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On unique and non-unique fixed point in parametric $\mathcal{N}_{\mathfrak{b}}-$metric spaces with application

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Abstract. We propose $\mathcal{SA}$, $\eta-\mathcal{SA}$, $\eta-\mathcal{SA}_{\min}$, and $\mathcal{SA}_{\eta,\delta,\zeta}-$contractions and notions of $\eta-$admissibility type $\mathfrak{b}$ and $\eta_{\mathfrak{b}}-$regularity in parametric $\mathcal{N}_{\mathfrak{b}}$-metric spaces to determine a unique fixed point, a unique fixed circle, and a greatest fixed disc. Further, we investigate the geometry of non-unique fixed points of a self mapping and demonstrate by illustrative examples that a circle or a disc in parametric $\mathcal{N}_{\mathfrak{b}}-$metric space is not necessarily the same as a circle or a disc in a Euclidean space. Obtained outcomes are extensions, unifications, improvements, and generalizations of some of the well-known previous results. We provide non-trivial illustrations to exhibit the importance of our explorations. Towards the end, we resolve the system of linear equations to demonstrate the significance of our contractions in parametric $\mathcal{N}_{\mathfrak{b}}-$metric space.

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1 Introduction and preliminaries

Banach [2] stated the first metric fixed point result in 1922. After this, enormous generalizations and extensions of Banach's result have been announced ([1], [4], [7], [13], [17], [23], [27] -[30], and so on). These essentially centred around two components: (i) by changing the structure and (ii) by changing the conditions on the mapping under consideration. One such interesting structure, parametric $\mathcal{N}_{\mathfrak{b}}$-metric spaces is recently introduced by Tas and Özgür [21]. It generalizes the metric space (Fréchet [5]), $\mathfrak{b}-$metric space (Bakhtin [1] and Czerwik [4]), $\mathcal{S}-$metric space (Sedghi et al. [17]), $\mathcal{S}_{\mathfrak{b}}-$metric space (Souayah and Mlaiki [19] and Sedghi et al. [16]), parametric $\mathcal{S}$-metric space (Tas and Özgür [20]), $A_{\mathfrak{b}}$-metric space (Ughade et al. [30]) and so on. It is worth to mention that Souayah et al. [19] used the symmetry condition, in addition to other conditions used by Sedghi et al. [16]. Motivated by the fact that the equations, obtained on modeling real-world problems may be solved using the fixed point technique and geometry of nonunique fixed points, we familiarize $\mathcal{SA}$, $\eta-\mathcal{SA}$, $\eta-\mathcal{SA}_{\min}$, $\mathcal{SA}_{\eta,\delta,\zeta}-$contractions and the notions of η-admissibility of type $\mathfrak{b}$ and $\eta_{\mathfrak{b}}-$ regularity in parametric $\mathcal{N}_{\mathfrak{b}}-$metric space to establish a unique fixed point, a unique fixed circle, and a greatest fixed disc. In the sequel, with the help of examples and remarks, we demonstrate that our contractions are incomparable over each one of those contractions wherein the continuity of mapping is presumed for the survival of a fixed point. Further, we investigate the geometry of non-unique fixed points in reference to fixed circle or greatest fixed disc problems and demonstrate by illustrative examples that a circle or a disc in parametric $\mathcal{N}_{\mathfrak{b}}-$metric space is not necessarily the same as a circle or a disc in a Euclidean space. We conclude the paper by resolving the system of linear equations to demonstrate the significance of our proposed contractions in parametric $\mathcal{N}_{\mathfrak{b}}-$metric space.

We denote $\mathcal{N}(\mathfrak{x},\mathfrak{x},\cdots,(\mathfrak{x})_{n-1},\mathfrak{y},\mathfrak{t})$ by $\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}$.

Definition 1 [21] *Let $\mathcal{X}\neq\emptyset$, $\mathfrak{b}\geq 1$ be a given real number, $\mathfrak{n}\in\mathbb{N}$. A distance function $\mathcal{N}:\mathcal{X}^{n}\times(0,\infty)\to[0,\infty)$ is a parametric $\mathcal{N}_{\mathfrak{b}}-$metric if*

(N1) $\mathcal{N}(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t})=0$ *iff* $\mathfrak{x}_1=\mathfrak{x}_2=\cdots=\mathfrak{x}_{n-1}=\mathfrak{x}_n$;

(N2) $\mathcal{N}(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t})\leq\mathfrak{b}[\mathcal{N}(\mathfrak{x}_1,\mathfrak{a},\mathfrak{t})+\mathcal{N}(\mathfrak{x}_2,\mathfrak{a},\mathfrak{t})+\cdots+\mathcal{N}(\mathfrak{x}_{n-1},\mathfrak{a},\mathfrak{t})+\mathcal{N}(\mathfrak{x}_n,\mathfrak{a},\mathfrak{t})]$,

$\mathfrak{t}>0$, *for every* $\mathfrak{a}$, $\mathfrak{x}_i\in\mathcal{X}$ *and* $i=1,2,\cdots,\mathfrak{n}$.

Example 1 [21] *Let* $\mathcal{X} = \{\mathcal{S}|\mathcal{S} : (0,\infty) \to \mathbb{R}\}$ *be the set of functions and* $\mathcal{N} : \mathcal{X}^3 \times (0,\infty) \to [0,\infty)$ *be*

$$\mathcal{N}(\mathcal{S}t, \mathcal{T}t, \mathcal{J}t, t) = \frac{1}{9}\left(|\mathcal{S}t - \mathcal{T}t| + |\mathcal{S}t - \mathcal{J}t| + |\mathcal{T}t - \mathcal{J}t|\right)^2,$$

$t > 0$, *for every* $\mathcal{S}$, $\mathcal{T}$, $\mathcal{J} \in \mathcal{X}$. *Noticeably,* $(\mathcal{X}, \mathcal{N})$ *is a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space with* $\mathfrak{n} = 3$ *and* $\mathfrak{b} = 4$.

Remark 1 *Noticeably, a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric is an improvement of a parametric* $\mathcal{S}-$*metric* [20] *because every parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric, for* $\mathfrak{b} = 1$ *and* $\mathfrak{n} = 3$, *is a parametric* $\mathcal{S}-$*metric. However, one may verify that a parametric* $\mathcal{S}-$*metric need not essentially be a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric.*

Lemma 1 [21] *In a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space* $(\mathcal{X}, \mathcal{N})$,

- **(i)** $\mathcal{N}_{\mathfrak{x},\mathfrak{y},t} \leq \mathfrak{b}\mathcal{N}_{\mathfrak{y},\mathfrak{x},t}$ and $\mathcal{N}_{\mathfrak{y},\mathfrak{x},t} \leq \mathfrak{b}\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}$,
- **(ii)** $\mathcal{N}_{\mathfrak{x},\mathfrak{y},t} \leq \mathfrak{b}[(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x},\mathfrak{z},t} + \mathcal{N}_{\mathfrak{y},\mathfrak{z},t}]$ *and* $\mathcal{N}_{\mathfrak{x},\mathfrak{y},t} \leq \mathfrak{b}[(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x},\mathfrak{z},t} + \mathfrak{b}\mathcal{N}_{\mathfrak{z},\mathfrak{y},t}]$,

$t > 0$ *and* $\mathfrak{x}$ $\mathfrak{y} \in \mathcal{X}$.

Definition 2 [21] *Let* $\{\mathfrak{x}_{\mathfrak{k}}\}$ *be a sequence in a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space* $(\mathcal{X}, \mathcal{N})$, *then*

- **(1)** $\{\mathfrak{x}_{\mathfrak{k}}\}$ *converges to* $\mathfrak{x}$, *if for* $\epsilon > 0$, *there exists an* $\mathfrak{n}_o \in \mathbb{N}$ *so that, we attain* $\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x},t} \leq \epsilon$, *i.e.,* $\lim_{k\to\infty} \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x},t} = 0$, $\mathfrak{l} \geq \mathfrak{n}_0$. *It is denoted* $\lim_{\mathfrak{k}\to\infty} \mathfrak{x}_{\mathfrak{k}} = \mathfrak{x}$;
- **(2)** $\{\mathfrak{x}_{\mathfrak{k}}\}$ *is a Cauchy sequence, if for each* $\epsilon > 0$, *there exists an* $\mathfrak{n}_o \in \mathbb{N}$ *so that, we attain* $\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x}_{\mathfrak{l}},t} \leq \epsilon$, *i.e.,* $\lim_{k\to\infty} \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x}_{\mathfrak{l}},t} = 0$, $\mathfrak{k}$, $\mathfrak{l} \geq \mathfrak{n}_o$;
- **(3)** $(\mathcal{X}, \mathcal{N})$ *is complete if every Cauchy sequence in* $(\mathcal{X}, \mathcal{N})$ *converges to a point in it.*

Lemma 2 [21] *If* $\{\mathfrak{x}_{\mathfrak{k}}\}$ *and* $\{\mathfrak{y}_{\mathfrak{k}}\}$ *are two sequences in a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space* $(\mathcal{X}, \mathcal{N})$ *that converge to* $\mathfrak{x}$ *and* $\mathfrak{y}$ *respectively in* $\mathcal{X}$ *then:*

- **(i)** $\mathfrak{x}$ *is unique,*
- **(ii)** $\{\mathfrak{x}_{\mathfrak{k}}\}$ *is a Cauchy sequence,*
- **(iii)** $\frac{1}{\mathfrak{b}^2}\mathcal{N}_{\mathfrak{x},\mathfrak{y},t} \leq \lim_{k\to\infty} \inf \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{y}_{\mathfrak{k}},t} \leq \lim_{k\to\infty} \sup \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{y}_{\mathfrak{k}},t} \leq \mathfrak{b}^2\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}$.

Lemma 3 [21] *If two sequences* $\{\mathfrak{x}_{\mathfrak{k}}\}$ *and* $\{\mathfrak{y}_{\mathfrak{k}}\}$ *in a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space* $(\mathcal{X}, \mathcal{N})$ *are such that*

$$\lim_{k\to\infty} \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{y}_{\mathfrak{k}},t} = 0,$$

when $\{\mathfrak{x}_{\mathfrak{k}}\}$ *is convergent and* $\lim_{k\to\infty} \mathfrak{x}_{\mathfrak{k}} = \mathfrak{x}_o$, $\mathfrak{x}_o \in \mathcal{X}$, *then* $\lim_{k\to\infty} \mathfrak{y}_{\mathfrak{k}} = \mathfrak{x}_o$.

2 Main results

I. Existence of a single fixed point

We define $\mathcal{SA}$, $\eta - \mathcal{SA}$, $\eta - \mathcal{SA}_{\min}$, and $\mathcal{SA}_{\eta,\delta,\zeta}-$contractive conditions in a parametric $\mathcal{N}_\mathfrak{b}$-metric space to prove a fixed point.

Definition 3 *A self mapping $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ in a parametric $\mathcal{N}_\mathfrak{b}-$metric space $(\mathcal{X}, \mathcal{N})$ with $\mathfrak{b} \geq 1$ is called an $\mathcal{SA}-$contraction if*

$$\begin{aligned}\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}} \leq \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2 \frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}} \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{1 + \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_3 \frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}} \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1 + \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} \\ + \mathfrak{a}_4 \frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}} \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{1 + \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_5 \frac{\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}} \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1 + \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}},\end{aligned} \tag{1}$$

where, $\sum_{i=1}^{5} \mathfrak{a}_i < 1$ and $\mathfrak{a}_1 + \mathfrak{a}_3 \mathfrak{b} < 1$, $(\mathfrak{a}_i, \quad i = 1$ to 5, are non-negative constants).

Definition 4 *A self mapping $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ in a parametric $\mathcal{N}_\mathfrak{b}-$metric space with $\mathfrak{b} \geq 1$ is called an $\eta - \mathcal{SA}-$contraction if*

$$\begin{aligned}\eta(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})\, \mathcal{N}_{\mathcal{S}x,\mathcal{S}\mathfrak{y},\mathfrak{t}} \leq \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2\, \phi\left(\max \left\{ \begin{array}{l} \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}, \\ \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}} + \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{2} \end{array} \right\} \right) \\ + \mathfrak{a}_3\, \phi\left(\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} [1 + \sqrt{\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}}]^2}{(1 + \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}})^2} \right),\end{aligned} \tag{2}$$

$\mathfrak{x}$, $\mathfrak{y} \in \mathcal{A}$, $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3 \geq 0$, $\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3 < 1$, $\mathfrak{a}_1 + \mathfrak{b}\mathfrak{a}_2 + \mathfrak{a}_3 < 1$, $0 \leq \mathfrak{a}_2 < \frac{1 - \mathfrak{a}_1 - \mathfrak{a}_3}{\mathfrak{b}^2 + \mathfrak{b}(\mathfrak{n} - 1)}$, η, $\phi : [0, \infty) \to [0, \infty)$ are increasing functions and $\phi(\mathfrak{t}) < \mathfrak{t}$, $\mathfrak{t} > 0$.

Definition 5 *A self mapping $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ in a parametric $\mathcal{N}_\mathfrak{b}$-metric space $(\mathcal{X}, \mathcal{N})$ with $\mathfrak{b} \geq 1$ is called an $\eta - \mathcal{SA}_{\min}$*

$$\begin{aligned}\eta(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})\, \phi(\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}) \leq \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2\, \phi\left(\min \left\{ \begin{array}{l} \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}, \\ \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}} + \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{2} \end{array} \right\} \right) \\ + \mathfrak{a}_3\, \phi\left(\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} [1 + \sqrt{\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}}]^2}{(1 + \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}})^2} \right),\end{aligned} \tag{3}$$

$\mathfrak{x}$, $\mathfrak{y} \in \mathcal{X}$ and $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3 \geq 0$ with $\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3 < 1$, η, $\phi : [0, \infty) \to [0, \infty)$ are increasing functions and $\phi(\mathfrak{t}) < \mathfrak{t}$, $\mathfrak{t} > 0$.

Definition 6 *A self mapping $\mathcal{S}:\mathcal{X}\to\mathcal{X}$ in a parametric $\mathcal{N}_{\mathfrak{b}}$-metric space $(\mathcal{X},\mathcal{N})$ with $\mathfrak{b}\geq 1$ is called an $\mathcal{SA}_{\eta,\delta,\zeta}-$ contraction if*

$$[\eta(\mathfrak{x},\mathfrak{y},t)-1+\delta]^{\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},t}}\leq \delta^{\zeta(\mathcal{N}_{\mathfrak{x},\mathfrak{y},t})\left(\max\left\{\mathcal{N}_{\mathfrak{x},\mathfrak{y},t},\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t},\mathrm{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},t},\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},t},\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t}+\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},t}}{2}\right\}\right)},\tag{4}$$

$\mathfrak{x}$, $\mathfrak{y}\in\mathcal{X}$, a constant $\delta\geq 1$, $\zeta:[0,\infty)\to[0,\frac{1}{\mathfrak{b}}]$ and $\eta:[0,\infty)\to[0,\infty)$ is a non-decreasing function.

Next, we prove our first main result for an $\mathcal{SA}-$contraction.

Theorem 1 *Let $\mathcal{S}:\mathcal{X}\to\mathcal{X}$ be an $\mathcal{SA}-$contraction (1) in a complete parametric $\mathcal{N}_{\mathfrak{b}}$-metric space $(\mathcal{X},\mathcal{N})$. Then, $\mathcal{S}$ has a unique fixed point in $\mathcal{X}$.*

Proof. Let us assume that $\mathfrak{x}_o\in\mathcal{X}$. Let a sequence $\{\mathfrak{x}_n\}$ be constructed as $\mathfrak{x}_{n+1}=\mathcal{S}\mathfrak{x}_n$. If, we have $\mathfrak{x}_{n_o}=\mathfrak{x}_{n_o+1}$ then $\mathfrak{x}_{n_o}=\mathfrak{x}_{n_o+1}=\mathcal{S}\mathfrak{x}_{n_o}$, $n_o\in\mathbb{N}$, i.e., we infer that $\mathfrak{x}_{n_o}$ is a fixed point of $\mathcal{S}$.
Let $\mathfrak{x}_{n_o}\neq\mathfrak{x}_{n_o+1}$, $n_0\in\mathbb{N}$. Using inequality (1), we attain

$$\begin{aligned}
\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}&=\mathcal{N}_{\mathcal{S}\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_n,t}\\
&\leq a_1\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+a_2\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_{n-1},t}\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_n,t}}{1+\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}+a_3\frac{\mathcal{N}_{x_{n-1},\mathcal{S}\mathfrak{x}_n,t}\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_{n-1},t}}{1+\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}\\
&\quad+a_4\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_{n-1},t}\mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_n,t}}{1+\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}+a_5\frac{\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_n,t}\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_{n-1},t}}{1+\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}\\
&=a_1\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+a_2\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{1+\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}+a_3\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},t}\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_n,t}}{1+\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}\\
&\quad+a_4\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},t}}{1+\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}+a_5\frac{\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_n,t}}{1+\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}\\
&\leq a_1\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+a_2\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}+a_4[\mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathfrak{b}^2\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}].
\end{aligned}$$

It follows that

$$(1-a_2-\mathfrak{b}^2a_4)\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}\leq\big(a_1+\mathfrak{b}(n-1)a_4\big)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.\tag{5}$$

Again using inequality (1), we obtain

$$\begin{aligned}
\mathcal{N}_{\mathfrak{x}_{n+1},\mathfrak{x}_n,t}&=\mathcal{N}_{\mathcal{S}\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_{n-1},t}\\
&\leq a_1\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}+a_2\frac{\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_n,t}\mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_{n-1},t}}{1+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}}+a_3\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_n,t}\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_{n-1},t}}{1+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}}
\end{aligned}$$

$$
\begin{aligned}
&+ \mathfrak{a}_4 \frac{\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_n,t}\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_{n-1},t}}{1+N_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}} + \mathfrak{a}_5 \frac{N_{\mathfrak{x}_{n-1},S\mathfrak{x}_{n-1},t}N_{\mathfrak{x}_{n-1},S\mathfrak{x}_n,t}}{1+N_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}} \\
&= \mathfrak{a}_1\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t} + \mathfrak{a}_2 \frac{\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}{1+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}} + \mathfrak{a}_3 \frac{\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_n,t}}{1+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}} \\
&+ \mathfrak{a}_4 \frac{\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_n,t}}{1+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}} + \mathfrak{a}_5 \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},t}}{1+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}} \\
&\leq \mathfrak{a}_1\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} + \mathfrak{a}_2\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} + \mathfrak{a}_5[\mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} + \mathfrak{b}^2\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}]. \quad (6)
\end{aligned}
$$

It follows that

$$(1-\mathfrak{a}_2-\mathfrak{b}^2\mathfrak{a}_5)\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \big(\mathfrak{a}_1+\mathfrak{b}(n-1)\mathfrak{a}_5\big)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}. \quad (7)$$

Adding inequalities (5) and (7), we obtain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \left(\frac{2\mathfrak{a}_1+\mathfrak{b}(n-1)(\mathfrak{a}_4+\mathfrak{a}_5)}{2-2\mathfrak{a}_2-\mathfrak{b}^2(\mathfrak{a}_4+\mathfrak{a}_5)}\right)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.$$

Let $\frac{2\mathfrak{a}_1+\mathfrak{b}(n-1)(\mathfrak{a}_4+\mathfrak{a}_5)}{2-2\mathfrak{a}_2-\mathfrak{b}^2(\mathfrak{a}_4+\mathfrak{a}_5)} = h$. In view of $\sum_1^5 \mathfrak{a}_i < 1$, $h \in (0,1)$. Then,

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.$$

Similarly,

$$\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} \leq h\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

So,

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h^2\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.$$

Following the same pattern, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h^n\mathcal{N}_{\mathfrak{x}_0,\mathfrak{x}_1,t}.$$

Since, $h \in (0,1)$, letting $n \to \infty$, $h^n \to 0$, we attain

$$\lim_{n\to\infty} \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} = 0. \quad (8)$$

Then, for $\mathfrak{l} > \mathfrak{k}$, $\mathfrak{k}, \mathfrak{l} \in \mathbb{N}$, using equation (8), condition (N2) and Lemma 1, we obtain

$$
\begin{aligned}
\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_\mathfrak{l},t} &\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}\mathcal{N}_{\mathfrak{x}_\mathfrak{l},\mathfrak{x}_{\mathfrak{k}+1},t} \leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^2\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_\mathfrak{l},t} \\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^3(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},t} + \mathfrak{b}^3\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_\mathfrak{l},t} \\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^3(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},t} + \mathfrak{b}^4\mathcal{N}_{\mathfrak{x}_\mathfrak{l},\mathfrak{x}_{\mathfrak{k}+2},t}
\end{aligned}
$$

$$\leq \mathfrak{b}(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{t}} + \mathfrak{b}^3(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{t}} + \mathfrak{b}^5(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_{\mathfrak{k}+3},\mathfrak{t}} + \mathfrak{b}^5\mathcal{N}_{\mathfrak{x}_\mathfrak{l},\mathfrak{x}_{\mathfrak{k}+3},\mathfrak{t}}$$
$$\leq \mathfrak{b}(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{t}} + \mathfrak{b}^3(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{t}} + \mathfrak{b}^5(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_{\mathfrak{k}+3},\mathfrak{t}} + \mathfrak{b}^7\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+3},\mathfrak{x}_{\mathfrak{k}+4},\mathfrak{t}} + \cdots .$$

Letting $\mathfrak{k}$, $\mathfrak{l} \to \infty$, we obtain

$$\lim_{\mathfrak{k},\ \mathfrak{l}\to\infty} \mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_\mathfrak{l},\mathfrak{t}} = 0,$$

i.e., $\{\mathfrak{x}_\mathfrak{n}\}$ is a Cauchy sequence. Using the completeness of the space, $\lim_{\mathfrak{k},\ \mathfrak{l}\to\infty} \mathfrak{x}_\mathfrak{k} = \mathfrak{x}, \mathfrak{x} \in \mathcal{X}$.

Assume $\mathfrak{x}$ is not a fixed point of $\mathcal{S}$. Applying inequality (1), we obtain

$$\begin{aligned}\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathcal{S}\mathfrak{x},\mathfrak{t}} = \mathcal{N}_{\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},\mathfrak{t}} &\leq \mathfrak{a}_1\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}} + \mathfrak{a}_2\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}} + \mathfrak{a}_3\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}}\\ &\quad + \mathfrak{a}_4\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}} + \mathfrak{a}_5\frac{\mathcal{N}_{x,\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}}\\ &= \mathfrak{a}_1\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}} + \mathfrak{a}_2\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x}_\mathfrak{k},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}} + \mathfrak{a}_3\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathfrak{x}_\mathfrak{k},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}}\\ &\quad + \mathfrak{a}_4\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x}_\mathfrak{k},\mathfrak{t}}\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}} + \mathfrak{a}_5\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathfrak{x}_\mathfrak{k},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}}.\end{aligned} \tag{9}$$

As $\mathfrak{k} \to \infty$, using condition (N1), we get $\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}} \leq 0$, i.e., $\mathcal{S}\mathfrak{x} = \mathfrak{x}$.

Presume that $\mathfrak{y}$ is one more fixed point of $\mathcal{S}$, then $\mathcal{S}\mathfrak{x} = \mathfrak{x}$ and $\mathcal{S}\mathfrak{y} = \mathfrak{y}$. Using inequality (1), we obtain

$$\begin{aligned}\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} = \mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}} &\leq \mathfrak{a}_1\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_3\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_4\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}}\\ &\quad + \mathfrak{a}_5\frac{\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} = (\mathfrak{a}_1 + \mathfrak{a}_3\mathfrak{b})\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, \text{ a contradiction.}\end{aligned}$$

Thus, $\mathfrak{x} = \mathfrak{y}$, i.e., a fixed point of $\mathcal{S}$ is unique. □

Next, we furnish a non-trivial illustration to exhibit the validity of the above outcome.

Example 2 *Let* $\mathcal{X} = \mathbb{R}^+ \cup \{0\}$. *Let a function* $\mathcal{N} : \mathcal{X}^3 \times (0,\infty) \to [0,\infty)$ *be*

$$\mathcal{N}(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{t}) = \begin{cases} 0, & \text{if } \mathfrak{x}=\mathfrak{y}=\mathfrak{z};\\ \mathfrak{t}^2\max\{\mathfrak{x},\mathfrak{y},\mathfrak{z}\}, & \text{otherwise,}\end{cases}$$

for each $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{X}$ *and* $\mathfrak{t} > 0$*. Then,* $(\mathcal{X}, \mathcal{N})$ *is a complete parametric* $\mathcal{N}_{\mathfrak{b}}$*−metric space for* $\mathfrak{b} = 2$ *and* $\mathfrak{n} = 3$*. Define* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *as*

$$\mathcal{S}\mathfrak{x} = \begin{cases} \frac{\mathfrak{x}^2}{16}, & \text{if } \mathfrak{x} \in [0, \mathfrak{a}); \\ \frac{\mathfrak{x}}{15}, & \text{if } \mathfrak{x} \in [\mathfrak{a}, \infty), \end{cases}, \quad \mathfrak{x} \in \mathcal{X}$$

with $\mathfrak{a} \in (\frac{1}{4}, 1)$*. Taking* $\mathfrak{a}_1 = \frac{1}{5} = \mathfrak{a}_2 = \mathfrak{a}_3 = \mathfrak{a}_4$ *and* $\mathfrak{a}_5 = \frac{1}{10}$*,* $\mathcal{S}$ *verifies the hypotheses of Theorem 1 and has a unique fixed point at* $\mathfrak{x} = 0$*.*

For $\mathfrak{a}_1 \in [0, 1)$ and $\mathfrak{a}_2 = \mathfrak{a}_3 = \mathfrak{a}_4 = \mathfrak{a}_5 = 0$, Theorem 1 is an extension and an improvement of Banach [2] to a parametric $\mathcal{N}_{\mathfrak{b}}$−metric space wherein the involved mapping is not necessarily continuous.

Following Sintunavarat [18], we familiarize η−admissibility of type $\mathfrak{b}$ and $\eta_{\mathfrak{b}}$−regularity to determine a fixed point in a parametric $\mathcal{N}_{\mathfrak{b}}$−metric space $(\mathcal{X}, \mathcal{N})$.

Definition 7 *A self mapping* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *is called* η*−admissible of type* $\mathfrak{b}$ *if there exists an* $\eta : \mathcal{X} \times \mathcal{X} \times (0, \infty) \to (0, \infty)$ *so that* $\eta(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) \geq \mathfrak{b}$ *implies that* $\eta(\mathcal{S}\mathfrak{x}, \mathcal{S}\mathfrak{y}, \mathfrak{t}) \geq \mathfrak{b}$*,* $\mathfrak{t} > 0$ *and* $\mathfrak{x}, \mathfrak{y} \in \mathcal{X}$*.*

Example 3 *Let* $\mathcal{X} = \{(0, 0), (1, 0), (1, 2), (1, 3), (1, 4)\}$ *be a subset of* $\mathbb{R}^2$*. Let* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *be*

$$\mathcal{S}\mathfrak{x} = \begin{cases} (1, 2), & \text{if } \mathfrak{x} \in \mathcal{X} \backslash \{(1, 4)\} \\ (1, 3), & \text{if } \mathfrak{x} = (1, 4). \end{cases}$$

Now, define an $\eta : \mathcal{X} \times \mathcal{X} \times (0, \infty) \to (0, \infty)$ *as*

$$\eta(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = \begin{cases} (1, 0), & \text{if } \mathfrak{x}, \mathfrak{y} \in \mathcal{X} \backslash \{(1, 4)\} \\ \frac{3}{2}, & \text{if } \mathfrak{x} = (1, 4). \end{cases}$$

In case $\mathfrak{x}, \mathfrak{y} \in \mathcal{X} \backslash \{(1, 4)\}$*, then* $\eta(\mathcal{S}\mathfrak{x}, \mathcal{S}\mathfrak{y}, \mathfrak{t}) = \eta((1, 2), (1, 2), \mathfrak{t}) = (1, 0)$*. If* $\mathfrak{x} = \mathfrak{y} = (1, 3)$*, then* $\eta(\mathcal{S}\mathfrak{x}, \mathcal{S}\mathfrak{y}, \mathfrak{t}) = \eta\big(\mathcal{S}(1, 4), \mathcal{S}(1, 4), \mathfrak{t})\big) = \eta((1, 3), (1, 3), \mathfrak{t})) = (1, 0)$*,* $\mathfrak{t} > 0$*. Hence,* $\mathcal{S}$ *is* η*−admissible of type* $\mathfrak{b}$*. One may verify that* $\mathcal{S}$ *is neither an* α*−admissible [14] nor an* α*−admissible type* $\mathcal{S}$ *[18].*

Definition 8 *Let* $\{\mathfrak{x}_{\mathfrak{n}}\}$ *be a sequence in* $\mathcal{X}$ *so that* $\eta(\mathfrak{x}_{\mathfrak{n}}, \mathfrak{x}_{\mathfrak{n}+1}, \mathfrak{t}) \geq \mathfrak{b}$*,* $\mathfrak{n} \in \mathbb{N} \cup \{0\}$*,* $\mathfrak{t} > 0$ *and* $\lim_{\mathfrak{n} \to \infty} \mathfrak{x}_{\mathfrak{n}} = \mathfrak{x} \in \mathcal{X}$*, then* $\mathcal{X}$ *is called* $\eta_{\mathfrak{b}}$*−regular if* $\eta(\mathfrak{x}_{\mathfrak{n}}, \mathfrak{x}, \mathfrak{t}) \geq \mathfrak{b}$*.*

Theorem 2 *Let* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *be an* $\eta - \mathcal{SA}-$ *contraction (2) in a complete parametric* $\mathcal{N}_{\mathfrak{b}}$*−metric space* $(\mathcal{X}, \mathcal{N})$ *with* $\mathfrak{b} \geq 1$ *satisfying*

(i) $\mathcal{S}$ *is* $\eta-$*admissible of type* $\mathfrak{b}$*;*

(ii) $\mathcal{X}$ *is* $\eta_{\mathfrak{b}}-$*regular;*

(iii) *There exists a* $\mathfrak{x}_o \in \mathcal{X}$ *so that* $\eta(x_0, \mathcal{S}\mathfrak{x}_o, \mathfrak{t}) \geq \mathfrak{b}$*, for* $\mathfrak{t} > 0$.

Then, $\mathcal{S}$ *has a unique fixed point.*

Proof. Consider $\mathfrak{x}_o \in \mathcal{X}$ so that $\eta(\mathfrak{x}_o, \mathcal{S}\mathfrak{x}_o, \mathfrak{t}) \geq \mathfrak{b}$, $\mathfrak{t} > 0$. Let a sequence $\{\mathfrak{x}_n\}$ be constructed as $\mathfrak{x}_{n+1} = \mathcal{S}\mathfrak{x}_n$, $n \in \mathbb{N} \cup \{0\}$. Since, $\eta(\mathfrak{x}_o, \mathfrak{x}_1, \mathfrak{t}) = \eta(\mathfrak{x}_o, \mathcal{S}\mathfrak{x}_o, \mathfrak{t}) \geq \mathfrak{b}$ and $\eta(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{t}) = \eta(\mathcal{S}x_0, \mathcal{S}\mathcal{S}\mathfrak{x}_o, \mathfrak{t}) \geq \mathfrak{b}$, using (ii). Following this pattern, we attain $\eta(\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{t}) \geq \mathfrak{b}$. In case, $\mathfrak{x}_n = \mathfrak{x}_{n+1}$, we conclude that $\mathfrak{x}_n$ is a fixed point of $\mathcal{S}$. Let $\mathfrak{x}_n \neq \mathfrak{x}_{n+1}$. Using inequality (2), we obtain

$$\begin{aligned}
\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t}} &= \mathcal{N}_{\mathcal{S}\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_n,\mathfrak{t}} \leq \eta(\mathfrak{x}_{n-1}, \mathfrak{x}_n, \mathfrak{t}) \mathcal{N}_{\mathcal{S}\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_n,\mathfrak{t}} \\
&\leq \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}} + \mathfrak{a}_2\, \phi\left(\max \begin{Bmatrix} \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_{n-1},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_{n-1},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_n,\mathfrak{t}}, \\ \mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_n,\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_{n-1},\mathfrak{t}} + \mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_n,\mathfrak{t}}}{2} \end{Bmatrix} \right) \\
&\quad + \mathfrak{a}_3\, \phi\left(\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{y},\mathfrak{t}} [1 + \sqrt{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}} \mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_{n-1},\mathfrak{t}}}]^2}{(1 + \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}})^2} \right) \\
&= \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}} + \mathfrak{a}_2\, \phi\left(\max \begin{Bmatrix} \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_n,\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t}}, \\ \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}} + \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t}}}{2} \end{Bmatrix} \right) \\
&\quad + \mathfrak{a}_3\, \phi\left(\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{y},\mathfrak{t}} [1 + \sqrt{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}} \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}}}]^2}{(1 + \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}})^2} \right) \\
&= \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}} + \mathfrak{a}_2\, \phi\left(\max \begin{Bmatrix} \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},\mathfrak{t}}, \\ \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}} + \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t}}}{2} \end{Bmatrix} \right) \\
&\quad + \mathfrak{a}_3\, \phi\left(\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{y},\mathfrak{t}} [1 + \sqrt{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}} \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}}}]^2}{(1 + \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}})^2} \right) \\
&\leq \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}} + \mathfrak{a}_2\, \phi\left(\max \begin{Bmatrix} \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},\mathfrak{t}}, \\ \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}} + \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t}}}{2} \end{Bmatrix} \right) \\
&\quad + \mathfrak{a}_3\, \phi(\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}}). \qquad (10)
\end{aligned}$$

If $\max\{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}} + \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t}}}{2}\} = \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}}$. Then, inequality (10) becomes

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t}} \leq \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}} + \mathfrak{a}_2 \phi(\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}}) + \mathfrak{a}_3 \phi(\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}}).$$

Since, $\phi(t) < t$, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} < (a_1 + a_2 + a_3)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.$$

Let $a_1 + a_2 + a_3 = h$, we have

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} < h\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.$$

Similarly,

$$\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} < h\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Therefore,

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} < h^2\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Following this pattern, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h^n\mathcal{N}_{\mathfrak{x}_o,\mathfrak{x}_1,t}.$$

Since, $h \in (0, 1)$, letting $n \to \infty$, $h^n \to 0$, we attain

$$\lim_{n\to\infty} \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} = 0. \tag{11}$$

If $\max\{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}, \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}, \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},t}, \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\} = \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}$. Then, inequality (10) becomes

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq a_1\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} + a_2\phi(\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}) + a_3\phi(\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}).$$

Since, $\phi(t) < t$, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \left(\frac{a_1 + a_3}{1 - a_2}\right)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.$$

Let $\left(\frac{a_1+a_3}{1-a_2}\right) = h$, we have

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.$$

Similarly,

$$\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} \leq h\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Therefore,

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h^2\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Following this pattern, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h^n\mathcal{N}_{\mathfrak{x}_o,\mathfrak{x}_1,t}.$$

Since, $h \in (0,1)$, letting $n \to \infty$, $h^n \to 0$, we attain

$$\lim_{n\to\infty} \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} = 0. \tag{12}$$

If $\max\{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}, N_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}, \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},t}, \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+N_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\} = \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},t}$. Then, inequality (10) becomes

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq a_1\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} + a_2\phi(\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},t}) + a_3\phi(\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}).$$

Utilizing Lemma 1 and the definition of ϕ , we attain

$$\begin{aligned}\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} &\leq (a_1+a_3)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} + a_2\phi\big(b(n-1)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} + b^2\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}\big)\\ &\leq \Big(\frac{a_1+a_3+a_2b(n-1)}{1-b^2a_2}\Big)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.\end{aligned}$$

Let $\frac{a_1+a_3+a_2b(n-1)}{1-b^2a_2} = h$, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.$$

Similarly,

$$\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} \leq h\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Therefore,

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h^2\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Following this pattern, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h^n\mathcal{N}_{\mathfrak{x}_0,\mathfrak{x}_1,t}.$$

Since, $0 \leq a_2 < \frac{1-a_1-a_3}{b^2+b(n-1)}$, $h \in (0,1)$, letting $n \to \infty$, $h^n \to 0$, we attain

$$\lim_{n\to\infty} \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} = 0. \tag{13}$$

If $\max\{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}, \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}, \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},t}, \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\} = \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}$. Then, inequality (10) becomes

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq a_1\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} + a_2\Big(\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\Big) + a_3\phi(\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}).$$

Since, $\phi(t) < t$, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq a_1\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} + a_2\Big(\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\Big) + a_3(\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t})$$

$$\leq \Big(\frac{2\mathfrak{a}_1+\mathfrak{a}_2+2\mathfrak{a}_3}{2-\mathfrak{a}_2}\Big)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.$$

Let $\left(\frac{2\mathfrak{a}_1+\mathfrak{a}_2+2\mathfrak{a}_3}{2-\mathfrak{a}_2}\right)=h$, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h\mathcal{N}_{x_{n-1},x_n,t}.$$

Similarly,

$$\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} \leq h\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Therefore,

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h^2\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Following this pattern, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h^n\mathcal{N}_{\mathfrak{x}_0,\mathfrak{x}_1,t}.$$

Since, $h \in (0,1)$, letting $n \to \infty$, $h^n \to 0$, we attain

$$\lim_{n\to\infty}\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} = 0. \tag{14}$$

Then, for $\mathfrak{k}$, $\mathfrak{l} \in \mathbb{N}$ so that $\mathfrak{l} > \mathfrak{k}$, using equation (14), condition (N2), and Lemma 1, we obtain

$$\begin{aligned}
\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_\mathfrak{l},t} &\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}\mathcal{N}_{\mathfrak{x}_\mathfrak{l},\mathfrak{x}_{\mathfrak{k}+1},t}\\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^2\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_\mathfrak{l},t}\\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^3(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},t} + \mathfrak{b}^3\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_\mathfrak{l},t}\\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^3(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},t} + \mathfrak{b}^4\mathcal{N}_{\mathfrak{x}_\mathfrak{l},\mathfrak{x}_{\mathfrak{k}+2},t}\\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^3(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},t} + \mathfrak{b}^5(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_{\mathfrak{k}+3},t}\\
&\quad + \mathfrak{b}^5\mathcal{N}_{\mathfrak{x}_\mathfrak{l},\mathfrak{x}_{\mathfrak{k}+3},t}\\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^3(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},t} + \mathfrak{b}^5(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_{\mathfrak{k}+3},t}\\
&\quad + \mathfrak{b}^7\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+3},\mathfrak{x}_{\mathfrak{k}+4},t} + \cdots .
\end{aligned}$$

Letting $\mathfrak{k}$, $\mathfrak{l} \to \infty$, we get $\lim_{\mathfrak{k},\,\mathfrak{l}\to\infty}\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_\mathfrak{l},t} = 0$, i.e., $\{\mathfrak{x}_n\}$ is a Cauchy sequence. Using the completeness hypotheses, $\lim_{\mathfrak{k},\mathfrak{l}\to\infty}\mathfrak{x}_\mathfrak{k} = \mathfrak{x}$, $\mathfrak{x} \in \mathcal{X}$.

Assume $\mathfrak{x}$ is not a fixed point of $\mathcal{S}$. Since, $\mathcal{X}$ is $\eta_\mathfrak{b}-$regular, then $\mathfrak{x}_n \to \mathfrak{x}$ as $n \to \infty$ and $\eta(\mathfrak{x}_n, \mathfrak{x}_{n+1}, t) \geq \mathfrak{b}$, which implies that $\eta(\mathfrak{x}_n, \mathfrak{x}, t) \geq \mathfrak{b}$, $n \in \mathbb{N} \cup \{0\}$. Using inequality (2), we attain

$$\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathcal{S}\mathfrak{x},t} = \mathcal{N}_{\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},t} \leq \eta(\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},t)\mathcal{N}_{\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},t}$$

$$\leq \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}} + \mathfrak{a}_2\, \phi\left(\max \begin{Bmatrix} \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \\ \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}+\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{2} \end{Bmatrix} \right)$$

$$+ \mathfrak{a}_3\, \phi\left(\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}[1+\sqrt{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}}]^2}{(1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}})^2}\right)$$

$$= \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}} + \mathfrak{a}_2\, \phi\left(\max \begin{Bmatrix} \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x}_{\mathfrak{k}},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x},\mathfrak{x}_{\mathfrak{k}},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \\ \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x}_{\mathfrak{k}},\mathfrak{t}}+\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{2} \end{Bmatrix} \right)$$

$$+ \mathfrak{a}_3\, \phi\left(\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}[1+\sqrt{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x}_{\mathfrak{k}},\mathfrak{t}}}]^2}{(1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}})^2}\right). \tag{15}$$

As $\mathfrak{k} \to \infty$, using Lemma 1 and condition (N1), we get $\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}} \leq 0$ which implies that $\mathcal{S}\mathfrak{x} = \mathfrak{x}$.
Let $\mathcal{S}$ has one more fixed point, i.e., $\mathcal{S}\mathfrak{x} = \mathfrak{x}$ and $\mathcal{S}\mathfrak{y} = \mathfrak{y}$, $(\mathfrak{x} \neq \mathfrak{y})$. Applying inequality (2), we obtain

$$\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} \leq \eta(\mathfrak{x},\mathfrak{y},\mathfrak{t})\mathrm{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}} \leq \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2\, \phi\left(\max \begin{Bmatrix} \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \\ \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}+\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{2} \end{Bmatrix} \right)$$

$$+ \mathfrak{a}_3\, \phi\left(\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}[1+\sqrt{\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}}]^2}{(1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}})^2}\right)$$

$$\leq \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2\, \phi\left(\max \begin{Bmatrix} \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, 0, \mathfrak{b}\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, \\ \frac{\mathfrak{b}\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}}{2} \end{Bmatrix} \right) + \mathfrak{a}_3\, \phi(\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}})$$

$$= \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2\, \phi\big(\mathfrak{b}\mathcal{N}_{\mathfrak{x},\mathfrak{v},\mathfrak{t}}\big) + \mathfrak{a}_3\, \phi(\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}).$$

Since, $\phi(\mathfrak{t}) < \mathfrak{t}$, $\mathfrak{t} > 0$, $\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} \leq (\mathfrak{a}_1 + \mathfrak{b}\mathfrak{a}_2 + \mathfrak{a}_3)\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}$, which is a contradiction. Thus, $\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} = 0$, i.e., $\mathfrak{x} = \mathfrak{y}$. So, a fixed point of $\mathcal{S}$ is unique. □

The next example is provided to justify Theorem 2.

Example 4 *Let* $\mathcal{X} = \mathbb{R}^+ \cup \{0\}$ *and function* $\mathcal{N} : \mathcal{X}^3 \times (0,\infty) \to [0,\infty)$ *be given by* $\mathcal{N}(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{t}) = \frac{1}{2}(|\mathfrak{x}-\mathfrak{y}| + |\mathfrak{x}-\mathfrak{z}| + |\mathfrak{y}-\mathfrak{z}|)^2$, *for every* $\mathfrak{t} > 0$*and* $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z} \in \mathcal{X}$. *Then,* $(\mathcal{X},\mathcal{N})$ *is a complete parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space with* $\mathfrak{b} = 2$ *and* $\mathfrak{n} = 3$. *Define* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *as*

$$\mathcal{S}\mathfrak{x} = \begin{cases} \frac{\mathfrak{x}^2}{16}, & \textit{if } \mathfrak{x} \in [0,\mathfrak{a}); \\ \frac{\mathfrak{x}}{15}, & \textit{if } \mathfrak{x} \in [\mathfrak{a},\infty), \end{cases}$$

$\mathfrak{x} \in \mathcal{X}$, $\mathfrak{a} \in (\frac{1}{4}, 1)$. *Now, define an* $\eta : \mathcal{X} \times \mathcal{X} \times (0,\infty) \to (0,\infty)$ *as*

$$\eta(\mathfrak{x},\mathfrak{y},\mathfrak{t}) = \begin{cases} 1, & \textit{if } \mathfrak{x},\ \mathfrak{y} \in \mathcal{X}; \\ \frac{3}{2}, & \textit{if otherwise} \end{cases}$$

and $\phi(\mathfrak{s}) = \frac{1}{2}\mathfrak{s}$. *Taking* $\mathfrak{a}_1 = \frac{1}{10} = \mathfrak{a}_2$ *and* $\mathfrak{a}_3 = \frac{1}{15}$, $\mathcal{S}$ *verifies the hypotheses of Theorem 2 and has a unique fixed point at* $\mathfrak{x} = 0$.

For $\mathfrak{a}_2 \in [0,1)$, $\mathfrak{a}_1 = \mathfrak{a}_3 = 0$ and $\eta(\mathfrak{x},\mathfrak{y},\mathfrak{t}) = 1$, Theorem 2 is an extension and an improvement of Ćirić [3] to a parametric $\mathcal{N}_{\mathfrak{b}}$–metric space wherein the involved mapping is not necessarily continuous.

The next result is slightly more interesting as here the max term is replaced by the min term in Theorem 2.10.

Theorem 3 *Let* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *be an* $\eta - \mathcal{SA}_{\min}$ *contraction* (3)) *in a complete parametric* $\mathcal{N}_{\mathfrak{b}}$–*metric space* $(\mathcal{X}, \mathcal{N})$ *with* $\mathfrak{b} \geq 1$ *satisfying*

- **(i)** $\mathcal{S}$ *is* η–*admissible of type* $\mathfrak{b}$;
- **(ii)** $\mathcal{X}$ *is* $\eta_{\mathfrak{b}}$–*regular;*
- **(iii)** *There exists a* $\mathfrak{x}_o \in \mathcal{X}$ *so that* $\eta(\mathfrak{x}_o, \mathfrak{f}\mathfrak{x}_o, \mathfrak{t}) \geq 1$, $\mathfrak{t} > 0$.

Then, $\mathcal{S}$ *has a unique fixed point in* $\mathcal{X}$.

Proof. The proof is easy and follows the pattern of Theorem 2. □

The following result is more interesting as a weaker control function ϕ is used with the η–admissibility of type $\mathfrak{b}$ function, without exploiting $\eta_{\mathfrak{b}}$–regularity, for a more general contractivity condition involving rational and irrational terms to establish a fixed point of discontinuous mapping.

Theorem 4 *Let* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *be a* $\mathcal{SA}_{\eta,\delta,\zeta}$– *contraction* (4) *in a complete parametric* $\mathsf{N}_{\mathfrak{b}}$*-metric space* $(\mathcal{X}, \mathcal{N})$ *satisfying*

- **(i)** $\mathcal{S}$ *is* η–*admissible of type* $\mathfrak{b}$;
- **(ii)** *There exists a* $\mathfrak{x}_o \in \mathcal{X}$ *so that* $\eta(\mathfrak{x}_o, \mathcal{S}\mathfrak{x}_o, \mathfrak{t}) \geq \mathfrak{b}$, $\mathfrak{t} > 0$.

Then, $\mathcal{S}$ *has a unique fixed point.*

Proof. Consider $\mathfrak{x}_o \in \mathcal{X}$ so that $\eta(\mathfrak{x}_o, \mathcal{S}\mathfrak{x}_o, \mathfrak{t}) \geq \mathfrak{b}$, $\mathfrak{t} > 0$. Let a sequence $\{\mathfrak{x}_n\}$ be constructed as $\mathfrak{x}_{n+1} = \mathcal{S}\mathfrak{x}_n$, $n \in \mathbb{N} \cup \{0\}$. As $\eta(\mathfrak{x}_o, \mathfrak{x}_1, \mathfrak{t}) = \eta(\mathfrak{x}_o, \mathcal{S}\mathfrak{x}_o, \mathfrak{t}) \geq \mathfrak{b}$ and $\eta(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{t}) = \eta(\mathcal{S}\mathfrak{x}_o, \mathcal{S}\mathcal{S}\mathfrak{x}_o) \geq \mathfrak{b}$, using (ii). Following the same pattern, we attain $\eta(\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{t}) \geq \mathfrak{b}$. If $\mathfrak{x}_n = \mathfrak{x}_{n+1}$, then we conclude that $\mathfrak{x}_n$ is a fixed point of $\mathcal{S}$.

Let $\mathfrak{x}_n \neq \mathfrak{x}_{n+1}$. Utilizing $\mathcal{SA}_{\eta\delta\zeta}$–contraction (4), we attain

$$\delta^{\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t}}} = \delta^{\mathcal{N}_{\mathcal{S}\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_n,\mathfrak{t}}} \leq [\eta(\mathfrak{x}_{n-1}, \mathfrak{x}_n, \mathfrak{t}) - 1 + \delta]^{\mathcal{N}_{\mathcal{S}\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_n,\mathfrak{t}}}$$

$$\leq \delta^{\zeta(\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t})\max\left\{\begin{array}{l}\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t},\mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_{n-1},t},\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_{n-1},t},\\ \mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_n,t},\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_{n-1},t}+\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_n,t}}{2}\end{array}\right\}}$$
$$= \delta^{\frac{1}{\mathfrak{b}}\max\{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t},\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t},\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\}}$$

Therefore,

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \frac{1}{\mathfrak{b}}\max\{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t},\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t},\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\}. \tag{16}$$

If $\max\{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t},\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t},\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\} = \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}$. Then, inequality (16) becomes

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \frac{1}{\mathfrak{b}}\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.$$

Similarly,

$$\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} \leq \frac{1}{\mathfrak{b}}\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Therefore,

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \frac{1}{\mathfrak{b}^2}\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Following the same pattern, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \frac{1}{\mathfrak{b}^n}\mathcal{N}_{\mathfrak{x}_0,\mathfrak{x}_1,t}.$$

Since, $\mathfrak{b} \geq 1$, letting $n \to \infty$, $\frac{1}{\mathfrak{b}^n} \to 0$, we attain

$$\lim_{n\to\infty}\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} = 0. \tag{17}$$

If $\max\{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t},\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t},\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\} = \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}$. Then, inequality (16) becomes

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \frac{1}{\mathfrak{b}}\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}, \quad \text{a contradiction.}$$

Therefore,

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} = 0.$$

If $\max\{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t},\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t},\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\} = \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}$. Then, inequality (16) becomes

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \left(\frac{1}{2\mathfrak{b}-1}\right)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.$$

Let $\left(\frac{1}{2\mathfrak{b}-1}\right) = \mathfrak{h}$, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \mathfrak{h}\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.$$

Similarly,

$$\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} \leq \mathfrak{h}\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Therefore,

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \mathfrak{h}^2\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Following the same pattern, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \mathfrak{h}^n\mathcal{N}_{\mathfrak{x}_0,\mathfrak{x}_1,t}.$$

Since, $\mathfrak{h} \in (0,1)$, letting $n \to \infty$, $\mathfrak{h}^n \to 0$, we attain

$$\lim_{n\to\infty} \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} = 0. \tag{18}$$

Now, claim that $\{\mathfrak{x}_n\}$ is a Cauchy sequence. Then, for $\mathfrak{k}$, $\mathfrak{l} \in \mathbb{N}$ so that $\mathfrak{l} > \mathfrak{k}$, using equation (18), condition (N2), and Lemma 1, we have

$$\begin{aligned}
\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_\mathfrak{l},t} &\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}\mathcal{N}_{\mathfrak{x}_\mathfrak{l},\mathfrak{x}_{\mathfrak{k}+1},t} \leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^2\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_\mathfrak{l},t} \\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^3(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},t} + \mathfrak{b}^3\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_\mathfrak{l},t} \\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^3(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},t} + \mathfrak{b}^4\mathcal{N}_{\mathfrak{x}_\mathfrak{l},\mathfrak{x}_{\mathfrak{k}+2},t} \\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^3(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},t} + \mathfrak{b}^5(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_{\mathfrak{k}+3},t} \\
&\quad + \mathfrak{b}^5\mathcal{N}_{\mathfrak{x}_\mathfrak{l},\mathfrak{x}_{\mathfrak{k}+3},t} \\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^3(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},t} + \mathfrak{b}^5(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_{\mathfrak{k}+3},t} \\
&\quad + \mathfrak{b}^7\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+3},\mathfrak{x}_{\mathfrak{k}+4},t} + \cdots .
\end{aligned}$$

Letting $\mathfrak{k}$, $\mathfrak{l} \to \infty$, we obtain

$$\lim_{\mathfrak{k},\mathfrak{l}\to\infty} \mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_\mathfrak{l},t} = 0.$$

Therefore, $\{\mathfrak{x}_n\}$ is a Cauchy sequence. Using the completeness hypotheses, $\lim_{\mathfrak{k}\to\infty} \mathfrak{x}_\mathfrak{k} = \mathfrak{x} \in \mathcal{X}$.

Assume $\mathfrak{x}$ is not a fixed point of $\mathcal{S}$. Applying inequality (4), we obtain

$$\begin{aligned}
\delta^{\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathcal{S}\mathfrak{x},t}} = \delta^{\mathcal{N}_{\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},t}} &\leq [\eta(\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},t) - 1 + \delta]^{\mathcal{N}_{\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},t}} \\
&\leq \delta^{\zeta(\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},t}) \max\left\{\begin{matrix} \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},t}, \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},t}, \mathcal{N}_{x,\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},t}, \\ \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t}, \frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},t}+\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t}}{2} \end{matrix}\right\}}
\end{aligned}$$

$$= \delta^{\frac{1}{\mathfrak{b}} \max\left\{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},t}, \mathcal{N}_{\mathfrak{x},\mathfrak{x}_{\mathfrak{k}+1},t}, \frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},t}+\mathcal{N}_{\mathfrak{x},\mathfrak{x}_{\mathfrak{k}+1},t}}{2}\right\}}. \tag{19}$$

As $\mathfrak{k} \to \infty$, using Lemma 1 and condition (N1), we get $\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t} \leq 0$, i.e., $\mathcal{S}\mathfrak{x} = \mathfrak{x}$.

Let $\mathcal{S}$ has one more fixed point, i.e., $\mathcal{S}\mathfrak{y} = \mathfrak{y}$, $(\mathfrak{x} \neq \mathfrak{y})$. Applying inequality (4), we obtain

$$\begin{aligned}
\delta^{\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}} &= \delta^{\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},t}} \leq [\eta(\mathfrak{x},\mathfrak{y},t) - 1 + \delta]^{\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},t}} \\
&\leq \delta^{\zeta(\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}) \max\left\{\begin{matrix} \mathcal{N}_{\mathfrak{x},\mathfrak{y},t}, \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t}, \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},t}, \\ \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},t}, \frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t}+\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},t}}{2} \end{matrix}\right\}} \\
&= \delta^{\frac{1}{\mathfrak{b}} \max\left\{\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}, \mathcal{N}_{\mathfrak{y},\mathfrak{x},t}, \frac{\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}+\mathcal{N}_{\mathfrak{y},\mathfrak{x},t}}{2}\right\}} \\
&= \delta^{\max\left\{\frac{1}{\mathfrak{b}}\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}, \mathcal{N}_{\mathfrak{x},\mathfrak{y},t}, \left(\frac{1+\mathfrak{b}}{2\mathfrak{b}}\right)\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}\right\}}.
\end{aligned}$$

Therefore,

$$\mathcal{N}_{\mathfrak{x},\mathfrak{y},t} \leq \max\{\frac{1}{\mathfrak{b}}\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}, \mathcal{N}_{\mathfrak{x},\mathfrak{y},t}, (\frac{1+\mathfrak{b}}{2\mathfrak{b}})\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}\}, \text{ a contradiction.}$$

Thus, $\mathcal{N}_{\mathfrak{x},\mathfrak{y},t} = 0$, i.e., $\mathfrak{x} = \mathfrak{y}$. Hence, a fixed point of $\mathcal{S}$ is unique. □

Next, we provide examples to demonstrate the authenticity of Theorem 4 besides exhibiting its supremacy over prior related outcomes.

Example 5 *Let $\mathcal{X}$ be the set of Lebesgue measurable functions on $[0,1]$ so that $\int_0^1 |\mathfrak{x}(t)|\partial t < 1$. Let $\mathcal{N} : \mathcal{X}^3 \times (0,\infty) \to [0,\infty)$ be*

$$\mathcal{N}(\mathfrak{x},\mathfrak{y},\mathfrak{z},t) = \frac{1}{3}\int_0^1 (|\mathfrak{x}-\mathfrak{y}| + |\mathfrak{x}-\mathfrak{z}| + |\mathfrak{y}-\mathfrak{z}|)^2 \partial t, \quad t > 0 \textit{ and } \mathfrak{x},\ \mathfrak{y},\ \mathfrak{z} \in \mathcal{X}.$$

Then, $(\mathcal{X},\mathcal{N})$ is a complete parametric $\mathcal{N}_{\mathfrak{b}}-$metric space with $\mathfrak{b} = 2$ and $\mathfrak{n} = 3$. Define $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ so that $\mathcal{S}\mathfrak{x} = \sin\mathfrak{x}$, $\mathfrak{x} \in \mathcal{X}$. Define $\eta : \mathcal{X}\times\mathcal{X}\times(0,\infty) \to (0,\infty)$ as $\eta(\mathfrak{x},\mathfrak{y},t) = e^{\mathfrak{x}+\mathfrak{y}+t}$. Let $\delta = 2$ and $\zeta : [0,\infty) \to (0,\frac{1}{2}]$ be given by $\zeta(\mathfrak{s}) = \frac{1}{2}$. Take $\mathfrak{x} = \frac{1}{2} = \mathfrak{y} = t$. Applying inequality (4), we get

$$[e^{\frac{3}{2}} - 1 + 2]^{\mathcal{N}_{\sin\frac{1}{2},\sin\frac{1}{2},\frac{1}{2}}} = [e^2 - 1 + 2]^0 = 1$$

$$\leq 2^{\zeta(\mathcal{N}_{\frac{1}{2},\frac{1}{2},\frac{1}{2}})\left(\max\left\{\mathcal{N}_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}, \mathcal{N}_{\frac{1}{2},\sin\frac{1}{2},\frac{1}{2}}, \mathcal{N}_{\frac{1}{2},\sin\frac{1}{2},\frac{1}{2}}, \mathcal{N}_{\frac{1}{2},\sin\frac{1}{2},\frac{1}{2}}, \frac{\mathcal{N}_{\frac{1}{2},\sin\frac{1}{2},\frac{1}{2}}+\mathcal{N}_{\frac{1}{2},\sin\frac{1}{2},\frac{1}{2}}}{2},\right\}\right)}.$$

Since, $\eta(\mathfrak{x},\mathfrak{y},t) = e^{\mathfrak{x}+\mathfrak{y}+t} > \mathfrak{b}$ implies that $\eta(\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},t) = e^{\mathcal{S}\mathfrak{x}+\mathcal{S}\mathfrak{y}+t} = e^{\sin\mathfrak{x}+\sin\mathfrak{y}+t} > \mathfrak{b}$. Therefore, $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ is $\eta-$admissible type $\mathfrak{b}$. Hence, $\mathcal{S}$ verifies the hypotheses of Theorem 4 and has a unique fixed point at $\mathfrak{x} = 0$. Clearly, $\mathcal{N}$ is not a parametric $\mathcal{S}-$metric.

Example 6 *Let $\mathcal{X} = \mathbb{R}^{+} \cup \{0\}$. Let a function $\mathcal{N} : \mathcal{X}^3 \times (0, \infty) \to [0, \infty)$ be*

$$\mathcal{N}(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, \mathfrak{t}) = \begin{cases} 0, & \text{if } \mathfrak{x} = \mathfrak{y} = \mathfrak{z}; \\ \frac{1}{3}, & \text{otherwise,} \end{cases},$$

for each $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z} \in \mathcal{X}$ and $\mathfrak{t} > 0$. Then, $(\mathcal{X}, \mathcal{N})$ is a complete parametric $\mathcal{N}_{\mathfrak{b}}-$metric space with $\mathfrak{b} = 2$ and $\mathfrak{n} = 3$. Define $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ as

$$\mathcal{S}\mathfrak{x} = \begin{cases} \frac{1}{3}, & \textit{if } \mathfrak{x} \in [0, \frac{1}{4}); \\ \frac{2}{3}, & \textit{if } \mathfrak{x} \in [\frac{1}{4}, \infty), \end{cases}$$

$\mathfrak{x} \in \mathcal{X}$. Define $\eta : \mathcal{X} \times \mathcal{X} \times (0, \infty) \to (0, \infty)$ as $\eta(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = 40\mathfrak{x} + \mathfrak{y} + \mathfrak{t}$, $\delta = 2$ and $\zeta : [0, \infty) \to (0, \frac{1}{2}]$ be given by $\zeta(\mathfrak{s}) = \frac{1}{2}$.

Case I. *If $\mathfrak{x} \in [0, \frac{1}{4})$ and $\mathfrak{x} = \mathfrak{y} = \mathfrak{t}$. Take $\mathfrak{x} = \frac{1}{5} = \mathfrak{y} = \mathfrak{t}$. Applying inequality (4), we get*

$$\begin{aligned} &[40\mathfrak{x} + \mathfrak{y} + \mathfrak{t} - \imath + 2]^{\mathcal{N}_{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}}} = [40\mathfrak{x} + \mathfrak{y} + \mathfrak{t} + \imath]^0 = 1 \\ &\leq 2^{\zeta\left(\mathrm{N}_{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}}\right)\left(\max\left\{\mathcal{N}_{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}}, \mathcal{N}_{\frac{1}{5}, \frac{1}{3}, \frac{1}{5}}, \mathcal{N}_{\frac{1}{5}, \frac{1}{3}, \frac{1}{5}}, \mathcal{N}_{\frac{1}{5}, \frac{1}{3}, \frac{1}{5}}, \frac{\mathcal{N}_{\frac{1}{5}, \frac{1}{3}, \frac{1}{5}} + \mathcal{N}_{\frac{1}{5}, \frac{1}{3}, \frac{1}{5}}}{2}\right\}\right)}. \end{aligned}$$

Case II. *If $\mathfrak{x} \in [\frac{1}{4}, \infty)$ and $\mathfrak{x} = \mathfrak{y} = \mathfrak{t}$. Take $\mathfrak{x} = \frac{1}{4} = \mathfrak{y} = \mathfrak{t}$. Applying inequality (4), we get*

$$\begin{aligned} &[40\mathfrak{x} + \mathfrak{y} + \mathfrak{t} - 1 + 2]^{\mathcal{N}_{\frac{2}{3}, \frac{2}{3}, \frac{2}{3}}} = [40\mathfrak{x} + \mathfrak{y} + \mathfrak{t} + 1]^0 = 1 \\ &\leq 2^{\zeta\left(\mathcal{N}_{\frac{2}{3}, \frac{2}{3}, \frac{2}{3}}\right)\left(\max\left\{\mathcal{N}_{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}}, \mathcal{N}_{\frac{1}{4}, \frac{2}{3}, \frac{1}{4}}, \mathcal{N}_{\frac{1}{4}, \frac{2}{3}, \frac{1}{4}}, \mathcal{N}_{\frac{1}{4}, \frac{2}{3}, \frac{1}{4}}, \frac{\mathcal{N}_{\frac{1}{4}, \frac{2}{3}, \frac{1}{4}} + \mathcal{N}_{\frac{1}{4}, \frac{2}{3}, \frac{1}{4}}}{2}\right\}\right)}. \end{aligned}$$

Case III. *If $\mathfrak{x} \in [0, \frac{1}{4})$, $\mathfrak{y} \in [\frac{1}{4}, \infty)$ and $\mathfrak{x} \neq \mathfrak{y} \neq \mathfrak{t}$. Take $\mathfrak{x} = \frac{1}{10}$, $\mathfrak{y} = \frac{1}{4}$ and $\mathfrak{t} = \frac{1}{9}$. Applying inequality (4), we get*

$$\begin{aligned} &[40\mathfrak{x} + \mathfrak{y} + \mathfrak{t} - 1 + 2]^{\mathcal{N}_{\frac{1}{3}, \frac{2}{3}, \frac{1}{9}}} \\ &\leq 2^{\zeta\left(\mathcal{N}_{\frac{1}{10}, \frac{1}{4}, \frac{1}{9}}\right)\left(\max\left\{\mathcal{N}_{\frac{1}{10}, \frac{1}{4}, \frac{1}{9}}, \mathcal{N}_{\frac{1}{10}, \frac{}{3}, \frac{1}{9}}, \mathcal{N}_{\frac{1}{4}, \frac{1}{3}, \frac{1}{9}}, \mathcal{N}_{\frac{1}{4}, \frac{2}{3}, \frac{1}{9}}, \frac{\mathcal{N}_{\frac{1}{10}, \frac{1}{3}, \frac{1}{9}} + \mathcal{N}_{\frac{1}{4}, \frac{2}{3}, \frac{1}{9}}}{2}\right\}\right)}. \end{aligned}$$

Now, $\eta(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = 40\mathfrak{x} + \mathfrak{y} + \mathfrak{t} > \mathfrak{b}$ implies that $\eta\big(\mathcal{S}\mathfrak{x}, \mathcal{S}\mathfrak{y}, \mathfrak{t}\big) = 40\mathcal{S}\mathfrak{x} + \mathcal{S}\mathfrak{y} + \mathfrak{t} > \mathfrak{b}$. Therefore, $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ is an $\eta-$admissible of type $\mathfrak{b}$. Hence, $\mathcal{S}$ verifies the hypotheses of Theorem 4 and has a unique fixed point at $\mathfrak{x} = \frac{2}{3}$.

For $\eta(\mathfrak{x},\mathfrak{y},\mathfrak{t}) = 1$, Theorem 4 is an extension and an improvement of Ćirić [3] to a parametric $\mathcal{N}_{\mathfrak{b}}-$metric space wherein the continuity of mapping is not essentially required.

Remark 2 (i) *If we take $\mathfrak{n} = 3$, $\mathfrak{b} = 1$ in Theorems 1, 2, 3 and 4, we get results in a parametric $\mathcal{S}-$metric space. Consequently, our outcomes generalize, improve, unify, and extend the known outcomes, choosing suitably the values of constants $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$, the functions ϕ and η (for instance: Bakhtin [1], Banach [2], Ćirić [3], Czerwik [4], Samet et al. [14], Sedghi et al. [15]-[17], Tas and Özgür [20, 21], Ughade et al. [30]). It is interesting to see that parametric $\mathcal{N}_{\mathfrak{b}}-$metric space is essentially greater, improved, and distinct than that of parametric $\mathcal{S}-$metric spaces or metric spaces due to the fact that it is defined on a domain with $\mathfrak{n}$ dimensions .*

(ii) *Clearly, $\mathcal{N}_{\mathfrak{b}}$ is not a parametric $\mathcal{S}-$metric and an underlying function is discontinuous in nature in the above Examples* 2.6, 2.11, *and* 2.15. *Consequently, our examples are not applicable to the recent and celebrated results existing in the literature wherein continuity of mapping is an essential condition and the underlying metric is other than the parametric $\mathcal{N}_{\mathfrak{b}}-$metric.*

(iii) *Theorems 1, 2, 3, and 4 along with the supporting Examples 2, 4, and 6, assert that continuity of self mapping is not a significant requirement for the survival of a unique fixed point of a $\mathcal{SA}$, $\eta - \mathcal{SA}$, $\eta - \mathcal{SA}_{\min}$, or $\mathcal{SA}_{\eta,\delta,\zeta}-$ contraction mapping in parametric $\mathcal{N}_{\mathfrak{b}}-$metric space. It is worth mentioning here that the continuity of a self mapping is an indispensable condition for proving a fixed point in most of the theorems existing in the literature (For a detailed discussion on the continuity, refer to Tomar and Karapinar [22]). Consequently, our outcomes reveal the prominence of novel contractions and mark supremacy.*

II. Existence of a unique fixed circle/fixed disc

Following Özgür [12], we introduce notions of the disc and fixed disc in parametric $\mathcal{N}_{\mathfrak{b}}-$ metric spaces and then apply our contractions to obtain a unique fixed circle/fixed disc. It is worth mentioning here that a fixed point of mapping is not always unique and the set of non-unique fixed points may form some geometrical shape like a circle or a disc or an ellipse or an elliptic disc. For more work on geometry, we may refer to [6]-[10], [25]-[26]. In the following, $(\mathcal{X},\mathcal{N})$ denotes the parametric $\mathcal{N}_{\mathfrak{b}}-$metric space.

Definition 9 [21] *A circle centred at $\mathfrak{x}_0$ having a radius $\mathfrak{r}$ in $(\mathcal{X}, \mathcal{N})$ is*

$$\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}} = \{\mathfrak{x} \in \mathcal{X} : \mathcal{N}_{\mathfrak{x},\mathfrak{x}_0,\mathfrak{t}} = \mathfrak{r}\}.$$

Definition 10 *We define a disc centred at $\mathfrak{x}_0$ having a radius $\mathfrak{r}$ in $(\mathcal{X}, \mathcal{N})$ as*

$$\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}} = \{\mathfrak{x} \in \mathcal{X} : \mathcal{N}_{\mathfrak{x},\mathfrak{x}_0,\mathfrak{t}} \leq \mathfrak{r}\}.$$

Definition 11 *For a self-mapping $\mathcal{S} : \mathcal{X} \longrightarrow \mathcal{X}$ in $(\mathcal{X}, \mathcal{N})$, if $\mathcal{S}\mathfrak{x} = \mathfrak{x}$, $\forall\ \mathfrak{x} \in \mathcal{C}^{\mathcal{X}_{\mathfrak{b}}}_{\mathfrak{x}_0,r}/\mathcal{D}^{\mathcal{X}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}}$, then $\mathcal{C}^{\mathcal{X}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}}/\mathcal{D}^{\mathcal{X}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}}$ is called a fixed circle/fixed disc of $\mathcal{S}$.*

Example 7 *Let $\mathcal{X} = \mathbb{R}^2$ and for $\mathfrak{n} = 3$, $\mathcal{N} : \mathcal{X}^3 \times (0, \infty) \longrightarrow \mathbb{R}^+$ be*

$$\mathcal{N}(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, \mathfrak{t}) = \mathfrak{t}^3(|\mathfrak{x} - \mathfrak{y}| + |\mathfrak{y} - \mathfrak{z}| + |\mathfrak{z} - \mathfrak{x}|)^2,$$

where $\mathfrak{x} = (\mathfrak{x}_1, \mathfrak{x}_2)$, $\mathfrak{y} = (\mathfrak{y}_1, \mathfrak{y}_2)$, $\mathfrak{z} = (\mathfrak{z}_1, \mathfrak{z}_2)$ and $|\mathfrak{x} - \mathfrak{y}| = |\mathfrak{x}_1 - \mathfrak{y}_1| + |\mathfrak{x}_2 - \mathfrak{y}_2|$. Obviously, $(\mathcal{X}, \mathcal{N})$ is a parametric $\mathcal{N}_{\mathfrak{b}}-$metric space with $\mathfrak{b} = 4$. Then, a circle centred at $\mathfrak{x}_0 = (0, 0)$ having a radius $\mathfrak{r} = 32$ is

$$\begin{aligned}
\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}} &= \{\mathfrak{x} \in \mathcal{X} : \mathcal{N}(\mathfrak{x}, \mathfrak{x}, \mathfrak{x}_0, \mathfrak{t}) = 32\} \\
&= \{\mathfrak{x} \in \mathcal{X} : \mathfrak{t}^3(|\mathfrak{x} - \mathfrak{x}| + |\mathfrak{x} - \mathfrak{x}_0| + |\mathfrak{x}_0 - \mathfrak{x}|)^2 = 32\} \\
&= \{\mathfrak{x} \in \mathcal{X} : 4\mathfrak{t}^3(|\mathfrak{x} - \mathfrak{x}_0|)^2 = 32\} \\
&= \{4\mathfrak{t}^3(|\mathfrak{x}_1| + |\mathfrak{x}_2|)^2 = 32\} \\
&= \{(|\mathfrak{x}_1| + |\mathfrak{x}_2|)^2 = \frac{8}{\mathfrak{t}^3}\}.
\end{aligned}$$

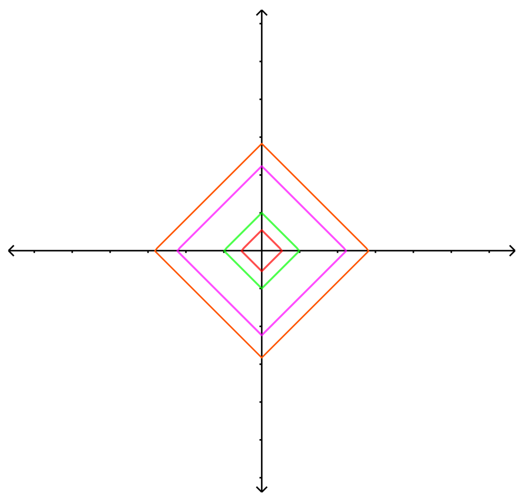

Fig. 1

Fig. 1 *Circles centred at* $(0,0)$ *with radius* 32 *for* $\mathfrak{t}=1,\ 1.18,\ 2,\ 3$ *are shown by the red, the green, the pink and the orange lines respectively.*

Similarly, a disc $\mathcal{D}^{\mathcal{N}_\mathfrak{b}}_{\mathfrak{x}_0,\mathfrak{r}}$ *centred at* $\mathfrak{x}_0=(0,0)$ *having radius* $\mathfrak{r}=32$ *is*

$$\mathcal{D}^{\mathcal{N}_\mathfrak{b}}_{\mathfrak{x}_0,\mathfrak{r}}=\{(|\mathfrak{x}_1|+|\mathfrak{x}_2|)^2\leq\frac{8}{\mathfrak{t}^3}\}$$

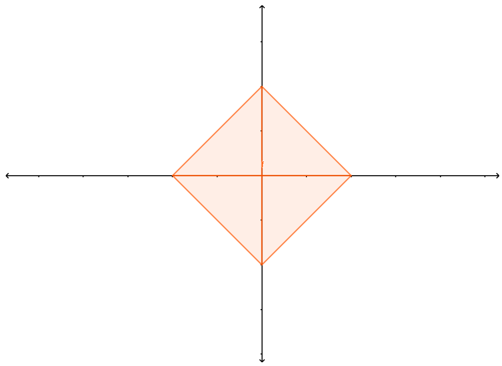

Fig. 2

Fig. 2 *Disc centred at* $(0,0)$ *with radius* 32 *for* $\mathfrak{t}=2$ *is shown by the pink shaded region.*

Now, we establish a unique fixed circle as an application of the $\mathcal{SA}-$contractive condition.

Theorem 5 *Let* $\mathcal{C}^{\mathcal{N}_\mathfrak{b}}_{\mathfrak{x}_0,\mathfrak{r}}$ *be a circle in* $(\mathcal{X},\mathcal{N})$*. Define* $\zeta:\mathbb{R}^+\cup\{0\}\longrightarrow\mathbb{R}$ *as:*

$$\zeta(\mathfrak{x})=\begin{cases}\mathfrak{x}-\mathfrak{r}, & \mathfrak{x}>0\\ 0, & \mathfrak{x}=0\end{cases}. \tag{20}$$

If a self mapping $\mathcal{S}:\mathcal{X}\longrightarrow\mathcal{X}$ *verifies*

(i) $\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathfrak{x}_0,\mathfrak{t}}=\mathfrak{r}$,

(ii) $\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}>\mathfrak{r},\ \mathfrak{x}\neq\mathfrak{y}$,

(iii) $\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}\leq\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}-\zeta(\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}),\ \mathfrak{x},\ \mathfrak{y}\in\mathcal{C}^{\mathcal{N}_\mathfrak{b}}_{\mathfrak{x}_0,\mathfrak{r}}$,

then $\mathcal{C}^{\mathcal{N}_\mathfrak{b}}_{\mathfrak{x}_0,\mathfrak{r}}$ *is a fixed circle of* $\mathcal{S}$*. Further if* $\mathcal{SA}-$*contractive condition* (1) *holds for* $\mathfrak{x}\in\mathcal{C}^{\mathcal{N}_\mathfrak{b}}_{\mathfrak{x}_0,\mathfrak{r}}$ *and* $\mathfrak{y}\in\mathcal{X}\setminus\mathcal{C}^{\mathcal{N}_\mathfrak{b}}_{\mathfrak{x}_0,\mathfrak{r}}$*, then* $\mathcal{C}^{\mathcal{N}_\mathfrak{b}}_{\mathfrak{x}_0,\mathfrak{r}}$ *is a unique fixed circle of* $\mathcal{S}$*.*

Proof. Let $\mathfrak{x} \in \mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}}$ be an arbitrary point. Using (i), $\mathcal{S}\mathfrak{x} \in \mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}}$. Now, we establish that $\mathfrak{x}$ is a fixed point of $\mathcal{S}$. Consider $\mathcal{S}\mathfrak{x} \neq \mathfrak{x}$. Taking $\mathfrak{y} = \mathcal{S}\mathfrak{x}$ in (ii)

$$\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}^2\mathfrak{x},\mathfrak{t}} > \mathfrak{r}. \tag{21}$$

Now, using (iii)

$$\begin{aligned} \mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}^2\mathfrak{x},\mathfrak{t}} &\leq \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}} - \zeta(\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}) \\ &= \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}} - \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}} + \mathfrak{r} \\ &= \mathfrak{r}, \end{aligned} \tag{22}$$

a contradiction. So a self mapping $\mathcal{S}$ fixes the circle $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}}$, i.e., a set of non-unique fixed points of $\mathcal{S}$ includes a circle.

Let there exist two fixed circles $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}_0}$ and $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_1,\mathfrak{r}_1}$ ($\mathfrak{r}_0 \neq \mathfrak{r}_1$) of $\mathcal{S}$, i.e., $\mathcal{S}$ satisfies the conditions (i) to (iii) for each of the circles $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}_0}$ and $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_1,\mathfrak{r}_1}$. Let $\mathfrak{x} \in \mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}}$ and $\mathfrak{y} \in \mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_1,\mathfrak{r}_1}$. Using (iv),

$$\begin{aligned} \mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}} = \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} &\leq \mathfrak{a}_1\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_3\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{x,y,\mathfrak{t}}} \\ &\quad + \mathfrak{a}_4\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_5\frac{\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} \\ &= \mathfrak{a}_1\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_3\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} \\ &\quad + \mathfrak{a}_4\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_5\frac{\mathcal{N}_{\mathfrak{y},\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} \\ &< \mathfrak{a}_1\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_3\mathcal{N}_{\mathfrak{y},\mathfrak{x},\mathfrak{t}} \\ &< (\mathfrak{a}_1 + \mathfrak{b}\mathfrak{a}_3)\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, \end{aligned}$$

a contradiction. Thus, $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}_0}$ is a unique fixed circle of $\mathcal{S}$. □

Example 8 *Let $\mathcal{X} = \mathbb{R}^2$ and for $\mathfrak{n} = 3$, $\mathcal{N} : \mathcal{X}^3 \times (0,\infty) \longrightarrow \mathbb{R}^+$ be*

$$\mathcal{N}(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{t}) = \mathfrak{t}^2\left(\left|\sin^{-1}\mathfrak{x} - \sin^{-1}\mathfrak{y}\right|^2 + \left|\sin^{-1}\mathfrak{y} - \sin^{-1}\mathfrak{z}\right|^2 + \left|\sin^{-1}\mathfrak{z} - \sin^{-1}\mathfrak{x}\right|^2\right),$$

where $\mathfrak{x} = (\mathfrak{x}_1,\mathfrak{x}_2)$, $\mathfrak{y} = (\mathfrak{y}_1,\mathfrak{y}_2)$, $\mathfrak{z} = (\mathfrak{z}_1,\mathfrak{z}_2)$ and

$$\left|\sin^{-1}\mathfrak{x} - \sin^{-1}\mathfrak{y}\right|^2 = \left|\sin^{-1}\mathfrak{x}_1 - \sin^{-1}\mathfrak{y}_1\right|^2 + \left|\sin^{-1}\mathfrak{x}_2 - \sin^{-1}\mathfrak{y}_2\right|^2.$$

Clearly, $(\mathcal{X},\mathcal{N})$ *is a parametric* $\mathcal{N}_{\mathfrak{b}}$*−metric space with* $\mathfrak{b}=4$*. Then, a circle centred at* $\mathfrak{x}_0=(0,0)$ *having radius* $\mathfrak{r}=8$ *is*

$$\begin{aligned}
\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}} &= \{\mathfrak{x}\in\mathcal{X}:\mathcal{N}(\mathfrak{x},\mathfrak{x},\mathfrak{x}_0,\mathfrak{t})=8\}\\
&= \Big\{\mathfrak{x}\in\mathcal{X}:\mathfrak{t}^2\Big(\left|\sin^{-1}\mathfrak{x}-\sin^{-1}\mathfrak{x}\right|^2+\left|\sin^{-1}\mathfrak{x}-\sin^{-1}\mathfrak{x}_0\right|^2\\
&\quad+\left|\sin^{-1}\mathfrak{x}_0-\sin^{-1}\mathfrak{x}\right|^2\Big)=8\Big\}\\
&= \Big\{2\mathfrak{t}^2\Big(\left|\sin^{-1}\mathfrak{x}_1\right|^2+\left|\sin^{-1}\mathfrak{x}_2\right|^2\Big)=8\Big\}\\
&= \Big\{\left|\sin^{-1}\mathfrak{x}_1\right|^2+\left|\sin^{-1}\mathfrak{x}_2\right|^2=\frac{4}{\mathfrak{t}^2}\Big\}.
\end{aligned}$$

For $\mathfrak{t}=2$*,*

$$\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}}=\{\mathfrak{x}\in\mathcal{X}:\left|\sin^{-1}\mathfrak{x}_1\right|^2+\left|\sin^{-1}\mathfrak{x}_2\right|^2=1\}. \tag{23}$$

Define a self mapping $\mathcal{S}:\mathcal{X}\longrightarrow\mathcal{X}$ *as* $\mathcal{S}\mathfrak{x}=\begin{cases}\mathfrak{x}, & \mathfrak{x}\in\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}}\\ (0,0.84), & \text{otherwise}\end{cases}$*. Then, a self mapping* $\mathcal{S}$ *verifies all the postulates of Theorem 5 and fixes a unique circle* $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}}$*, i.e., the set of non-unique fixed points of a self mapping* $\mathcal{S}$ *contains a unique fixed circle* $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}}$*.*

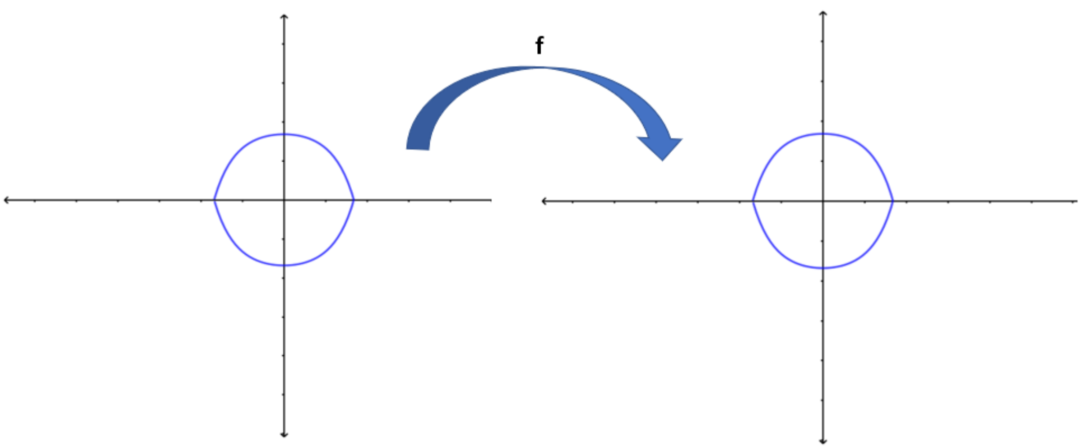

Fig. 3

Fig. 3 *The blue lines demonstrate a circle 23 which is fixed by a function* $\mathcal{S}$*.*

Theorem 6 *Let $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ be a disc in $(\mathcal{X},\mathcal{N})$. Define $\zeta:\mathbb{R}^+\cup\{0\}\longrightarrow\mathbb{R}$ as in Equation (20). If a self mapping $\mathcal{S}:\mathcal{X}\longrightarrow\mathcal{X}$ verifies*

(i) $\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathfrak{x}_o,\mathfrak{t}}\leq\mathfrak{r}$,

(ii) $\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}>\mathfrak{r},\ \mathfrak{x}\neq\mathfrak{y}$,

(iii) $\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}\leq\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}-\zeta(\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}),\ \mathfrak{x},\ \mathfrak{y}\in\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$,

then $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ is a fixed disc of $\mathcal{S}$.

(iv) *Further if, $\mathcal{S}\mathcal{A}$−contractive condition (1) holds for $\mathfrak{x}\in\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ and $\mathfrak{y}\in\mathcal{X}\setminus\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$, then $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ is a disc of maximum radius $\mathfrak{r}$, i.e., there is no fixed disc $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ of $\mathcal{S}$ having a radius greater than $\mathfrak{r}$.*

Proof. Following the pattern of Theorem 5, we can easily show that $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ is a fixed disc of $\mathcal{S}$.
Let there exist two fixed discs $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}_o}$ and $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_1,\mathfrak{r}_1}$; $\mathfrak{r}_o<\mathfrak{r}_1$ of $\mathcal{S}$; i.e., $\mathcal{S}$ satisfies the conditions $(\mathfrak{i})$ to $(\mathfrak{iii})$ for each of the discs $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}_o}$ and $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_1,\mathfrak{r}_1}$. Let $\mathfrak{x}\in\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}_o}$ and $\mathfrak{y}\in\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_1,\mathfrak{r}_1}$ such that $\mathfrak{y}\notin\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}_o}$. Using $(\mathfrak{iv})$,

$$\begin{aligned}
\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}=\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}&\leq\mathfrak{a}_1\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}+\mathfrak{a}_2\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}}+\mathfrak{a}_3\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{x,y,\mathfrak{t}}}+\mathfrak{a}_4\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}}\\
&\quad+\mathfrak{a}_5\frac{\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}}\\
&=\mathfrak{a}_1\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}+\mathfrak{a}_2\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}}+\mathfrak{a}_3\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}}+\mathfrak{a}_4\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}}\\
&\quad+\mathfrak{a}_5\frac{\mathcal{N}_{\mathfrak{y},\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}}\\
&<\mathfrak{a}_1\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}+\mathfrak{a}_3\mathcal{N}_{\mathfrak{y},\mathfrak{x},\mathfrak{t}}\\
&<(\mathfrak{a}_1+\mathfrak{b}\mathfrak{a}_3)\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}},
\end{aligned}$$

a contradiction. Hence, $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}_o}$ is a fixed disc of $\mathcal{S}$ having a maximum radius $\mathfrak{r}$. □

Example 9 *If in Example 8, a disc centred at $\mathfrak{x}_o=(0,0)$ having radius $\mathfrak{r}=8$*

is

$$\begin{aligned}\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}} &= \{\mathfrak{x}\in\mathcal{X}:\mathcal{N}(\mathfrak{x},\mathfrak{x},\mathfrak{x}_o,\mathfrak{t})\le 8\}\\ &= \Big\{\mathfrak{x}\in\mathcal{X}:\mathfrak{t}^2\Big(\left|\sin^{-1}\mathfrak{x}-\sin^{-1}\mathfrak{x}\right|^2+\left|\sin^{-1}\mathfrak{x}-\sin^{-1}\mathfrak{x}_0\right|^2\\ &\quad+\left|\sin^{-1}\mathfrak{x}_0-\sin^{-1}\mathfrak{x}\right|^2\Big)\le 8\Big\}\\ &= \Big\{2\mathfrak{t}^2\Big(\left|\sin^{-1}\mathfrak{x}_1\right|^2+\left|\sin^{-1}\mathfrak{x}_2\right|^2\Big)\le 8\Big\}\\ &= \Big\{\left|\sin^{-1}\mathfrak{x}_1\right|^2+\left|\sin^{-1}\mathfrak{x}_2\right|^2\le\frac{4}{\mathfrak{t}^2}\Big\}.\end{aligned}$$

For $\mathfrak{t}=2$,

$$\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}} = \{\mathfrak{x}\in\mathcal{X}:\left|\sin^{-1}\mathfrak{x}_1\right|^2+\left|\sin^{-1}\mathfrak{x}_2\right|^2\le 1\}. \tag{24}$$

Define a self mapping $\mathcal{S}:\mathcal{X}\longrightarrow\mathcal{X}$ *as* $\mathcal{S}\mathfrak{x}=\begin{cases}\mathfrak{x}, & \mathfrak{x}\in\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}\\ (0,0.84), & \textit{otherwise}\end{cases}$. *Then, a self mapping* $\mathcal{S}$ *verifies all the postulates of Theorem 6 except (iv) and fixes a disc* $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$, *i.e., the set of non-unique fixed points of a self mapping* $\mathcal{S}$ *contains a fixed disc* $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$.

Remark 3 (i) *Following a similar pattern, we may establish a unique fixed circle (greatest fixed disc) using* $\eta-\mathcal{SA}$, $\eta-\mathcal{SA}_{\min}$ *and* $\mathcal{SA}_{\eta,\delta,\zeta}-$ *contractions.*

(ii) *It is fascinating to see that the shape of a circle or a disc may change on changing the radius, the centre, or the involved metric (refer to figures 1 and 3).*

(iii) *It is not necessary that a circle or a disc in a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space is the same as a circle or a disc in a Euclidean space.*

(iv) *Noticeably, the radius of a fixed circle or a fixed disc does not depend on a centre and may not be maximal.*

(v) $\mathcal{S}\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}=\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ *or* $\mathcal{S}\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}=\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ *does not imply that* $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ *or* $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ *is a fixed circle or a fixed disc of* $\mathcal{S}$.

3 An application

Motivated by the fact that the theory of linear systems is the foundation of numerical linear algebra, which performs a significant role in chemistry, physics,

computer science, engineering, and economics, we resolve the system of linear equations in parametric $\mathcal{N}_b-$metric space using $\mathcal{SA}-$contraction condition (1).
Let $\mathcal{X} = \mathbb{R}^m$and $\mathcal{N} : \mathcal{X}^m \times (0, \infty) \to [0, \infty)$ be

$$\mathcal{N}(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, t) = t^3 \big(\Sigma_{i=1}^m |\mathfrak{x}_i - \mathfrak{y}_i| + \Sigma_{i=1}^m |\mathfrak{y}_i - \mathfrak{z}_i| + \Sigma_{i=1}^m |\mathfrak{z}_i - \mathfrak{x}_i| \big)^2,$$

where $\mathfrak{x} = (\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_m)$, $\mathfrak{y} = (\mathfrak{y}_1, \mathfrak{y}_2, \ldots, \mathfrak{y}_m)$, $\mathfrak{z} = (\mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_m) \in \mathbb{R}^m$. Obviously, $(\mathcal{X}, \mathcal{N})$ is a parametric $\mathcal{N}_b-$metric space with $b = 4$, $n = 3$.

Theorem 7 *The system of linear equations*

$$\begin{aligned} \mathfrak{c}_{11}\mathfrak{x}_1 + \mathfrak{c}_{12}\mathfrak{x}_2 + \cdots + \mathfrak{c}_{1m}\mathfrak{x}_m &= \mathfrak{d}_1 \\ \mathfrak{c}_{21}\mathfrak{x}_1 + \mathfrak{c}_{22}\mathfrak{x}_2 + \cdots + \mathfrak{c}_{2m}\mathfrak{x}_m &= \mathfrak{d}_2 \\ &\cdots \\ \mathfrak{c}_{m1}\mathfrak{x}_1 + \mathfrak{h}_{m2}\mathfrak{x}_2 + \cdots + \mathfrak{c}_{mm}\mathfrak{x}_m &= \mathfrak{d}_m, \end{aligned} \tag{25}$$

where $\mathfrak{c}_{ij}$, $\mathfrak{d}_i \in \mathbb{R}$, $i, j = 1, 2, \ldots, m$, *have a unique solution if* $\max_{j=1}^m \big(\Sigma_{i=1}^m |\mathfrak{c}_{ij}| \big)^2 < \lambda < 1$.

Proof. Define a self mapping $\mathcal{S} : \mathcal{X} \longrightarrow \mathcal{X}$ as $\mathcal{S}\mathfrak{x} = \mathcal{C}\mathfrak{x} + \mathfrak{d}$, $\mathfrak{x}$, $\mathfrak{d} \in \mathbb{R}^m$ and $\mathcal{C} = [\mathfrak{c}_{ij}]_{m\times m}$. First, we show that the self-mapping S satisfies Theorem 1. Then, the unique fixed point of the operator $\mathcal{S}$ is the unique solution of a system of linear equations (25). For $\mathfrak{x}$, $\mathfrak{y} \in \mathbb{R}^n$

$$\begin{aligned} \mathcal{N}(\mathcal{S}\mathfrak{x}, \mathcal{S}\mathfrak{x}, \mathcal{S}\mathfrak{y}, t) &= t^3 \big(\Sigma_{i=1}^m |\mathcal{S}\mathfrak{x}_i - \mathcal{S}\mathfrak{x}_i| + \Sigma_{i=1}^m |\mathcal{S}\mathfrak{x}_i - \mathcal{S}\mathfrak{y}_i| + \Sigma_{i=1}^m |\mathcal{S}\mathfrak{y}_i - \mathcal{S}\mathfrak{x}_i| \big)^2 \\ &= 4t^3 \big(\Sigma_{i=1}^m |\mathcal{S}\mathfrak{x}_i - \mathcal{S}\mathfrak{y}_i| \big)^2 \\ &= 4t^3 (\Sigma_{i=1}^m |\Sigma_{j=1}^m \mathfrak{c}_{ij}(\mathfrak{x}_j - \mathfrak{y}_j)|)^2 \\ &\leq 4t^3 \big(\Sigma_{i=1}^m (\Sigma_{j=1}^m |\mathfrak{c}_{ij}|^2 |\mathfrak{x}_j - \mathfrak{y}_j|^2) \big) \\ &\leq 4t^3 \Big(\max_{j=1}^m \Sigma_{i=1}^m |\mathfrak{c}_{ij}|^2 \Big) \Big(\Sigma_{j=1}^m |\mathfrak{x}_j - \mathfrak{y}_j|^2 \Big) \\ &< 4t^3 \Sigma_{j=1}^m |\mathfrak{x}_j - \mathfrak{y}_j|^2 \\ &= t^3 \mathcal{N}(\mathfrak{x}, \mathfrak{x}, \mathfrak{y}, t) \\ &\leq \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x},\mathfrak{y},t} + \mathfrak{a}_2 \frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t} \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},t}}{1 + \mathcal{N}_{\mathfrak{x},\mathfrak{y},t}} + \mathfrak{a}_3 \frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},t} \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},t}}{1 + \mathcal{N}_{\mathfrak{x},\mathfrak{y},t}} \\ &\quad + \mathfrak{a}_4 \frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t} \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},t}}{1 + \mathcal{N}_{\mathfrak{x},\mathfrak{y},t}} + \mathfrak{a}_5 \frac{\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},t} \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},t}}{1 + \mathcal{N}_{\mathfrak{x},\mathfrak{y},t}}, \end{aligned}$$

then $\mathcal{S}$ satisfies $\mathcal{SA}$−contraction (1) for $\mathfrak{a}_1 = \mathfrak{a}_3 = \mathfrak{a}_5 = \frac{1}{5}$ and $\mathfrak{a}_2 = \mathfrak{a}_4 = \frac{1}{10}$. Thus, Theorem 1 is verified and consequently, $\mathcal{S}$ has a unique fixed point. Hence, a system of linear equations (25) has a unique solution. □

Remark 4 *Similarly, we may apply* $\eta - \mathcal{SA}$, $\eta - \mathcal{SA}_{\min}$ *and* $\mathcal{SA}_{\eta,\delta,\zeta}$−*contractions to resolve a system of linear equations arising from modeling real-world problems. It is worth mentioning here that to model real-life or scientific problems by means of algebra we transform the known situation into mathematical assertions so that it evidently explains the correlation between the unknowns and the known information.*

4 Conclusion

We have established a unique fixed point, a unique fixed circle, and a greatest fixed disc for the $\mathcal{SA}$, $\eta - \mathcal{SA}$, $\eta - \mathcal{SA}_{\min}$, and $\mathcal{SA}_{\eta,\delta,\zeta}$− contractions in parametric $\mathcal{N}_{\mathfrak{b}}$−metric spaces, which is fascinating, generalized, and distinct than a usual metric space due to the fact that it is defined on a domain with $\mathfrak{n}$ dimensions. In the sequel, we have explored a new direction in the geometry of non-unique fixed points of discontinuous mapping in parametric $\mathcal{N}_{\mathfrak{b}}$−metric spaces. It is interesting to mention here that a circle or a disc in parametric $\mathcal{N}_{\mathfrak{b}}$−metric space changes its shape by changing the centre, the radius, or the metric under consideration. Our theorems are refined and extended variants of the well-known results. The examples furnished display an interesting characteristic of novel contractions that continuity of mappings is not mandatory for the survival of a fixed point. The paper is concluded by resolving the system of linear equations as an application to demonstrate the significance of our contractions in parametric $\mathcal{N}_{\mathfrak{b}}$−metric space. Essentially, these investigations unlock a distinct era in parametric $\mathcal{N}_{\mathfrak{b}}$−metric spaces.

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