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Grüss-type fractional inequality via Caputo-Fabrizio integral operator

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Abstract. In this article, the main objective is to establish the Grüss-type fractional integral inequalities by employing the Caputo-Fabrizio fractional integral.

1 Introduction

Grüss inequality which establishes a connection between the integral of the product of two functions and the product of the integrals of the two functions. In 1935, G. Grüss proved the following well known classical integral inequality, see [24, 27].

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Theorem 1 [27] *Let* $f, g : [a, b] \to R$ *be two integrable functions such that* $\phi \leq f(x) \leq \Phi$ *and* $\gamma \leq g(x) \leq \Gamma$ *for all* $x \in [a, b]$*;* ϕ, Φ, γ *and* Γ *are constant, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx.\frac{1}{b-a}\int_a^b g(x)dx\right| \leq \frac{1}{4}(\Gamma-\gamma)(\Phi-\phi), \tag{1}$$

where the constant $\frac{1}{4}$ *is sharp.*

During the last few years, several numerous generalizations, variants and extensions of the Grüss inequality have appeared in the literature, see [18, 19, 20, 22, 24, 25, 26, 27, 29, 30, 31, 40] and the references cited therein. Chinchane and Pachpatte [10], investigated some new fractional integral inequalities of the Grüss-type by considering the Saigo fractional integral operator. In [1, 21, 34, 35, 36] authors obtained some the Grüss-type inequalities by using different types of fractional integral operators. Fractional calculus is generalization of traditional calculus into non-integer differential and integral order. Fractional calculus is very important due to it's various application in field of science and technology, see [2, 4, 32, 37].

In [5, 6], Caputo and Fabrizio introduced a new fractional derivative and application of new time and spatial fractional derivative with exponential kernels. In literature very little work is reported on fractional integral inequalities using Caputo and Caputo-Fabrizio integral operator. Wang et al. [39] presented some properties of Caputo–Fabrizio fractional integral operator in the setting of-convex function. Recently, Nchama and et al. [28], proposed some fractional integral inequalities using the Caputo-Fabrizio fractional integral.

Recently, many researchers have worked on fractional integral inequalities using the Riemann-Liouville, Hadamard and q-fractional integral, see [3, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 23, 38]. In [16], Dahmani and et al. gave the following fractional integral inequality using the Riemann-Liouville fractional integral.

Motivated from [5, 6, 10, 16, 28, 39], our purpose in this paper is to propose some new results using the Caputo-Fabrizio integral operator. The paper has been organized as follows, in Section 2, we recall some auxiliary results related to the Caputo-Fabrizio integral operator. In Section 3, we investigate the Grüss-type fractional integral inequality using the Caputo-Fabrizio integral operator, in Section 4, we give the concluding remarks.

2 Preliminaries

In this section, we give some auxiliary results of fractional calculus that will be useful in this paper.

Definition 1 [6, 28] *Let $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. The Caputo-Fabrizio fractional integral of order α of a function f is defined by*

$$\mathcal{I}_{0,x}^{\alpha} f(x) = \frac{1}{\alpha} \int_0^x e^{-(\frac{1-\alpha}{\alpha})(x-s)} f(s) ds. \tag{2}$$

Definition 2 [6, 28] *Let $\alpha, a \in \mathbb{R}$ such that $0 < x < 1$. The Caputo-Fabrizio fractional derivative of order α of a function f is defined by*

$$\mathcal{I}_{a,x}^{\alpha} f(x) = \frac{1}{1-\alpha} \int_a^x e^{\frac{-\alpha}{1-\alpha}(x-s)} f'(s) ds. \tag{3}$$

Definition 3 *Let $\alpha > 0, \beta, \eta \in \mathbb{R}$, then the Saigo fractional integral $\mathcal{I}_{0,x}^{\alpha,\beta,\eta}[f(x)]$ of order α for a real valued continuous function $f(x)$ is defined by*

$$\mathcal{I}_{0,x}^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{\tau}{x}) f(\tau) d\tau, \tag{4}$$

where the function $F_1(-)$ is the Gaussian hypergeometric function defined by

$$F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{(x)^n}{n!},$$

and $(a)_n$ is the pochhammer symbol

$$(a)_n = a(a+1)...(a+n-1), (a)_0 = 1.$$

Definition 4 *The Hadamard fractional integral is defined by*

$${}^{H}\mathcal{I}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_1^x (\log \frac{x}{\tau})^{\alpha-1} f(\tau) \frac{d\tau}{\tau} \text{ for } \mathrm{Re}(\alpha) > 0,\ x > 1. \tag{5}$$

Definition 5 *The Riemann-Liouville fractional integral is defined by*

$$\mathcal{I}_{0,x}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau) d\tau. \tag{6}$$

3 Grüss-type fractional integral inequality

In this section, we investigate Grüss-type fractional integral inequalities involving the Caputo-Fabrizio fractional integer operator, for which assume that
(H_1) There exist two integrable function $\Phi_1(x)$, $\Phi_2(x)$ on $[0,\infty[$, such that

$$\Phi_1(x) \le u(x) \le \Phi_2(x), \ \text{ for all } \ x \in [0,\infty[.$$

(H_2) There exist two integrable function $\Psi_1(x)$,$\Psi_2(x)$ on $[0,\infty[$, such that

$$\Psi_1(x) \le v(x) \le \Psi_2(x), \ \text{ for all } \ x \in [0,\infty[.$$

Theorem 2 *Suppose that* u *be an integrable function defined on* $[0,\infty[$*, consider the condition* (H_1) *hold. Then for all* $x > 0$*,* $\alpha, \beta > 0$*, we have*

$$\begin{aligned} &\mathcal{I}_{0,t}^{\beta}\Phi_1(x)\mathcal{I}_{0,t}^{\alpha}u(x) + \mathcal{I}_{0,t}^{\alpha}\Phi_2(x)\mathcal{I}_{0,t}^{\beta}u(x) \ge \\ &\mathcal{I}_{0,t}^{\alpha}\Phi_2(x)\mathcal{I}_{0,t}^{\beta}\Phi_1(x) + \mathcal{I}_{0,t}^{\alpha}u(x)\mathcal{I}_{0,t}^{\beta}u(x). \end{aligned} \tag{7}$$

Proof. From condition (H_1), for all $\rho, \sigma \ge 0$, we obtain

$$(\Phi_2(\rho) - u(\rho))\,(u(\sigma) - \Phi_1(\sigma)) \ge 0. \tag{8}$$

that is

$$\Phi_2(\rho)u(\sigma) - \Phi_2(\rho)\Phi_1(\sigma) - u(\rho)u(\sigma) + u(\rho)\Phi_1(\sigma) \ge 0, \tag{9}$$

which implies that

$$\Phi_2(\rho)u(\sigma) + u(\rho)\Phi_1(\sigma) \ge \Phi_2(\rho)\Phi_1(\sigma) + u(\rho)u(\sigma). \tag{10}$$

Multiplying (10) by $\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}$, which is positive because $\rho \in (0,x)$, $x > 0$.

$$\begin{aligned} &u(\sigma)\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}\Phi_2(\rho) + \Phi_1(\sigma)\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}u(\rho) \\ &\ge \Phi_1(\sigma)\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}\Phi_2(\rho) + u(\sigma)\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}u(\rho). \end{aligned} \tag{11}$$

Now, integrating (11) with respect to ρ from 0 to x, we have

$$\begin{aligned} &\frac{u(\sigma)}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}\Phi_2(\rho)d\rho + \frac{\Phi_1(\sigma)}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}u(\rho)d\rho \\ &\ge \frac{\Phi_1(\sigma)}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}\Phi_2(\rho)d\rho + \frac{u(\sigma)}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}u(\rho)d\rho, \end{aligned} \tag{12}$$

therefore

$$\begin{aligned} & u(\sigma)\mathcal{I}_{0,t}^{\alpha}\Phi_2(x) + \Phi_1(\sigma)\mathcal{I}_{0,t}^{\alpha}u(x) \\ & \geq \Phi_1(\sigma)\mathcal{I}_{0,t}^{\alpha}\Phi_2(x) + u(\sigma)\mathcal{I}_{0,t}^{\alpha}u(x). \end{aligned} \tag{13}$$

Now, multiplying (13) by $\frac{1}{\beta}e^{-(\frac{1-\beta}{\beta})(x-\sigma)}$, which is positive because $\sigma \in (0, x)$, $x > 0$. Then integrating obtained result with respective to σ from 0 to x, we obtain

$$\begin{aligned} & \frac{\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)}{\beta}\int_0^x e^{-(\frac{1-\beta}{\beta})(x-\sigma)}u(\sigma)d\sigma + \frac{\mathcal{I}_{0,x}^{\alpha}u(x)}{\beta}\int_0^x e^{-(\frac{1-\beta}{\beta})(x-\sigma)}\Phi_1(\sigma)d\sigma \\ & \geq \frac{\mathcal{I}_{0,t}^{\alpha}\Phi_2(x)}{\beta}\int_0^x e^{-(\frac{1-\beta}{\beta})(x-\sigma)}\Phi_1(\sigma)d\sigma + \frac{\mathcal{I}_{0,t}^{\alpha}u(x)}{\beta}\int_0^x e^{-(\frac{1-\beta}{\beta})(x-\sigma)}u(\sigma)d\sigma. \end{aligned} \tag{14}$$

This completes the proof. □

Remark 1 *If* u *be an integrable function defined on* $[0, \infty[$*, such that* $\gamma \leq u(x) \leq \Gamma$*, for all* $x \in [0, \infty[$ *and* $\gamma, \Gamma \in \mathbb{R}$*. Then for all* $x > 0$ *and* $\alpha, \beta > 0$*, we have*

$$\begin{aligned} & \gamma\left(\frac{1}{1-\beta}\left[1 - e^{-(\frac{1-\beta}{\beta})x}\right]\right)\mathcal{I}_{0,x}^{\alpha}u(x) + \left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}_{0,x}^{\beta}u(x) \\ & \geq \Gamma\gamma\left(\frac{1}{1-\beta}\left[1 - e^{-(\frac{1-\beta}{\beta})x}\right]\right)\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right) + \mathcal{I}_{0,x}^{\alpha}u(x)\mathcal{I}_{0,x}^{\beta}u(x). \end{aligned} \tag{15}$$

Theorem 3 *If* u *and* v *be two integrable functions defined on* $[0, \infty[$*, Suppose that* (H_1) *and* (H_2) *holds. Then for all* $x > 0$*,* $\alpha, \beta > 0$*, the following inequalities satisfied*

$$\begin{aligned} & (h1)\ \mathcal{I}_{0,x}^{\beta}\Psi_1(x)\,\mathcal{I}_{0,x}^{\alpha}u(x) + \mathcal{I}_{0,x}^{\alpha}\Phi_2(x)\,\mathcal{I}_{0,x}^{\beta}v(x) \\ & \geq \mathcal{I}_{0,x}^{\beta}\Psi_1(x)\,\mathcal{I}_{0,x}^{\alpha}\Phi_2(x) + \mathcal{I}_{0,x}^{\alpha}u(x)\,\mathcal{I}_{0,x}^{\beta}v(x). \\ & (h2)\ \mathcal{I}_{0,x}^{\beta}\Phi_1(x)\,\mathcal{I}_{0,x}^{\alpha}v(x) + \mathcal{I}_{0,x}^{\alpha}\Psi_2(x)\,\mathcal{I}_{0,x}^{\beta}u(x) \\ & \geq \mathcal{I}_{0,x}^{\beta}\Phi_1(x)\,\mathcal{I}_{0,x}^{\alpha}\Psi_2(x) + \mathcal{I}_{0,x}^{\beta}u(x)\,\mathcal{I}_{0,x}^{\alpha}v(x). \\ & (h3)\ \mathcal{I}_{0,x}^{\alpha}\Phi_2(x)\,\mathcal{I}_{0,x}^{\beta}\Psi_2(x) + \mathcal{I}_{0,x}^{\alpha}u(x)\,\mathcal{I}_{0,x}^{\beta}v(x) \\ & \geq \mathcal{I}_{0,x}^{\alpha}\Phi_2(x)\,\mathcal{I}_{0,x}^{\beta}v(x) + \mathcal{I}_{0,x}^{\beta}\Psi_2(x)\,\mathcal{I}_{0,x}^{\alpha}u(x). \\ & (h4)\ \mathcal{I}_{0,x}^{\alpha}\Phi_1(x)\,\mathcal{I}_{0,x}^{\beta}\Psi_1(x) + \mathcal{I}_{0,x}^{\alpha}u(x)\,\mathcal{I}_{0,x}^{\beta}v(x) \\ & \geq \mathcal{I}_{0,x}^{\alpha}\Phi_1(x)\,\mathcal{I}_{0,x}^{\beta}v(x) + \mathcal{I}_{0,x}^{\beta}\Psi_1(x)\,\mathcal{I}_{0,x}^{\alpha}u(x). \end{aligned} \tag{16}$$

Proof. To prove (h1), we use condition (H_1) and (H_2), for all $x \in [0, \infty[$, we have

$$(\Phi_2(\rho) - u(\rho))\,(v(\sigma) - \Psi_1(\sigma)) \geq 0, \tag{17}$$

which implies that

$$\Phi_2(\rho)v(\sigma) + u(\rho)\Psi_1(\sigma) \geq \Phi_2(\rho)\Psi_1(\sigma) + u(\rho)u(\sigma). \tag{18}$$

Multiplying (18) by $\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}$, which is positive because $\rho \in (0, x)$, $x > 0$

$$\begin{aligned} &v(\sigma)\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}\Phi_2(\rho) + \Psi_1(\sigma)\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}u(\rho) \\ &\geq \Psi_1(\sigma)\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}\Phi_2(\rho) + v(\sigma)\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}u(\rho). \end{aligned} \tag{19}$$

Integrating (19) with respect to ρ from 0 to x, we get

$$\begin{aligned} &\frac{v(\sigma)}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}\Phi_2(\rho)d\rho + \frac{\Psi_1(\sigma))}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}u(\rho)d\rho \\ &\geq \frac{\Psi_1(\sigma)}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}\Phi_2(\rho)d\rho + \frac{v(\sigma)}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}u(\rho)d\rho, \end{aligned} \tag{20}$$

therefore

$$v(\sigma)\mathcal{I}^{\alpha}_{0,x}\Phi_2(x) + \Psi_1(\sigma)\mathcal{I}^{\alpha}_{0,x}u(x) \geq \Psi_1(\sigma)\mathcal{I}^{\alpha}_{0,x}\Phi_2(x) + v(\sigma)\mathcal{I}^{\alpha}_{0,x}u(x). \tag{21}$$

Multiplying both sides of (21) by $\frac{1}{\beta}e^{-(\frac{1-\beta}{\beta})(x-\sigma)}$, which is positive because $\sigma \in (0, x)$, $x > 0$. Then integrating resulting identity with respective σ over 0 to x, we obtain

$$\begin{aligned} &\frac{\mathcal{I}^{\alpha}_{0,x}\Phi_2(x)}{\beta}\int_0^x e^{-(\frac{1-\beta}{\beta})(x-\sigma)}v(\sigma)d\sigma + \frac{\mathcal{I}^{\alpha}_{0,x}u(x)}{\beta}\int_0^x e^{-(\frac{1-\beta}{\beta})(x-\sigma)}\Psi_1(\sigma)d\sigma \\ &\geq \frac{\mathcal{I}^{\alpha}_{0,x}\Phi_2(x)}{\beta}\int_0^x e^{-(\frac{1-\beta}{\beta})(x-\sigma)}v\Psi_1(\sigma)d\sigma + \frac{\mathcal{I}^{\alpha}_{0,x}u(x)}{\beta}\int_0^x e^{-(\frac{1-\beta}{\beta})(x-\sigma)}v(\sigma)d\sigma. \end{aligned} \tag{22}$$

This gives desired inequality (h1).
To prove (h2)-(h4), we use the following inequalities respectively
$(\Psi_2(\rho) - v(\rho))\,(u(\sigma) - \Phi_1(\sigma)) \geq 0.$
$(\Phi_2(\rho) - u(\rho))\,(v(\sigma) - \Psi_2(\sigma)) \leq 0.$
$(\Phi_2(\rho) - u(\rho))\,(v(\sigma) - \Psi_1(\sigma)) \leq 0.$ □

Remark 2 *If* u *and* v *be two integrable function defined on* $[0,\infty[$, *Assume that* (H_3) *There exist real constant* $\Gamma,\gamma,\Gamma^{'},\gamma^{'}$ *such that*

$$\gamma \le u(x) \le \Gamma \ \text{and}\ \gamma^{'} \le v(x) \le \Gamma^{'} \ \ \forall x \in [0,\infty[. \tag{23}$$

Then for all $x>0$, $\alpha,\beta>0$, *the following inequalities satisfied*

$$\begin{aligned}
&(h_1)\ \left(\gamma^{'}\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}u(x)+\left(\Gamma\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\beta}_{0,x}v(x)\\
&\ge\left(\gamma^{'}\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\left(\Gamma\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)+\mathcal{I}^{\alpha}_{0,x}u(x)\mathcal{I}^{\beta}_{0,x}v(x).\\
&(h_2)\ \left(\gamma\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}v(x)+\left(\Gamma^{'}\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\beta}_{0,x}u(x)\\
&\ge\left(\gamma\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\left(\Gamma^{'}\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)+\mathcal{I}^{\beta}_{0,x}u(x)\mathcal{I}^{\alpha}_{0,x}v(x).\\
&(h_3)\ \left(\Gamma\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\left(\Gamma^{'}\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)+\mathcal{I}^{\alpha}_{0,x}u(x)\mathcal{I}^{\beta}_{0,x}v(x)\\
&\ge\left(\Gamma\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\beta}_{0,x}v(x)+\left(\gamma\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}u(x).\\
&(h_4)\ \left(\gamma\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\left(\gamma^{'}\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)+\mathcal{I}^{\alpha}_{0,x}u(x)\mathcal{I}^{\beta}_{0,x}v(x)\\
&\ge\left(\gamma\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\mathcal{I}^{\beta}_{0,x}v(x)+\left(\gamma^{'}\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}u(x).
\end{aligned} \tag{24}$$

Lemma 1 *If* u *be an integrable function on* $[0,\infty)$, *and* $\Phi_1(x)$, $\Phi_2(x)$ *be two integrable functions on* $[0,\infty)$. *Assume that the condition* H_1 *holds. Then for all* $x>0$, $\alpha>0$, *we have*

$$\begin{aligned}
&\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}u^2(x)-(\mathcal{I}^{\alpha}_{0,x}u(x))^2\\
&=(\mathcal{I}^{\alpha}_{0,x}\Phi_2(x)-\mathcal{I}^{\alpha}_{0,x}u(x))\,(\mathcal{I}^{\alpha}_{0,x}u(x)-\mathcal{I}^{\alpha}_{0,x}\Phi_1(x))\\
&-\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\left[(\Phi_2(x)-u(x))(u(x)-\Phi_1(x))\right]\\
&+\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\Phi_1u(x)-\mathcal{I}^{\alpha}_{0,x}\Phi_1(x)\mathcal{I}^{\alpha}_{0,x}u(x)\\
&+\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\Phi_2u(x)-\mathcal{I}^{\alpha}_{0,x}\Phi_2(x)\mathcal{I}^{\alpha}_{0,x}u(x)
\end{aligned} \tag{25}$$

$$+\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)-\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}_{0,x}^{\alpha}\Phi_1\Phi_2(x).$$

Proof. Since u be an integrable function on $[0,\infty)$. For all ρ, $\sigma>0$, we have

$$\begin{aligned}&(\Phi_2(\sigma)-u(\sigma))\,(u(\rho)-\Phi_1(\rho))+(\Phi_2(\rho)-u(\rho))\,(u(\sigma)-\Phi_1(\sigma))\\&-(\Phi_2(\rho)-u(\rho))\,(u(\rho)-\Phi_1(\rho))-(\Phi_2(\sigma)-u(\sigma))\,(u(\sigma)-\Phi_1(\sigma))\\&=u^2(\rho)+u^2(\sigma)-2u(\tau)u(\rho)+\Phi_2(\sigma)u(\rho)+\Phi_1(\rho)u(\sigma)-\Phi_1(\rho)\Phi_2(\sigma)\\&+\Phi_1(\rho)u(\sigma)+\Phi_1(\sigma)u(\rho)-\Phi_1(\sigma)\Phi_2(\rho)-\Phi_2(\rho)u(\rho)+\Phi_1(\rho)\Phi_2(\rho)\\&-\Phi_1(\rho)u(\rho)-\Phi_2(\sigma)u(\sigma)+\Phi_1(\sigma)\Phi(\sigma)-\Phi_1(\sigma)u(\sigma).\end{aligned}\tag{26}$$

Multiplying both sides of (26) by $\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}$, which is positive because $\rho\in(0,x)$, $x>0$, integrating obtained result with respect to ρ from 0 to x, we have

$$\begin{aligned}&(\Phi_2(\sigma)-u(\sigma))\,(\mathcal{I}_{0,x}^{\alpha}u(x)-\mathcal{I}_{0,x}^{\alpha}\Phi_1(x))\\&+(\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)-\mathcal{I}_{0,x}^{\alpha}u(x))\,(u(\sigma)-\Phi_1(\sigma))\\&-\mathcal{I}_{0,x}^{\alpha}\left[(\Phi_2(x)-u(x))\,(u(x)-\Phi_1(x))\right]-\frac{(\ln x)^{\alpha}}{\Gamma(\alpha+1)}\,(\Phi_2(\sigma)\\&-u(\sigma))\,(u(\sigma)-\Phi_1(\sigma))\\&=\mathcal{I}_{0,x}^{\alpha}u^2(x)+u^2(\sigma)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\\&-2u(\sigma)\mathcal{I}_{0,x}^{\alpha}u(x)+\Phi_2(\sigma)\mathcal{I}_{0,x}^{\alpha}u(x)\\&+u(\sigma)\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)-\Phi_2(\sigma)\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)+u(\sigma)\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)\\&+\Phi_1(\sigma)\mathcal{I}_{0,x}^{\alpha}u(x)-\Phi_1(\sigma)\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)-\mathcal{I}_{0,x}^{\alpha}\Phi_2u(x)\\&+\mathcal{I}_{0,x}^{\alpha}\Phi_1\Phi_2(x)-\mathcal{I}_{0,x}^{\alpha}\Phi_1u(x)-\Phi_2(\sigma)u(\sigma)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\\&+\Phi_1(\sigma)\Phi_2(\sigma)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\\&-\Phi_1(\sigma)u(\sigma)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right).\end{aligned}\tag{27}$$

Again, multiplying (27) by $\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\sigma)}$, which is positive because $\sigma\in(0,x)$,

$x > 0$, integrating obtained result with respect to ρ from 0 to x, we have

$$\begin{aligned}
&(\mathcal{I}^{\alpha}_{0,x}\Phi_2(x) - \mathcal{I}^{\alpha}_{0,x}u(x))\,(\mathcal{I}^{\alpha}_{0,x}u(x) - \mathcal{I}^{\alpha}_{0,x}\Phi_1(x))\\
&+ (\mathcal{I}^{\alpha}_{0,x}\Phi_2(x) - \mathcal{I}^{\alpha}_{0,x}u(x))\,(\mathcal{I}^{\alpha}_{0,x}u(x) - \mathcal{I}^{\alpha}_{0,x}\Phi_1(x))\\
&- \left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\left[(\Phi_2(x) - u(x))\,(u(x) - \Phi_1(x))\right]\\
&- \left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\left[(\Phi_2(x) - u(x))\,(u(x) - \Phi_1(x))\right]\\
&= \mathcal{I}^{\alpha}_{0,x}u^2(x)\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right) + \mathcal{I}^{\alpha}_{0,x}u^2(x)\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\\
&- 2\mathcal{I}^{\alpha}_{0,x}u(x)\mathcal{I}^{\alpha}_{0,x}u(x) + \mathcal{I}^{\alpha}_{0,x}\Phi_2(x)\mathcal{I}^{\alpha}_{0,x}u(x) + \mathcal{I}^{\alpha}_{0,x}u(x)\mathcal{I}^{\alpha}_{0,x}\Phi_1(x)\\
&- \mathcal{I}^{\alpha}_{0,x}\Phi_2(x)\mathcal{I}^{\alpha}_{0,x}\Phi_1(x)\\
&+ \mathcal{I}^{\alpha}_{0,x}u(x)\,{}_{H}D^{-\alpha}_{1,x}\Phi_2(x) + \mathcal{I}^{\alpha}_{0,x}\Phi_1(x)\mathcal{I}^{\alpha}_{0,x}u(x) - \mathcal{I}^{\alpha}_{0,x}\Phi_1(x)\mathcal{I}^{\alpha}_{0,x}\Phi_2(x)\\
&- \mathcal{I}^{\alpha}_{0,x}\Phi_2 u(x)\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right) + \mathcal{I}^{\alpha}_{0,x}\Phi_1\Phi_2(x)\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\\
&- \mathcal{I}^{\alpha}_{0,x}\Phi_1 u(x)\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\\
&- \mathcal{I}^{\alpha}_{0,x}\Phi_2 u(x)\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right) + \mathcal{I}^{\alpha}_{0,x}\Phi_1\Phi_2(x)\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\\
&- \mathcal{I}^{\alpha}_{0,x}\Phi_1 u(x)\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right),
\end{aligned} \tag{28}$$

which implies that

$$\begin{aligned}
&2\,(\mathcal{I}^{\alpha}_{0,x}\Phi_2(x) - \mathcal{I}^{\alpha}_{0,x}u(x))\,(\mathcal{I}^{\alpha}_{0,x}u(x) - \mathcal{I}^{\alpha}_{0,x}\Phi_1(x))\\
&- 2\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\left[(\Phi_2(x) - u(x))\,(u(x) - \Phi_1(x))\right]\\
&= 2\mathcal{I}^{\alpha}_{0,x}u^2(x)\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right) - 2(\mathcal{I}^{\alpha}_{0,x}u(x))^2 + 2\mathcal{I}^{\alpha}_{0,x}\Phi_2(x)\mathcal{I}^{\alpha}_{0,x}u(x)\\
&+ 2\mathcal{I}^{\alpha}_{0,x}u(x)\mathcal{I}^{\alpha}_{0,x}\Phi_1(x) - 2\mathcal{I}^{\alpha}_{0,x}\Phi_2(x)\mathcal{I}^{\alpha}_{0,x}\Phi_1(x)\\
&- 2\mathcal{I}^{\alpha}_{0,x}\Phi_2 u(x)\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right) + 2\mathcal{I}^{\alpha}_{0,x}\Phi_1\Phi_2(x)\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\\
&- 2\mathcal{I}^{\alpha}_{0,x}\Phi_1 u(x)\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right).
\end{aligned} \tag{29}$$

This completes the proof. □

If $\Phi_1(x) = \gamma$ and $\Phi_2(x) = \Gamma$; $\gamma, \Gamma \in \mathbb{R}$ for all $x \in [0, \infty)$, then inequality (25) reduces to following lemma.

Lemma 2 *If $\gamma, \Gamma \in \mathbb{R}$, and $u(x)$ be an integrable function on $[0, \infty)$ and satisfying the condition $\gamma \le u(x) \le \Gamma$. Then for all $x > 0$, $\alpha > 0$, we have*

$$\begin{aligned}
&\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}u^2(x) - (\mathcal{I}^{\alpha}_{0,x}u(x))^2 \\
&= \left(\Gamma\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right) - \mathcal{I}^{\alpha}_{0,x}u(x)\right) \times \\
&\left(\mathcal{I}^{\alpha}_{0,x}u(x) - \left(\gamma\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\right) \\
&- \left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\left((\Gamma - u(x))(u(x) - \gamma)\right).
\end{aligned} \tag{30}$$

Theorem 4 *Let u and v be two integrable functions on $[0, \infty)$, and $\Phi_1(x)$, $\Phi_2(x), \Psi_1(x)$ and $\Psi_2(x)$ are four integrable functions on $[0, \infty)$ satisfying the conditions H_1 and H_2 on $[0, \infty)$. Then for all $x > 0$, $\alpha > 0$, we have*

$$\begin{aligned}
&\left|\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}uv(x) - \mathcal{I}^{\alpha}_{0,x}u(x)\mathcal{I}^{\alpha}_{0,x}v(x)\right| \\
&\le \sqrt{R(u, \Phi_1(x), \Phi_2(x))R(v, \Psi_1(x), \Psi_2(x))}.
\end{aligned} \tag{31}$$

where $R(a, b, c)$ is defined by

$$\begin{aligned}
R(a, b, c) = {}& (\mathcal{I}^{\alpha}_{0,x}c(x) - \mathcal{I}^{\alpha}_{0,x}a(x))\,(\mathcal{I}^{\alpha}_{0,x}a(x) - \mathcal{I}^{\alpha}_{0,x}b(x)) \\
&+ \left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}ba(x) - \mathcal{I}^{\alpha}_{0,x}b(x)\mathcal{I}^{\alpha}_{0,x}a(x) \\
&+ \left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}ca(x) - \mathcal{I}^{\alpha}_{0,x}c(x)\mathcal{I}^{\alpha}_{0,x}a(x) \\
&+ \mathcal{I}^{\alpha}_{0,x}b(x)\mathcal{I}^{\alpha}_{0,x}c(x) + \left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}bc(x).
\end{aligned} \tag{32}$$

Proof Let u and v be two functions defined on $[0, \infty)$ satisfying the condition H_1 and H_2. Define

$$H(\rho, \sigma) := (u(\rho) - u(\sigma))\,(v(\rho) - v(\sigma))\,; \rho, \sigma \in (0, x), x > 0, \tag{33}$$

it follows that

$$H(\rho,\sigma) := u(\rho)v(\rho) - u(\rho)v(\sigma) - u(\sigma)v(\rho) + u(\sigma)v(\sigma). \tag{34}$$

Now, multiplying (34) by $\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}$, which is positive because $\rho \in (0,x)$, $x > 0$, integrating obtained result with respect to ρ from 0 to x, we have

$$\begin{aligned}&\frac{1}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}H(\rho,\sigma)d\rho\\ &= \mathcal{I}^{\alpha}_{0,x}uv(x) - u(\tau)\mathcal{I}^{\alpha}_{0,x}u(x) - u(\sigma)\mathcal{I}^{\alpha}_{0,x}v(x) + u(\sigma)v(\sigma)\mathcal{I}^{\alpha}_{0,x}(1).\end{aligned} \tag{35}$$

Multiplying (35) by $\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\sigma)}$, which is positive because $\sigma \in (0,x)$, $x > 0$, integrating obtained result with respect to σ from 0 to x, we have

$$\begin{aligned}&\frac{1}{\alpha^2}\int_0^t\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}e^{-(\frac{1-\alpha}{\alpha})(x-\sigma)}H(\rho,\sigma)d\rho d\sigma\\ &= 2\left(\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}uv(x) - \mathcal{I}^{\alpha}_{0,x}u(x)\mathcal{I}^{\alpha}_{0,x}u(x)\right).\end{aligned} \tag{36}$$

Applying the Cauchy-Schwarz inequality to (36), we have

$$\begin{aligned}&\left(\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}uv(x) - \mathcal{I}^{\alpha}_{0,x}u(x)\mathcal{I}^{\alpha}_{0,x}v(x)\right)^2 \leq\\ &\left(\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}u^2(x) - (\mathcal{I}^{\alpha}_{0,x}u(x))^2\right) \times\\ &\left(\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}v^2(x) - \mathcal{I}^{\alpha}_{0,x}v(x)^2\right),\end{aligned} \tag{37}$$

since $(\Phi_2(x) - u(t))(u(t) - \Phi_1(x)) \geq 0$ and $(\Psi_2(x) - v(t))(v(t) - \Psi_1(x)) \geq 0$, we have

$$\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}(\Phi_2(x) - u(t))(u(t) - \Phi_1(x)) \geq 0, \tag{38}$$

and

$$\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}(\Psi_2(x) - v(t))(v(t) - \Psi_1(x)) \geq 0. \tag{39}$$

Thus, we have

$$\begin{aligned}
&\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}_{0,x}^{\alpha}u^2(x)-(\mathcal{I}_{0,x}^{\alpha}u(x))^2\\
&\leq (\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)-\mathcal{I}_{0,x}^{\alpha}u(x))(\mathcal{I}_{0,x}^{\alpha}u(x)-\mathcal{I}_{0,x}^{\alpha}\Phi_1(x))\\
&+\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}_{0,x}^{\alpha}\Phi_1u(x)-\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)\mathcal{I}_{0,x}^{\alpha}u(x)\\
&+\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}_{0,x}^{\alpha}\Phi_2u(x)-\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)\mathcal{I}_{0,x}^{\alpha}u(x)\\
&+\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)-\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}_{0,x}^{\alpha}\Phi_1\Phi_2(x)\\
&=R(u,\Phi_1,\Phi_2),
\end{aligned}\tag{40}$$

and

$$\begin{aligned}
&\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}_{0,x}^{\alpha}u^2(x)-(\mathcal{I}_{0,x}^{\alpha}u(x))^2\\
&\leq (\mathcal{I}_{0,x}^{\alpha}\Psi_2(x)-\mathcal{I}_{0,x}^{\alpha}u(x))(\mathcal{I}_{0,x}^{\alpha}u(x)-\mathcal{I}_{0,x}^{\alpha}\Psi_1(x))\\
&+\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}_{0,x}^{\alpha}\Psi_1u(x)-\mathcal{I}_{0,x}^{\alpha}\Psi_1(x)\mathcal{I}_{0,x}^{\alpha}u(x)\\
&+\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}_{0,x}^{\alpha}\Psi_2u(x)-\mathcal{I}_{0,x}^{\alpha}\Psi_2(x)\mathcal{I}_{0,x}^{\alpha}u(x)\\
&+\mathcal{I}_{0,x}^{\alpha}\Psi_1(x)\mathcal{I}_{0,x}^{\alpha}\Psi_2(x)-\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}_{0,x}^{\alpha}\Psi_1\Psi_2(x)\\
&=R(u,\Psi_1,\Psi_2).
\end{aligned}\tag{41}$$

Combining (37), (40) and (41), we get (31).

4 Concluding Remarks

Nchama et al. [28], investigated some integral inequalities by considering Caputo-Fabrizio fractional integral operator. In [6] Caputo and Farbrizio introduced a new fractional differential and integral operator. Motivated by the above work, here we studied Grüss-type inequalities and other fractional inequalities by considering Caputo-Fabrizio fractional integral operator. By the help of this study we establish more general inequalities than in the classical cases. The inequalities investigated in this paper give some contribution to the fields of fractional calculus and Caputo-Fabrizio fractional integral operator.

These inequalities are expected to lead to some application for finding bounds and uniqueness of solutions in fractional differential equations.

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