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A short note on Layman permutations

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Abstract. A permutation p of $[k] = \{1, 2, 3, \ldots, k\}$ is called Layman permutation iff $i+p(i)$ is a Fibonacci number for $1 \leq i \leq k$. This concept is introduced by Layman in the A097082 entry of the Encyclopedia of Integers Sequences, that is the number of Layman permutations of $[n]$. In this paper, we will study Layman permutations. We introduce the notion of the Fibonacci complement of a natural number, that plays a crucial role in our investigation. Using this notion we prove some results on the number of Layman permutations, related to a conjecture of Layman that is implicit in the A097083 entry of OEIS.

1 Introduction

Sequence $(F_i)_{i=0}^{\infty}$ is the Fibonacci sequence ([9] A000045) defined as $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ with $F_0 = 0$ and $F_1 = 1$. We refer to $F_2 < F_3 < F_4 < \ldots$ as Fibonacci numbers. These numbers

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

are the initial object of essential mathematical research. Also, many deep results in mathematics use them to solve central open problems. For example, the solution of Hilbert's tenth problem [8], or designing complex data structures for important algorithms [4] rely on properties of Fibonacci numbers.

Many mathematical concepts are related to Fibonacci numbers. Enumerating special permutations leads to the sequence $(F_i)_{i=0}^{\infty}$: The set of permutations

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with $|\sigma(i) - i| \leq 1$ for all $i = 1, \dots, n$ is called the set of Fibonacci permutations. Investigating these has proved very fruitful (see for example [1] and [3]). Permutation polynomials can also be linked to Fibonacci numbers (see [2]).

Our motivation is different from the above. Layman introduced a special property of permutations (hereafter referred to as *Layman's property*) which is also related to Fibonacci numbers. Such permutations are henceforth called *Layman permutations.*

Definition 1 (Layman (2004) [7]) *A permutation* p *of* $[k] = \{1, 2, 3, \dots, k\}$ *is called Layman permutation iff* $i + p(i)$ *is a Fibonacci number for all* $1 \leq i \leq k$.

The following permutations are Layman permutations

$$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1&2\\2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&3&2&4\end{pmatrix}.$$

We use the two-line notation to represent permutations. The last one denotes $\pi : 1 \mapsto 1$, $2 \mapsto 3$, $3 \mapsto 2$, $4 \mapsto 4$, i.e. $\pi(1) = 1$, $\pi(2) = 3$, $\pi(3) = 2$, $\pi(4) = 4$. The Layman's property means that the column sums in these permutations are Fibonacci numbers.

Layman also submitted the sequence of "the number of Layman permutations of $[n]$" to OEIS (entry A097082). The first few terms suggest that for all positive integer n the set $[n]$ has Layman permutation. Also, infinitely often $[n]$ has unique Layman permutation. The sequence of these positive integers is submitted as the A097083 entry of OEIS. These entries of the Encyclopedia do not have any mathematical content. The statements in A097083 are all hypothetical ones, they are conjectures.

The main reason for this paper is to establish some mathematical results on these sequences. Our main results are two claims. The first one is an easy observation.

Observation 1 *For all positive natural number* n *the set* $[n]$ *has a Layman permutation.*

For the second results we need to introduce the sequence

$$M_m(n) = \sum_{2 \leq i \leq n, i \equiv n \pmod m} F_i = F_n + F_{n-m} + F_{n-2m} + \dots.$$

We can consider $M_m(2)$ as the initial term of the sequence or start the sequence with $M_m(0) = M_m(1) = 0$ (the value of the empty sum).

$$M_4(n) = \sum_{2 \le i \le n, i \equiv n \pmod 4} F_i = F_n + F_{n-4} + F_{n-8} + \dots$$

plays a very important role in our discussion.

Theorem 1 *If $n \in \mathbb{N}_+$ is not in the sequence $(M_4(k))_{k=2}^{\infty}$ then $[n]$ has at least two Layman permutations.*

The entry A097083 of OEIS suggests the following conjecture.

Conjecture 1 *For $n \in \mathbb{N}_+$ that set $[n]$ has a unique Layman permutation if and only if n is in the sequence $(M_4(k))_{k=2}^{\infty}$.*

We established one direction of the conjecture.

In section 2 we introduce the notion of the Fibonacci complement of a positive integer. Using the properties of this notion in section 3 we prove our main results.

Throughout the paper the set $\{0, 1, 2, 3, \dots\}$, i.e. the set of natural numbers is denoted as $\mathbb{N}$. $\mathbb{N}_+$ denotes the set of positive integers. The intervals are always intervals of $\mathbb{Z}$, so $]2, 6] = (2, 6] = \{3, 4, 5, 6\}$. $A \dot{\cup} B$ denotes $A \cup B$ and contains the extra information that A and B are disjoint.

2 Fibonacci complement of positive integers

Definition 2 *Let $n \in \mathbb{N}_+$ be a positive integer. $\nu \in \mathbb{N}$ is the Fibonacci complement of n iff $1 \le \nu \le n$ and $n + \nu$ is a Fibonacci number.*

We will use F-complement as an abbreviation of Fibonacci complement.

Observation 2 *Every positive number has one or two F-complements.*

Proof. Let F_ℓ be the minimal Fibonacci number, that is larger than n: $F_{\ell-1} \le n < F_\ell$. The F-complements are the terms of $F_\ell - n < F_{\ell+1} - n < F_{\ell+2} - n < \dots$, that are at most n.

$$n < F_\ell < F_\ell + (F_{\ell+1} - n) = F_{\ell+2} - n,$$

hence we have only two options left: $F_\ell - n$ and $F_{\ell+1} - n$.

$$F_\ell = F_{\ell-1} + F_{\ell-2} \le n + n,$$

so $F_\ell - n \le n$ is an F-complement indeed. □

Notation 1 *Using the notation of the proof of the above observation we write* $\overline{n}^F$ *for* $F_\ell - n$*, i.e.* $\overline{n}^F$ *is the only F-complement of* n *or the smaller of the two ones.*

The final result of this section describes which case occurs for each natural number n. For this, we need some preparations.

Recall, that

$$M_3(k) = F_k + F_{k-3} + F_{k-6} + \dots.$$

Lemma 1 $M_k(3)$ *is the largest natural number* t*, that satisfies* $2t < F_{k+2}$.

Proof. It is well-known that $2F_k = F_{k+2} - F_{k-1}$ (see [6]). So

$$\begin{aligned} 2M_3(k) =& 2F_k + 2F_{k-3} + 2F_{k-6} + \dots \\ =& (F_{k+2} - F_{k-1}) + (F_{k-1} - F_{k-4}) + (F_{k-4} - F_{k-7}) + \dots. \end{aligned}$$

The last term is $2F_2 = F_4 - 1$ or $2F_3 = F_5 - 1$ or $2F_4 = F_6 - 2$. Depending on the parity of F_{k+2} (F_s is even iff s is divisible by 3, see [6]) we get $F_{k+2} - 1$ (when F_{k+2} is odd) or $F_{k+2} - 2$ (when F_{k+2} is even). After collapsing the telescopic sum we get $F_{k+2} - 1$ or $F_{k+2} - 2$, that proves the claim. □

Recall that

$$M_2(k) = F_k + F_{k-2} + F_{k-4} + \dots = F_{k+1} - 1,$$

where the last equality is a well-known, easy fact on Fibonacci numbers (see [6]). Using Lemma 1 we get the following important claim.

Lemma 2 *For all* $\ell \in \mathbb{N}_+$ *any number* $n \in [M_3(\ell-1)+1, M_2(\ell-1)] = [M_3(\ell-1)+1, F_\ell[$ *has two F-complements. If* $n \in [F_{\ell-1}, M_3(\ell-1)]$ *for any* $\ell \in \mathbb{N}_+$*, then it has exactly one F-complement.*

Note that $[F_{\ell-1}, M_3(\ell-1)] \dot\cup]M_3(\ell-1), F_\ell[$ covers all integers in $[F_{\ell-1}, F_\ell[$, furthermore these intervals partition $\mathbb{N}_+$.

Proof. Take an arbitrary natural number n from $[F_{\ell-1}, F_\ell[$. Note that our notation coincides with the notation of the proof of Observation 2: F_ℓ is the minimal Fibonacci number, that is larger than n: $F_{\ell-1} \le n < F_\ell$.

From the proof of Observation 2 we know that n has two F-complements iff $F_{\ell+1} - n \le n$, i.e. $F_{\ell+1} \le 2n$.

Lemma 1 says that $M_3(\ell-1)$ satisfies $2M_3(\ell-1) < F_{\ell+1}$. Hence the elements of $[F_{\ell-1}, M_3(\ell-1)]$, i.e. $M_3(\ell-1)$ and smaller numbers from our interval have unique F-complement.

Lemma 1 also says that $2(M_3(\ell-1)+1) \ge F_{\ell+1}$. Hence $n \in]M_3(\ell-1), F_\ell[$ has two F-complements. □

3 Layman permutations

Notation 2 *Let $\mathcal{L}_n$ be denote the set of Layman permutations.*

We will use the so-called two-line notation to describe permutations. A $2 \times n$ matrix visualizes the permutation. The Layman property is equivalent to that each column sum is a Fibonacci number. Examples for Layman permutations:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 5 & 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 4 & 3 & 7 & 6 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 2 & 4 & 8 & 7 & 6 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 1 & 10 & 9 & 8 & 2 & 6 & 5 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 6 & 10 & 9 & 8 & 7 & 1 & 5 & 4 & 3 \end{pmatrix}.$$

Observation 3 *For any positive integer n the set $\mathcal{L}_n$ is not empty.*

Our previous examples prove that the claim is true for $n \leq 8$. If $\overline{n}^F = 1$ (equivalently $\overline{n}^{F-} := \overline{n}^F - 1 = 0$) then the reverse permutation exhibits the truth of Observation 3.

One can prove Observation 3 by induction: If $\overline{n}^F - 1 > 0$ take $p \in \mathcal{L}_{\overline{n}^{F-}}$ and extend it with

$$\begin{pmatrix} \overline{n}^F & \overline{n}^F + 1 & \dots & n-1 & n \\ n & n-1 & \dots & \overline{n}^F + 1 & \overline{n}^F \end{pmatrix}.$$

The argument, proving Observation 3, immediately gives us the following claim.

Observation 4 *If n has two F-complements then $\mathcal{L}_n$ has more than one element.*

Indeed. We have already constructed one. In that we used $\overline{n}^F$ to cut $[n]$ into two blocks and apply induction plus a reverse permutation. We can do the same with a second F-complement.

Observe that $M_4(\ell - 1) \in [F_{\ell-1}, M_3(\ell - 1)]$, hence $M_4(\ell - 1)$ has a unique F-complement.

Also understanding the simple proof leads to the following definition.

Definition 3 *Let* $(s_i^{(n)})$ *the following finite, decreasing sequence of positive integers:* $s_0 = n$*, furthermore if* s_i *exists and* $\overline{s_i}^{F-}$ *is positive then* s_{i+1} *exists too and* $s_{i+1} = \overline{s_i}^{F-}$.

Based on $(s_i^{(n)})_{i=0}^k$ we can explicitly describe the permutation produced by the above recursion: Partition $\{1, 2, \dots, n\}$ into blocks

$$\{1, 2, \dots, s_k\} \cup \{s_k + 1, \dots, s_{k-1}\} \cup \dots \cup \{s_2 + 1, \dots, s_1\} \cup \{s_1 + 1, \dots, s_0\}$$

and reverse the order of each block (note that $\overline{s_k}^F = 1$ and $\overline{s_{i-1}}^F = s_i + 1$ for $i = 1, 2, \dots, k-1$).

Let us see a few examples (each arrow denotes the application of the mapping $x \mapsto \overline{x}^{F-}$):

$$s^{(2021)} : 2021 \to 562 \to 47 \to 8 \to 4,$$
$$s^{(1869)} : 1869 \to 714 \to 272 \to 104 \to 59 \to 13 \to 5 \to 2, s^{(14)} : 14 \to 6 \to 1,$$
$$s^{10} : 10 \to 2, s^{(9)} : 9 \to 3 \to 1, s^{(8)} : 8 \to 4, s^{(7)} : 7, s^{(6)} : 6 \to 1.$$

Corollary 1 *Assume that for* $n \in \mathbb{N}_+$ *in the sequence* $(s_i^{(n)})_{i=0}^k$ *we have the element* 6 *or* 10 *or a number with two F-complements. Then* $\mathcal{L}_n$ *has more than one element.*

For example, $\mathcal{L}_6$ and $\mathcal{L}_{10}$ have more than 1 permutation (see of our previous examples). $\mathcal{L}_7$ has more than 1 permutation since 7 has two F-complements (1 and 6). $\mathcal{L}_{2021}$ has more than one permutation since 47 is in its s-sequence and 47 has two F-complements (8 and 42): For example, we obtain two elements of $\mathcal{L}_{2021}$ we start with two elements of $\mathcal{L}_{47}$ based on the two F-complements of 47 and extend them by

$$\begin{pmatrix} 48 & 49 & \dots & 561 & 562 & 563 & \dots & 2021 \\ 562 & 561 & \dots & 49 & 48 & 2021 & \dots & 563 \end{pmatrix}.$$

Note that in the case of $n = M_4(\ell)$ the corresponding s-sequence is

$$M_4(\ell), M_4(\ell - 2), M_4(\ell - 4), \dots$$

a sequence ending with $M_4(3) = 2$ or with $M_4(2) = 1$. Indeed $M_4(\ell) + M_4(\ell - 2) = M_2(\ell) = F_{\ell+1} - 1$, i.e. $\overline{M_4(\ell)}^{F-} = M_4(\ell - 2)$.

A simple consequence of Corollary 1 is the following Theorem.

Theorem 2 *Assume that* $n \in \mathbb{N}_+$ *is a number not in the form* $M_4(\ell)$*. Then* $\mathcal{L}_n$ *has more than one element.*

Proof. $n \in [F_{\ell-1}, F_\ell[$ for a unique ℓ. We are going to prove our claim, by induction on ℓ.

The claim is easy for $\ell = 3, 4, 5, 6, 7$. For the induction step, assume that

$$n \in [F_{\ell-1}, F_\ell[\setminus\{M_4(\ell-1)\} = [F_{\ell-1}, M_3(\ell-1)]\setminus\{M_4(\ell-1)\}\dot\cup[M_3(\ell-1)+1, M_2(\ell-1)]$$

If $n \in [M_3(\ell-1)+1, M_2(\ell-1)]$, them we are done since $n = s_0^{(n)}$ has two F-Complements. If $k \in [F_{\ell-1}, M_3(\ell-1)]$, then

$$s_1^{(k)} = \overline{k}^{F-} = F_\ell - k - 1 \in [F_\ell - M_3(\ell-1) - 1, F_\ell - F_{\ell-1} - 1].$$

Remember, that $\overline{M_4(\ell-1)}^{F-} = M_4(\ell-3)$.

So if $n \in [F_{\ell-1}, M_3(\ell-1)] \setminus \{M_4(\ell-1)\}$ then

$$s_1^{(n)} = \overline{(n)}^{F-} = F_\ell - n - 1 \in [F_\ell - M_3(\ell-1) - 1, M_4(\ell-3) - 1]\dot\cup[M_4(\ell-3)+1, F_{\ell-2} - 1].$$

Easy to check that $M_4(\ell-4) < F_\ell - M_3(\ell-1) - 1$ hence the right hand side does not contain any number of the form $M_4(m)$. The Theorem is proved. □

The proof really gave us the claim, that if n is not of the form $M_4(\ell)$, then the assumption of Corollary 1 holds.

So the hardness of Layman's conjecture (Conjecture 1) is to prove that for $n = M_4(\ell)$ we have a unique Layman permutation.

4 Conclusion

We consider Conjecture 1 as a nice, important conjecture. It has a graph theoretical interpretation about bipartite graphs with a unique perfect matching. The investigation of bipartite graphs with unique perfect matching ([5]) is independent of our motivation. The conjecture connects two different lines of research. We made the first step to settle the conjecture. We need further effort to understand Layman permutations.

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