq 2$ *and* p *is prime, then* $\phi(p^n) = p^n - p^{n-1}$.

We begin with the following observation.

Theorem 1 *The graph* $G_{nu}(\mathbb{Z}_p) \cong (\frac{p-1}{2})K_2$*, where* $p \geq 3$ *is a prime number.*

Proof. Since all nonzero elements of the ring $\mathbb{Z}_p$ are units, so the vertex set of $G_{nu}(\mathbb{Z}_p)$ is V$= \{1, 2, 3, \ldots, , p-1\}$. Partition the vertex set V into two disjoint subsets V_1 and V_2, where $V_1 = \{1, 2, \ldots, \frac{p-1}{2}\}$ and $V_2 = \{\frac{p+1}{2}, \ldots, p-1\}$. Let x and y be any two elements in V_1. Then, clearly $x+y \leq p-2$, implying that the sum of any two elements in V_1 is a unit. Therefore, no two vertices in V_1 are adjacent. Now, let x and y be any elements in V_2. Clearly, $p+2 \leq x+y \leq 2p-3$, so that $x+y$ is a unit, which implies that there is no edge in V_2. Also, for every element $k \in V_1$, $1 \leq k \leq \frac{p-1}{2}$, there is exactly one element $p-k$ in V_2 such that $k+p-k = p$ is a non unit. Hence the graph $G_{nu}(\mathbb{Z}_p)$ is bipartite and contains $\frac{p-1}{2}$ copies of K_2, that is $G_{nu}(\mathbb{Z}_p) \cong (\frac{p-1}{2})K_2$. □

Remark. From Theorem 1, we observe that the independence number of $G_{nu}(\mathbb{Z}_p)$ is equal to $\frac{\phi(p)}{2}$.

Theorem 2 *For prime* $p \geq 5$ *and* $n \geq 2$*, the graph* $G_{nu}(\mathbb{Z}_{p^n})$ *is* p^{n-1}*- regular graph.*

Proof. By Euler's ϕ- function , $\phi(p^n) = p^n - p^{n-1}$. So the order of the graph $G_{nu}(\mathbb{Z}_{p^n})$ is $\phi(p^n)$. Let $V = \{1, 2, \ldots, p-1, p+1, \ldots, 2p-1, 2p+1, \ldots, p^n-1\}$ be the vertex set of $G_{nu}(\mathbb{Z}_{p^n})$. It is clear that V has no vertex of the type np^α. As $|V| = \phi(p^n)$, so the number of non units in $\mathbb{Z}_{p^n}$ is $p^n - \phi(p^n) = p^n - p^n + p^{n-1} = p^{n-1}$. Let $D = \{np^\alpha : n, \alpha \in \mathbb{N}\}$ be the set of non units in $G_{nu}(\mathbb{Z}_{p^n})$, so that $|D| = p^{n-1}$. Consider the set $S = \{np^\alpha - k : k \in V\}$. Clearly, each vertex of V is adjacent to every vertex of S, since for every fixed $k \in V$ and $np^\alpha - k \in S$, we have $k + np^\alpha - k = np^\alpha \notin U(\mathbb{Z}_{p^n})$. Define a mapping $f : D \to S$ by $f(np^\alpha) = np^\alpha - k$. Clearly, f is bijective, so it follows that $|S| = p^{n-1}$. As each vertex of $G_{nu}(\mathbb{Z}_{p^n})$ is adjacent to every vertex of S, so degree of every vertex of $v \in V = |S|$. Thus, $G_{nu}(\mathbb{Z}_{p^n})$ is $p^{n-1} - \text{regular}$. □

Example 1 *Let* $p = 5$ *and* $n = 2$*. The graph* $G_{nu}(\mathbb{Z}_{5^2})$ *is* $5^{2-1} = 5$*—regular, as shown in Figure* 1.

Theorem 3 *The graph* $G_{nu}(\mathbb{Z}_p \times \mathbb{Z}_q)$ *is* $(\phi(p) + \phi(q) - 1)$*— regular, where both* p *and* q *are distinct odd primes with* $p < q$*. Further, the domination number* $\gamma(G_{nu}(\mathbb{Z}_p \times \mathbb{Z}_q)) = \phi(p)$ *and* $gr(\gamma(G_{nu}(\mathbb{Z}_p \times \mathbb{Z}_q)) = 3$.

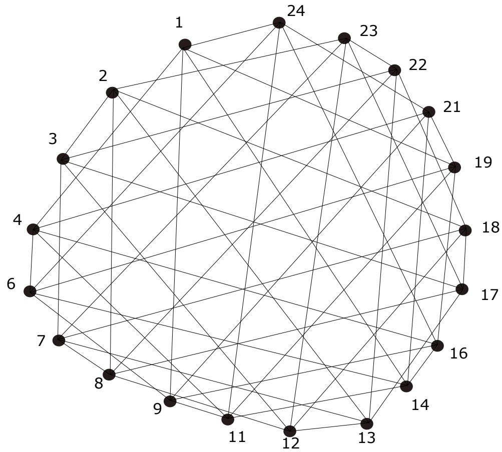

Figure 1: $G_{nu}(\mathbb{Z}_{5^2})$

Proof. Since p and q are odd primes with $p < q$, the number of units in the rings $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are $\phi(p)$ and $\phi(q)$, respectively. So the order of the graph $G_{nu}(\mathbb{Z}_p \times \mathbb{Z}_q)$ is $\phi(p)\phi(q)$. Now, let

$$\begin{aligned} V =&\{(u_1, v_1), (u_1, v_2), \dots (u_1, v_{q-1}), (u_2, v_1), \dots (u_2, v_{q-1}), \dots \\ &(u_{p-1}, v_1), (u_{p-1}, v_2), \dots, (u_{p-1}, v_{q-1})\} \end{aligned}$$

be the vertex set of $G_{nu}(\mathbb{Z}_p \times \mathbb{Z}_q)$, where $\{u_i : 1 \le i \le (p-1)\}$ and $\{v_j : 1 \le j \le (q-1)\}$ are the set of units in $\mathbb{Z}_p$ and $\mathbb{Z}_q$, respectively. Partition vertex set V into $\phi(p)$ disjoint subsets, each having cardinality $\phi(q)$, which are given by

$$B_1 = \{(u_1, v_1), (u_1, v_2), (u_1, v_3), \dots, (u_1, v_{q-1})\}$$

$$B_2 = \{(u_2, v_1), (u_2, v_2), (u_2, v_3), \dots, (u_2, v_{q-1})\}$$

$$B_3 = \{(u_3, v_1), (u_3, v_2), (u_3, v_3), \dots, (u_3, v_{q-1})\}$$

$$\vdots$$

$$B_{\phi(p)} = \{(u_{p-1}, v_1), (u_{p-1}, v_2), (u_{p-3}, v_3), \dots, (u_{p-1}, v_{q-1})\}.$$

Choose some arbitrary subset, say B_i, $1 \le i \le \phi(p)$. We show that each vertex in B_i has degree $\phi(p) + \phi(q) - 1$. Let $(i, x) \in B_i$ be an arbitrary vertex, where $1 \le x \le \phi(q)$. Obviously, (i, x) is adjacent to every vertex in $B_{\phi(p)+1-i}$, and (i, x) is adjacent to exactly one vertex in the remaining subsets. So the degree of (i, x) is $\phi(p) + \phi(q) - 1$, proving first part of the result.

The vertices of V are $\{(u_1, v_1), (u_1, v_2), (u_1, v_3), \ldots, (u_1, v_{q-1}), (u_2, v_1), (u_2, v_2), (u_2, v_3), \ldots, (u_2, v_{q-1}), \ldots (u_{p-1}, v_1), (u_{p-1}, v_2), (u_{p-3}, v_3), \ldots, (u_{p-1}, v_{q-1})$. The vertices of the type (u_1, v_i), where $1 \leq i \leq q-1$, are adjacent to the vertex (u_{p-1}, v_1). Similarly, the vertices of the type (u_2, v_i), where $1 \leq i \leq q-1$, are adjacent to the vertex (u_{p-2}, v_1). In this way, the vertices (u_{p-1}, v_i), where $1 \leq i \leq q-1$, are adjacent to the vertex of the type (u_1, v_1). Now, form the subset of the vertex set V, say D, where D contains vertices of the type $\{(u_1, v_1), (u_2, v_1), (u_3, v_1), \ldots, (u_{p-1}, v_1)\}$. We have $|D| = p - 1 = \phi(p)$. Also, each vertex of $V \setminus D$ is adjacent to at least one vertex of D. We show that D is minimal with the above conditions. From D, if we remove any number of the vertices of the type (u_x, v_1), where $1 \leq x \leq p-1$, then there exist vertices of the type (u_{p-x}, v_i) in $V \setminus D$, which are not adjacent to any vertex in D. It follows that D is a minimal dominating set and $\gamma(G_{nu}(\mathbb{Z}_p \times \mathbb{Z}_q)) = |D| = p - 1 = \phi(p)$. The subset $\{(u_{p-1}, v_1), (u_1, v_{q-1}), (u_{p-1}, v_{q-1})\}$ of the vertex set V, forms an induced subgraph which is complete. Hence it follows that $gr(\gamma(G_{nu}(\mathbb{Z}_p \times \mathbb{Z}_q)) = 3$. □

Theorem 4 *If* $n = 2m$*, then the graph* $G_{nu}(\mathbb{Z}_{2m})$ *is complete.*

Proof. We know that the number of units of the ring $\mathbb{Z}_{2m}$ is $\phi(2m)$. Also, if $a \in \mathbb{Z}_{2m}$ is a unit then $(a, 2m) = 1$. So the vertex set of $G_{nu}(\mathbb{Z}_{2m})$ contains only odd integers, while as all even integers are nonunits. Let $V = \{v_{\alpha_1}, v_{\alpha_2}, v_{\alpha_3}, \ldots, v_{\alpha_{\phi(2m)}}\}$ be the vertex set of $G_{nu}(\mathbb{Z}_{2m})$, where the set $\{v_{\alpha_i} | i = 1, 2, 3, \ldots, \phi(2m)\}$ is the set of units of $\mathbb{Z}_{2m}$. Clearly, every v_{α_i} in V is an odd integer. As sum of two odd integers is even, therefore, every two vertices in V are adjacent. Thus, $G_{nu}(\mathbb{Z}_{2m})$ forms a complete graph. □

Theorem 5 *Let* $R \cong \mathbb{Z}_3 \times \mathbb{Z}_p$*, where* p *is odd prime. Then the graph* $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)$ *is a connected* $p-$*regular graph and* $\gamma(G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)) = \frac{\phi(p)}{2}$, $gr(G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)) = 3$, $cl(G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)) = 4$.

Proof. As the units of the ring $\mathbb{Z}_3$ are $\{1, 2\}$ and units of the ring $\mathbb{Z}_p$ are $\{1, 2, 3, 4, \ldots, p-1\}$, so the vertex set for the graph $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)$ is

$$V = \{(1, 1), (1, 2), \ldots, (1, p-1), (2, 1), (2, 2), \ldots, (2, p-1)\}.$$

Partition vertex set V into two disjoint sets V' and V'' such that $V' = \{(1, i) | 1 \leq i \leq p-1\}$ and $V'' = \{(2, j) | 1 \leq j \leq p-1\}$. Then $|V'| = p - 1$ and $|V''| = p - 1$. By definition, each vertex of V of the type $(2, u)$ is adjacent to every vertex of V of the type $(1, v)$, where u and v are units in the ring

$\mathbb{Z}_p$, since $(1, v) + (2, u) = (3, u') \notin U(\mathbb{Z}_3 \times \mathbb{Z}_p)$, where $u' = u + v \in \mathbb{Z}_p$. It follows that $K_{p-1,p-1}$ is an induced subgraph of $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)$. Also, in V', corresponding to each vertex of the type $(1, u_r)$, there exists a unique vertex $(1, u_{p-r})$ in V' such that $(1, r) \sim (1, p - r)$. The same argument holds for V''. Therefore, the degree of each vertex in both the sets V' and V'' is equal to $p - 1 + 1 = p$. Hence the graph $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)$ is p-regular.

Every vertex of V' is adjacent to every vertex of V'', since $(1, u) + (2, u) = (3, u') \notin U(\mathbb{Z}_3 \times \mathbb{Z}_p)$. So there exists a path between every pair of vertices $\{(1, u), (2, v)\}$, where $u, v \in \mathbb{Z}_p$. From the above discussion, for partite sets V' and V'', vertices of the type $(1, u_{p-r})$ are adjacent to the vertices of the type $(1, u_r)$ in V', and vertices of the type $(2, u_{p-r})$ are adjacent to vertices of the type $(2, u_r)$ in V''. So, it follows that there is a path between every pair of vertices in V, see Figure 2. Thus, $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)$ is connected.

As each vertex of the type $(2, u_i)$ is adjacent to the vertex $(1, 1)$, and the vertex $(1, u_{p-1})$ is adjacent to $(1, 1)$, so the remaining vertices are of the type $\{(1, u_j) | 2 \le j \le p - 2\}$. Now, corresponding to each vertex of the type $(1, u_j)$ in V, where $2 \le j \le p - 2$, there exist a vertex of the type $(1, u_{p-j})$, where $2 \le j \le p - 2$, such that $(1, u_j) + (1, u_{p-j}) \notin U(\mathbb{Z}_3 \times \mathbb{Z}_p)$. Let D be a subset of the vertex set V defined as $D = \{(1, u_j) : 1 \le j \le \frac{p-1}{2}\}$. Now, each vertex of $V \setminus D$ is adjacent to at least one vertex of D, since each vertex of the type $(2, u_i)$ is adjacent to every vertex of the type $(1, u_i)$. Also, half of the vertices of the type $(1, u_i)$, where $1 \le i \le \frac{p-1}{2}$, are adjacent to other half of the vertices of the type $(1, u_k)$, where $\frac{p+1}{2} \le k \le p - 1$. From D, if we remove vertices of the type $\{(1, u_r)\}$, where $1 \le r \le \frac{p-1}{2}$, then those vertices go to set $V \setminus D$. Therefore, there exist vertices in $V \setminus D$ of the type $(1, u_{p-r})$, $r = 1, 2, 3, \ldots \frac{p-1}{2}$, which are not adjacent to any vertex in D. So $D \setminus \{(1, u_r)\}$ does not form a dominating set for $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)$. Therefore, it follows that D is a dominating set for $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)$ and $|D| = \frac{p-1}{2} = \frac{\phi(p)}{2}$. So $\gamma(G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)) = \frac{\phi(p)}{2}$.

Let $S = \{(1, u_1), (1, u_{p-1}), (2, u_i)\}$ be the subset of V, where u_i, $1 \le i \le p-1$, are units in the ring $\mathbb{Z}_p$. The induced subgraph $< S >$ is a cycle of length 3, so it follows that $gr(G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)) = 3$. Again, let $S' = \{(1, 1), (1, p - 1), (2, 1), (2, p - 1)\}$ be the subset of the vertex set V. Then the induced subgraph $< S' >$ is a complete subgraph. This induced subgraph is maximal, in $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)$. To see this, if we add any vertex either of the type $(1, u_i)$, where $u_i \ne 1, p-1$, or of the type $(2, u_j)$, where $u_j \ne 1, p-1$, to S', then the following three possibilities arise. (i) If we add vertex $(1, u_i)$, where $2 \le i \le p - 2$, to S', then this vertex is not adjacent to the vertices $(1, 1), (1, p - 1)$. So the induced subgraph $< S' + (1, u_i) >$ is not complete. (ii) If we add vertex $(2, u_i)$

to S', where $2 \le i \le p-2$, then this vertex is not adjacent to the vertices $(2,1),(2,p-1)$. So the induced subgraph $< S' + (2,u_i) >$ is not complete. (iii) If we add both types of vertices as in (i) and (ii), then in this case the induced subgraph $< S' + (1,u_i) + (2,u_i) >$ is not complete. Thus the subgraph induced by $< S' >$ forms a clique in $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)$ and therefore $cl(G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)) = |S'| = 4$. The graph $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)$ can be seen in Figure 2. □

Theorem 6 *If $R \cong \mathbb{Z}_{2^n} \times \mathbb{Z}_m$, then the graph $G_{nu}(\mathbb{Z}_{2^n} \times \mathbb{Z}_m)$ is complete.*

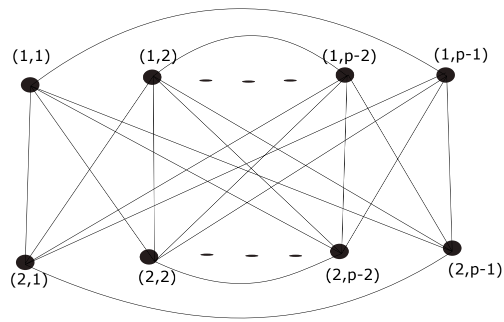

Figure 2: $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)$

Proof. It is easy to see that the set of units for the ring $\mathbb{Z}_{2^n}$ are $\{2k+1; k \in \mathbb{Z}\}$. Let V be the vertex set for the graph $G_{nu}(\mathbb{Z}_{2^n} \times \mathbb{Z}_m)$. Then, $V = \{(x_i, y_j) : 1 \le i \le \phi(2^n), 1 \le j \le \phi(m)\}$, where x_i and y_i are units in the rings $\mathbb{Z}_{2^n}$ and $\mathbb{Z}_m$, respectively. Since each x_i, $1 \le i \le \phi(2^n)$, is an odd integer, therefore, $(x_i, y_j) + (x_r, y_s) \notin U(\mathbb{Z}_{2^n} \times \mathbb{Z}_m)$, as $x_i + x_r$ is always even. Thus, each vertex of V is adjacent to every vertex of V. Thus the graph $G_{nu}(\mathbb{Z}_{2^n} \times \mathbb{Z}_m)$ is a complete graph. □

Theorem 6 can be generalized as follows, the proof of which is similar to that of Theorem 6.

Theorem 7 *If $R \cong \mathbb{Z}_{2^n} \times \mathbb{Z}_{\alpha_1} \times \mathbb{Z}_{\alpha_2} \times \cdots \times \mathbb{Z}_{\alpha_m}$, then the graph $G_{nu}(\mathbb{Z}_{2^n} \times \mathbb{Z}_{\alpha_1} \times \mathbb{Z}_{\alpha_2} \times \cdots \times \mathbb{Z}_{\alpha_m})$ is complete.*

Theorem 8 *Let $n \in \mathbb{N}$ and p be a prime. Then $G_{nu}(\mathbb{Z}_{p^n})$ is complete if and only if $p = 2$ and $G_{nu}(\mathbb{Z}_{p^n})$ is complete bipartite if and only if $p = 3$. Moreover, if $p > 3$, then $G_{nu}(\mathbb{Z}_{p^n})$ has $\frac{p-1}{2}$ components each being a complete bipartite graph isomorphic to $K_{r,r}$, where $r = p^n - 1$.*

Proof. Partition vertex set $V(G_{nu}((\mathbb{Z}_{p^n})))$ into subsets V_1, V_2, ..., V_{p-1}, where $V_i = \{pk - i : k \in \mathbb{N} \text{ and } pk - i < p^n\}$, $1 \leq i \leq p - 1$. Then, as $|V(G_{nu}((\mathbb{Z}_{p^n})))| = (p-1)p^{n-1}$, we have $|V_i| = p^{n-1}$, for each i. Moreover, each V_i is an independent set for all $p \geq 3$.
If $p = 2$, then $V(G_{nu}((\mathbb{Z}_{2^n}))) = V_1 = \{1, 3, 5, \dots, 2^n - 1\}$ and so $G_{nu}((\mathbb{Z}_{2^n}))$ is complete. For $p = 3$, $V(G_{nu}((\mathbb{Z}_{3^n}))) = V_1 \cup V_2$, where $V_1 = \{3k - 1 : k \in \mathbb{N} \text{ and } 3k - 1 < 3^n\}$ and $V_2 = \{3k - 2 : k \in \mathbb{N} \text{ and } 3k - 2 < 3^n\}$. Then, for any $x \in V_1$ and $y \in V_2$, we have $x + y \notin U(\mathbb{Z}_{3^n})$. Thus, $G_{nu}(\mathbb{Z}_{3^n})$ is isomorphic to $K_{3^{n-1},3^{n-1}}$. Now, for $p > 3$, let $x \in V_t$ and $y \in V_s$, $1 \leq t, s \leq p - 1$. Then x and y are adjacent in $G_{nu}(\mathbb{Z}_{p^n})$ if and only if $t + s = p$. Thus, we partition the set $\{V_1, V_2, \dots, V_{p-1}\}$ into the $(p - 1)/2$ sets, namely, $V_{j,p-j} = \{V_j, V_{p-j}\}$, $1 \leq j \leq (p - 1)/2$. Then each $V_{j,p-j}$ induces a complete bipartite graph $K_{r,r}$, where $r = |V_i| = p^{n-1}$. □

Conclusion For a finite commutative ring R we associated a co-unit graph, denoted by $G_{nu}(R)$, with vertex set $V(G_{nu}(R)) = U(R)$, where $U(R)$ is the set of units of R, and two distinct vertices x, y of $U(R)$ being adjacent if and only if $x + y \notin U(R)$. We investigated some basic properties of $G_{nu}(R)$, where R is the ring of integers modulo n, for different values of n. We obtained the domination number, the clique number and the girth of $G_{nu}(R)$. For the future work, we need to investigate several other graph invariants of $G_{nu}(R)$, for any ring R. Also, there is scope to study the line graph of the co-unit graph, in analogy to the line graph of the unit graph, see [10]. Further directions to study in co-unit graphs can be metric dimension and spectra, for instance like in [3, 9, 10, 11, 12].

References

[1] M. Afkhami and F. Khosh-Ahang, Unit graphs of rings of polynomials and power series, *Arab. J. Math.* **2,3** (2013), 233–246.

[2] N. Ashrafi, H. R. Maimani, M. R. Pournaki and S. Yassemi, Unit graphs associated to rings, *Comm. Algebra* **38,8** (2010), 2851–2871.

[3] M. Imran Bhat and S. Pirzada, On strong metric dimension of zero divisor graphs of rings, *The Korean J. Math.* **27,3** (2019), 563–580.

[4] R. P. Grimaldi, Graphs from rings, Proceedings of the 20th Southeastern Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1989). Congr. Numer. **71** (1990), 95–103.

[5] S. Kiani, H. R. Maimani, M. R. Pournaki and S. Yassemi, Classification of rings with unit graphs having domination number less than four, *Rend. Sem. Mat. Univ. Padova* **133** (2015), 173–195.

[6] H. R. Maimani, M. R. Pournaki and S. Yassemi, Necessary and sufficient conditions for unit graphs to be Hamiltonian, *Pacific J. Math.* **249,2** (2011), 419–429.

[7] S. Pirzada, *An Introduction to Graph Theory*, Universities Press, Orient Blakswan, Hyderabad (2012).

[8] S. Pirzada and Aaqib Altaf, Line graphs of unit graphs associated with the direct product of rings, *The Korean J. Math.* **30,1**, (2022) 65-72.

[9] S. Pirzada, M. Aijaz and Shane Redmond, Upper dimension and bases of zero divisor graphs of commutative rings, *AKCE International J. Graphs Comb.* **17,1**, (2020) 168–173.

[10] S. Pirzada and M. Aijaz, Metric and upper dimension of zero divisor graphs associated to commutative rings, *Acta Univ. Sapientiae, Informatica* **12,1** (2020), 84–101.

[11] S. Pirzada and M. Imran Bhat, Computing metric dimension of compressed zero divisor graphs associated to rings, *Acta Univ. Sap. Mathematica* **10,2** (2018), 298–318.

[12] S. Pirzada and Rameez Raja, On the metric dimension of a zero divisor graph, *Communications Algebra* **45,4** (2017), 1399–1408.

[13] H. Su, G. Tang and Y. Zhou, Rings whose unit graphs are plannar, *Publ. Math. Debrecen* **86,3-4** (2015), 363–376.

[14] H. Su and Y. Wei, The diameter of unit graphs of rings, *Taiwanese J. Math.* **23,1** (2019), 1–10.

[15] H. Su and Y. Zhou, On the girth of the unit graph of a ring, *J. Algebra Appl.* **13,2** (2014), 1350082, 12 pp.

Received: December 14, 2021