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Partial sums of the Rabotnov function

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Abstract. This article deals with the ratio of normalized Rabotnov function $\mathbb{R}_{\alpha,\beta}(z)$ and its sequence of partial sums $(\mathbb{R}_{\alpha,\beta})_m(z)$. Several examples which illustrate the validity of our results are also given.

1 Introduction

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.
Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ which consists of univalent functions in $\mathcal{U}$. Consider the function $R_{\alpha,\beta}(z)$ defined by

$$R_{\alpha,\beta}(z) = z^{\alpha} \sum_{n=0}^{\infty} \frac{\beta^n}{\Gamma((n+1)(1+\alpha))} z^{n(1+\alpha)} \tag{2}$$

where Γ stands for the Euler gamma function and $\alpha \geq 0$, $\beta \in \mathbb{C}$ and $z \in \mathcal{U}$. This function was introduced by Rabotnov in 1948 [14] and is therefore known

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as the Rabotnov function.
The function defined by (2) does not belong to the class $\mathcal{A}$. Therefore, we consider the following normalization of the Rabotnov function $R_{\alpha,\beta}(z)$: for $z \in \mathcal{U}$,

$$\mathbb{R}_{\alpha,\beta}(z) = \Gamma(1+\alpha)\, z^{1/(1+\alpha)} R_{\alpha,\beta}\left(z^{1/(1+\alpha)}\right) = \sum_{n=0}^{\infty} \frac{\beta^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))} z^{n+1} \quad (3)$$

where $\alpha \geq 0$ and $\beta \in \mathbb{C}$.
Note that some special cases of $\mathbb{R}_{\alpha,\beta}(z)$ are:

$$\begin{cases} \mathbb{R}_{0,-\frac{1}{3}}(z) = ze^{-\frac{z}{3}} \\ \mathbb{R}_{1,\frac{1}{2}}(z) = \sqrt{2z}\sinh\sqrt{\frac{z}{2}} \\ \mathbb{R}_{1,-\frac{1}{4}}(z) = 2\sqrt{z}\sin\frac{\sqrt{z}}{2} \\ \mathbb{R}_{1,1}(z) = \sqrt{z}\sinh\sqrt{z} \\ \mathbb{R}_{1,2}(z) = \frac{\sqrt{2z}\sinh\sqrt{2z}}{2}. \end{cases} \quad (4)$$

For various interesting developments concerning partial sums of analytic univalent functions, the reader may be (for examples) refered to the works of Kazımoğlu et al. [7], Çağlar and Orhan [1], Lin and Owa [9], Deniz and Orhan [3, 4], Owa et al. [13], Sheil-Small [17], Silverman [18] and Silvia [20]. Recently, some researchers have studied on partial sums of special functions (see [2, 7, 8, 12, 16, 22]).
In this paper, we investigate the ratio of normalized Rabotnov function $R_{\alpha,\beta}(z)$ and its derivative defined by (3) to their sequences of partial sums

$$\begin{cases} (\mathbb{R}_{\alpha,\beta})_0(z) = z \\ (\mathbb{R}_{\alpha,\beta})_m(z) = z + \sum\limits_{n=1}^{m} A_n z^{n+1}, \ m \in \mathbb{N} = \{1,2,3,\ldots\}, \end{cases} \quad (5)$$

where

$$A_n = \frac{\beta^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))}, \ \alpha \geq 0 \text{ and } \beta \in \mathbb{C}.$$

We obtain lower bounds on ratios like

$$\Re\left\{\frac{\mathbb{R}_{\alpha,\beta}(z)}{(\mathbb{R}_{\alpha,\beta})_m(z)}\right\}, \ \Re\left\{\frac{(\mathbb{R}_{\alpha,\beta})_m(z)}{\mathbb{R}_{\alpha,\beta}(z)}\right\}, \ \Re\left\{\frac{\mathbb{R}'_{\alpha,\beta}(z)}{(\mathbb{R}_{\alpha,\beta})'_m(z)}\right\}, \ \Re\left\{\frac{(\mathbb{R}_{\alpha,\beta})'_m(z)}{\mathbb{R}'_{\alpha,\beta}(z)}\right\}.$$

Several examples will be also given.
Results concerning partial sums of analytic functions may be found in [5, 15].

2 Main results

In order to prove our results we need the following lemma.

Lemma 1 *Let* $\alpha \geq 0$ *and* $\beta \in \mathbb{C}$. *Then the function* $\mathbb{R}_{\alpha,\beta}(z)$ *satisfies the following inequalities:*

$$|\mathbb{R}_{\alpha,\beta}(z)| \leq e^{\frac{|\beta|}{1+\alpha}} \quad (z \in \mathcal{U}) \tag{6}$$

$$\left|\mathbb{R}'_{\alpha,\beta}(z)\right| \leq \left(1 + \frac{|\beta|}{1+\alpha}\right) e^{\frac{|\beta|}{1+\alpha}} \quad (z \in \mathcal{U}). \tag{7}$$

Proof. By using the inductive method, we easily see that

$$(1+\alpha)^n (n)!\Gamma(1+\alpha) \leq \Gamma((1+\alpha)(n+1))$$

and thus

$$\frac{\Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))} \leq \frac{1}{(1+\alpha)^n (n)!}, \quad \alpha \geq 0, \ n \in \mathbb{N}. \tag{8}$$

Making use of (8) and also the well-known triangle inequality, for $z \in \mathcal{U}$, we have

$$\begin{aligned} |\mathbb{R}_{\alpha,\beta}(z)| &= \left| z + \sum_{n=1}^{\infty} \frac{\beta^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))} z^{n+1} \right| \leq 1 + \sum_{n=1}^{\infty} \frac{|\beta|^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))} \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{|\beta|^n}{(1+\alpha)^n (n)!} = e^{\frac{|\beta|}{1+\alpha}} \end{aligned}$$

and thus, inequality (6) is proved.
To prove (7), using again (8) and the triangle inequality, for $z \in \mathcal{U}$, we obtain

$$\begin{aligned} \left|\mathbb{R}'_{\alpha,\beta}(z)\right| &= \left| 1 + \sum_{n=1}^{\infty} \frac{(n+1)\beta^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))} z^n \right| \leq 1 + \sum_{n=1}^{\infty} \frac{(n+1)|\beta|^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))} \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{(n+1)|\beta|^n}{(1+\alpha)^n (n)!} = \left(1 + \frac{|\beta|}{1+\alpha}\right) e^{\frac{|\beta|}{1+\alpha}} \end{aligned}$$

and thus, inequality (7) is proved. □

Let $w(z)$ be an analytic function in $\mathcal{U}$. In the sequel, we will frequently use the following well-known result:

$$\Re\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0, \ z \in \mathcal{U} \text{ if and only if } |w(z)| < 1, \ z \in \mathcal{U}.$$

Theorem 1 *Let* $\alpha \geq 0$ *and* $0 < |\beta| \leq (1+\alpha)\ln 2$. *Then*

$$\Re\left\{\frac{\mathbb{R}_{\alpha,\beta}(z)}{(\mathbb{R}_{\alpha,\beta})_m(z)}\right\} \geq 2 - e^{\frac{|\beta|}{1+\alpha}}, \ z \in \mathcal{U} \tag{9}$$

and

$$\Re\left\{\frac{(\mathbb{R}_{\alpha,\beta})_m(z)}{\mathbb{R}_{\alpha,\beta}(z)}\right\} \geq e^{\frac{-|\beta|}{1+\alpha}}, \ z \in \mathcal{U}. \tag{10}$$

Proof. From inequality (6) we get

$$1 + \sum_{n=1}^{\infty} A_n \leq e^{\frac{|\beta|}{1+\alpha}}, \text{where } A_n = \frac{\beta^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))}.$$

The last inequality is equivalent to

$$\left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}} - 1}\right)\sum_{n=1}^{\infty} A_n \leq 1.$$

In order to prove the inequality (9), we consider the function $w(z)$ defined by

$$\frac{1+w(z)}{1-w(z)} = \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}} - 1}\right)\frac{\mathbb{R}_{\alpha,\beta}(z)}{(\mathbb{R}_{\alpha,\beta})_m(z)} - \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}} - 1} - 1\right)$$

and, thus we have

$$\frac{1+w(z)}{1-w(z)} = \frac{1 + \sum_{n=1}^{m} A_n z^n + \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n z^n}{1 + \sum_{n=1}^{m} A_n z^n}. \tag{11}$$

From (11), we obtain

$$w(z) = \frac{\left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n z^n}{2 + 2\sum_{n=1}^{m} A_n z^n + \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n z^n}$$

and

$$|w(z)| \leq \frac{\left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n}{2 - 2\sum_{n=1}^{m} A_n - \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n}.$$

Now, $|w(z)| \leq 1$ if and only if

$$\left(\frac{2}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n \leq 2-2\sum_{n=1}^{m} A_n$$

which is equivalent to

$$\sum_{n=1}^{m} A_n + \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n \leq 1. \tag{12}$$

To prove (12), it suffices to show that its left-hand side is bounded above by

$$\left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=1}^{\infty} A_n$$

which is equivalent to

$$\left(\frac{2-e^{\frac{|\beta|}{1+\alpha}}}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=1}^{m} A_n \geq 0.$$

The last inequality holds true for $0<|\beta| \leq (1+\alpha)\ln 2$.
We use the same method to prove the inequality (10). Consider the function $w(z)$ given by

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}+1\right)\frac{\mathbb{R}_{\alpha,\beta}(z)}{\left(\mathbb{R}_{\alpha,\beta}\right)_m(z)} - \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right) \\ &= \frac{1+\sum_{n=1}^{m} A_n z^n - \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n z^n}{1+\sum_{n=1}^{m} A_n z^n}.\end{aligned}$$

From the last equality we get

$$w(z) = \frac{-\left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}+1\right)\sum_{n=m+1}^{\infty} A_n z^n}{2+2\sum_{n=1}^{m} A_n z^n - \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}-1\right)\sum_{n=m+1}^{\infty} A_n z^n}$$

and

$$|w(z)| \leq \frac{\left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}+1\right)\sum_{n=m+1}^{\infty} A_n}{2-2\sum_{n=1}^{m} A_n - \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}-1\right)\sum_{n=m+1}^{\infty} A_n}.$$

Then, $|w(z)| \leq 1$ if and only if

$$\sum_{n=1}^{m} A_n + \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}} - 1}\right) \sum_{n=m+1}^{\infty} A_n \leq 1. \tag{13}$$

Since the left-hand side of (13) is bounded above by

$$\left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}} - 1}\right) \sum_{n=1}^{\infty} A_n,$$

we have that the inequality (10) holds true. Now, the proof of our theorem is completed. □

Theorem 2 *Let* $\alpha \geq 0$ *and* $1 < \left(1 + \frac{|\beta|}{1+\alpha}\right) e^{\frac{|\beta|}{1+\alpha}} \leq 2$. *Then*

$$\Re\left\{\frac{\mathbb{R}'_{\alpha,\beta}(z)}{\left(\mathbb{R}_{\alpha,\beta}\right)'_m(z)}\right\} \geq 2 - \left(1 + \frac{|\beta|}{1+\alpha}\right) e^{\frac{|\beta|}{1+\alpha}}, \; z \in \mathcal{U} \tag{14}$$

and

$$\Re\left\{\frac{\left(\mathbb{R}_{\alpha,\beta}\right)'_m(z)}{\mathbb{R}'_{\alpha,\beta}(z)}\right\} \geq \left(\frac{1+\alpha}{1+\alpha+|\beta|}\right) e^{\frac{-|\beta|}{1+\alpha}}, \; z \in \mathcal{U}. \tag{15}$$

Proof. From (7) we have

$$1 + \sum_{n=1}^{\infty} (n+1) A_n \leq \mathcal{L}_{\alpha,\beta},$$

where $A_n = \frac{\beta^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))}$, $\mathcal{L}_{\alpha,\beta} = \left(1 + \frac{|\beta|}{1+\alpha}\right) e^{\frac{|\beta|}{1+\alpha}}$, $\alpha \geq 0$, $\beta \in \mathbb{C}$ and $n \in \mathbb{N}$. The above inequality is equivalent to

$$\frac{1}{\mathcal{L}_{\alpha,\beta} - 1} \sum_{n=1}^{\infty} (n+1) A_n \leq 1.$$

To prove (14), define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{1}{\mathcal{L}_{\alpha,\beta} - 1} \frac{\mathbb{R}'_{\alpha,\beta}(z)}{\left(\mathbb{R}_{\alpha,\beta}\right)'_m(z)} - \frac{2 - \mathcal{L}_{\alpha,\beta}}{\mathcal{L}_{\alpha,\beta} - 1}$$

which gives

$$w(z)=\frac{\frac{1}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_nz^n}{2+2\sum_{n=1}^{m}(n+1)A_nz^n+\frac{1}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_nz^n}$$

and

$$|w(z)|\leq\frac{\frac{1}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_n}{2-2\sum_{n=1}^{m}(n+1)A_n-\frac{1}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_n}.$$

The condition $|w(z)|\leq 1$ holds true if and only if

$$\sum_{n=1}^{m}(n+1)A_n+\frac{1}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_n\leq 1. \tag{16}$$

The left-hand side of (16) is bounded above by

$$\frac{1}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=1}^{\infty}(n+1)A_n$$

which is equivalent to

$$\frac{2-\mathcal{L}_{\alpha,\beta}}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=1}^{m}(n+1)A_n\geq 0$$

which holds true for $1<\left(1+\frac{|\beta|}{1+\alpha}\right)e^{\frac{|\beta|}{1+\alpha}}\leq 2$.
The proof of (15) follows the same pattern. Consider the function $w(z)$ given by

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= \frac{\mathcal{L}_{\alpha,\beta}}{\mathcal{L}_{\alpha,\beta}-1}\frac{\mathbb{R}'_{\alpha,\beta}(z)}{(\mathbb{R}_{\alpha,\beta})'_m(z)}-\frac{1}{\mathcal{L}_{\alpha,\beta}-1}\\ &= \frac{1+\sum_{n=1}^{m}(n+1)A_nz^n-\frac{1}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_nz^n}{1+\sum_{n=1}^{\infty}(n+1)A_nz^n}.\end{aligned}$$

Consequently, we have that

$$w(z)=\frac{-\frac{\mathcal{L}_{\alpha,\beta}}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_nz^n}{2+2\sum_{n=1}^{m}(n+1)A_nz^n-\frac{2-\mathcal{L}_{\alpha,\beta}}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_nz^n}$$

and

$$|w(z)| \leq \frac{\frac{\mathcal{L}_{\alpha,\beta}}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_n}{2-2\sum_{n=1}^{m}(n+1)A_n-\frac{2-\mathcal{L}_{\alpha,\beta}}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_n}.$$

The last inequality implies that $|w(z)| \leq 1$ if and only if

$$\frac{2}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_n \leq 2-2\sum_{n=1}^{m}(n+1)A_n$$

or equivalently

$$\sum_{n=1}^{m}(n+1)A_n+\frac{1}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_n \leq 1. \tag{17}$$

It remains to show that the left-hand side of (17) is bounded above by

$$\frac{1}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=1}^{\infty}(n+1)A_n.$$

This is equivalent to

$$\frac{2-\mathcal{L}_{\alpha,\beta}}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=1}^{m}(n+1)A_n \geq 0,$$

which holds true for $1 < \left(1+\frac{|\beta|}{1+\alpha}\right)e^{\frac{|\beta|}{1+\alpha}} \leq 2$. Now, the proof of our theorem is completed. □

3 Illustrative examples and image domains

In this section, we present several illustrative examples along with the geometrical descriptions of the image domains of the appropriately chosen disk by the partial sums which we considered in our main theorems in Sections 2. From Theorem 1 and Theorem 2, we obtain the following corollaries for special cases of α and β.

Corollary 1 *If we take $\alpha = 0$ and $\beta = -\frac{1}{3}$, we have*

$$\mathbb{R}_{0,-\frac{1}{3}}(z) = ze^{-\frac{z}{3}}, \quad \mathbb{R}'_{0,-\frac{1}{3}}(z) = -\frac{1}{3}e^{-\frac{z}{3}}(z-3)$$

and for $m = 0$ *we get*

$$\left(\mathbb{R}_{0,-\frac{1}{3}}(z)\right)_0(z) = z, \quad \left(\mathbb{R}'_{0,-\frac{1}{3}}(z)\right)_0(z) = 1,$$

so,

$$\begin{aligned} \Re\left\{e^{-\frac{z}{3}}\right\} &\geq 2 - e^{\frac{1}{3}} \approx 0.60439, \quad z \in \mathcal{U}, \\ \Re\left\{e^{\frac{z}{3}}\right\} &\geq e^{-\frac{1}{3}} \approx 0.71653, \quad z \in \mathcal{U}, \\ \Re\left\{-\frac{1}{3}e^{-\frac{z}{3}}(z-3)\right\} &\geq 2 - \frac{4}{3}e^{\frac{1}{3}} \approx 0.13918, \quad z \in \mathcal{U}, \\ \Re\left\{-\frac{3e^{\frac{z}{3}}}{z-3}\right\} &\geq \frac{3}{4}e^{-\frac{1}{3}} \approx 0.5374, \quad z \in \mathcal{U}. \end{aligned}$$

Corollary 2 *For* $\alpha = 1$ *and* $\beta = \frac{1}{2}$*, we obtain*

$$\mathbb{R}_{1,\frac{1}{2}}(z) = \sqrt{2z}\sinh\sqrt{\frac{z}{2}}, \quad \mathbb{R}'_{1,\frac{1}{2}}(z) = \frac{1}{2}\cosh\sqrt{\frac{z}{2}} + \frac{\sinh\sqrt{\frac{z}{2}}}{\sqrt{2z}}$$

and for $m = 0$ *we have*

$$\left(\mathbb{R}_{1,\frac{1}{2}}(z)\right)_0(z) = z, \quad \left(\mathbb{R}'_{1,\frac{1}{2}}(z)\right)_0(z) = 1,$$

so,

$$\begin{aligned} \Re\left\{\sqrt{\frac{2}{z}}\sinh\sqrt{\frac{z}{2}}\right\} &\geq 2 - e^{\frac{1}{4}} \approx 0.71597, \quad z \in \mathcal{U}, \\ \Re\left\{\sqrt{\frac{z}{2}}\operatorname{csch}\sqrt{\frac{z}{2}}\right\} &\geq e^{-\frac{1}{4}} \approx 0.7788, \quad z \in \mathcal{U}, \\ \Re\left\{\frac{1}{2}\cosh\sqrt{\frac{z}{2}} + \frac{\sinh\sqrt{\frac{z}{2}}}{\sqrt{2z}}\right\} &\geq 2 - \frac{5}{4}e^{\frac{1}{4}} \approx 0.39497, \quad z \in \mathcal{U}, \\ \Re\left\{\frac{2}{\cosh\sqrt{\frac{z}{2}} + \frac{\sqrt{2}\sinh\sqrt{\frac{z}{2}}}{\sqrt{z}}}\right\} &\geq \frac{4}{5}e^{-\frac{1}{4}} \approx 0.62304, \quad z \in \mathcal{U}. \end{aligned}$$

Setting $m = 0$, $\alpha = 1$ and $\beta = -\frac{1}{4}$ in Theorem 1 and Theorem 2 respectively, we obtain the next result involving the function $\mathbb{R}_{1,-\frac{1}{4}}(z)$, defined by (4), and its derivative.

Corollary 3 *The following inequalities hold true:*

$$\Re\left\{\frac{1}{\sqrt{z}}\sin\frac{\sqrt{z}}{2}\right\} \geq \frac{2-e^{\frac{1}{8}}}{2} \approx 0.43343, \quad z \in \mathcal{U},$$

$$\Re\left\{\sqrt{z}\csc\frac{\sqrt{z}}{2}\right\} \geq 2e^{-\frac{1}{8}} \approx 1.765, \quad z \in \mathcal{U},$$

$$\Re\left\{\cos\frac{\sqrt{z}}{2}+\frac{2\sin\frac{\sqrt{z}}{2}}{\sqrt{z}}\right\} \geq 4-\frac{9}{4}e^{\frac{1}{8}} \approx 1.4504, \quad z \in \mathcal{U},$$

$$\Re\left\{\frac{1}{\cos\frac{\sqrt{z}}{2}+\frac{2\sin\frac{\sqrt{z}}{2}}{\sqrt{z}}}\right\} \geq \frac{4}{9}e^{-\frac{1}{8}} \approx 0.39222, \quad z \in \mathcal{U}.$$

Example 1 *The image domains of* $f_1(z) = \frac{1}{\sqrt{z}}\sin\frac{\sqrt{z}}{2}$, $f_2(z) = \sqrt{z}\csc\frac{\sqrt{z}}{2}$, $f_3(z) = \cos\frac{\sqrt{z}}{2}+\frac{2\sin\frac{\sqrt{z}}{2}}{\sqrt{z}}$ *and* $f_4(z) = \frac{1}{\cos\frac{\sqrt{z}}{2}+\frac{2\sin\frac{\sqrt{z}}{2}}{\sqrt{z}}}$ *are shown in Figure 1.*

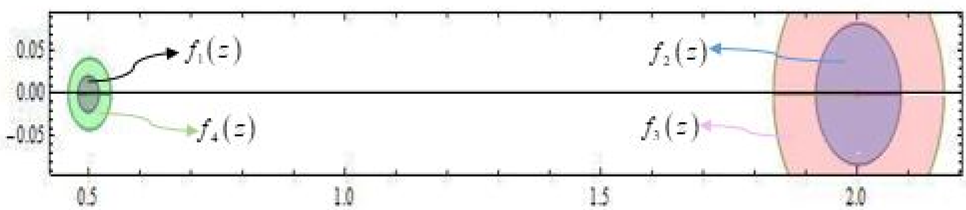

Figure 1.

It is therefore of interest to determine the largest disk $\mathcal{U}_\rho$ in which the partial sums $f_n = z + \sum_{k=1}^{n} a_k z^{k+1}$ of the functions $f \in \mathcal{A}$ are univalent, starlike, convex and close-to-convex. Recently, Ravichandran also wrote a survey [15] on geometric properties of partial sums of univalent functions. By the Noshiro-Warschowski Theorem (see [6]) for $m = 0$ in the inequality (14) of Theorem 2, we conclude that the function $\mathbb{R}_{\alpha,\beta}$ is univalent and also close-to-convex under the condition $1 < \left(1+\frac{|\beta|}{1+\alpha}\right)e^{\frac{|\beta|}{1+\alpha}} \leq 2$. Noshiro [11] showed that the radius of starlikeness of f_n partial sums of the functions $f \in \mathcal{A}$ is $1/M$ if satisfies the inequality $|f'(z)| \leq M$. Therefore if we consider the inequality (7) in Lemma 1, we conclude that the radius of starlikeness of $(\mathbb{R}_{\alpha,\beta})_m$ is $\left(\frac{1+\alpha}{1+\alpha+|\beta|}\right)e^{\frac{-|\beta|}{1+\alpha}}$. For functions whose derivatives has positive real part $(\Re(f'(z)) > 0)$, Silverman [19] and Singh [21] proved that f_n is univalent in $|z| < r_n$, where r_n is the smallest positive root of the equation $1 - r - 2r^n = 0$ and convex in $|z| < 1/4$, respectively. In light of these results, for $m = 0$ in the inequality (14) of Theorem 2, $(\mathbb{R}_{\alpha,\beta})_m$ is univalent in $|z| < r_n$ and convex in $|z| < 1/4$. According to the result of Miki [10], from (14), $(\mathbb{R}_{\alpha,\beta})_m$ is close-to-convex in $|z| < 1/4$. The results are all sharp.

References

[1] M. Çağlar and H. Orhan, On neighborhood and partial sums problem for generalized Sakaguchi type functions, *The Scientifc Annals of Al.I. Cuza University of Iasi*, **1** (2014), 17–28.

[2] M. Çağlar and E. Deniz, Partial sums of the normalized Lommel functions, *Math. Inequal. Appl.*, **18** (2015), 1189–1199.

[3] E. Deniz and H. Orhan, Some Properties Of certain subclasses of analytic functions with negative coefficients by using generalized Ruscheweyh derivative operator, *Czech. Math. J.*, **60** (2010), 699–713.

[4] E. Deniz and H. Orhan, Certain subclasses of multivalent functions defined by new multiplier transformations, *Arab. J. Sci. Eng.*, **36** (2011), 1091–1112.

[5] B. A. Frasin, Generalization of partial sums of certain analytic and univalent functions, *Appl. Math. Lett.*, **21** (2008), 735–741.

[6] A. W. Goodman, Univalent Functions, *Vol I. Mariner Publ. Comp., Tampa*, Florida, 1984.

[7] S. Kazımoğlu, E. Deniz and M. Çağlar, Partial Sums of The Bessel-Struve Kernel Function *3rd International Conference on Mathematical and Related Sciences: Current Trend and Developments*, (2020), 267–275.

[8] S. Kazımoğlu, Partial Sums of The Miller-Ross Function, *Turk. J. Sci.*, **6** (2021), 167–173.

[9] L.J. Lin and S. Owa, On partial sums of the Libera integral operator, *J. Math. Anal. Appl.*, **213** (1997), 444–454.

[10] Y. Miki, A note on close-to-convex functions, *J. Math. Soc. Japan*, **8** (1956), 256–268.

[11] K. Noshiro, On the starshaped mapping by an analytic function, *Proc. Imp. Acad.*, **8** (1932), 275-277.

[12] H. Orhan and N. Yağmur, Partial Sums of generalized Bessel functions, *J. Math. Inequal.*, **8** (2014), 863–877.

[13] S. Owa, H.M. Srivastava and N. Saito, Partial sums of certain classes of analytic functions, *Int. J. Comput. Math.*, **81** (2014), 1239–1256.

[14] Y. N. Rabotnov, Equilibrium of elastic media with an aftereffect, *Prikl. Matem. Mekh.*, **12** (1948), 53–62.

[15] V. Ravichandran, Geometric properties of partial sums of univalent functions, *Math. Newslett.*, **22** (2012), 208–221.

[16] M. S. Rehman, Q. Z. Ahmad, H. M. Srivastava, B. Khan and N. Khan, Partial sums of generalized $q-$Mittag-Leffler functions, *Aims Math.*, **5** (2019), 408–420.

[17] T. Sheil-Small, A note on partial sums of convex schlicht functions, *Bull. London Math. Soc.*, **2** (1970), 165–168.

[18] H. Silverman, Partial sums of starlike and convex functions, *J. Math. Anal. Appl.*, **209** (1997), 221–227.

[19] H. Silverman, Partial sums of a class of univalent functions, *Tamkang J. Math.*, **29** (1998), 171-174.

[20] E.M. Silvia, On partial sums of convex functions of order α, *Houston J. Math.*, **11** (1985), 397–404.

[21] R. Singh, Radius of convexity of partial sums of a certain power series, *J. Austral. Math. Soc.*, **11** (1970), 407-410.

[22] N. Yağmur, H. Orhan, Partial sums of generalized Struve functions, *Miskolc Math. Notes*, **17** (2016), 657–670.

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