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Two generalizations of dual-complex Lucas-balancing numbers

Dorota Bród
The Faculty of Mathematics and Applied Physics,
Rzeszow University of Technology,
al. Powstańców Warszawy 12
35-959 Rzeszów, Poland
email: dorotab@prz.edu.pl

Anetta Szynal-Liana
The Faculty of Mathematics and Applied Physics,
Rzeszow University of Technology,
al. Powstańców Warszawy 12
35-959 Rzeszów, Poland
email: aszynal@prz.edu.pl

Iwona Włoch
The Faculty of Mathematics and Applied Physics,
Rzeszow University of Technology,
al. Powstańców Warszawy 12
35-959 Rzeszów, Poland
email: iwloch@prz.edu.pl

Abstract. In this paper, we study two generalizations of dual-complex Lucas-balancing numbers: dual-complex k-Lucas balancing numbers and dual-complex k-Lucas-balancing numbers. We give some of their properties, among others the Binet formula, Catalan, Cassini, d'Ocagne identities.

1 Introduction

The sequence of balancing numbers, denoted by $\{B_n\}$, was introduced by Behera and Panda in [4]. In [9], Panda introduced the sequence of Lucas-balancing numbers, denoted by $\{C_n\}$ and defined as follows: if B_n is a balancing number, the number $C_n = \sqrt{8B_n^2 + 1}$ is called a Lucas-balancing number.

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Recall that a balancing number n with balancer r is the solution of the Diophantine equation

$$1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+r). \qquad (1)$$

The balancing and Lucas-balancing numbers fulfill the following recurrence relations

$$B_n = 6B_{n-1} - B_{n-2} \text{ for } n \geq 2, \text{ with } B_0 = 0, B_1 = 1,$$

$$C_n = 6C_{n-1} - C_{n-2} \text{ for } n \geq 2, \text{ with } C_0 = 1, C_1 = 3.$$

The Table 1 includes initial terms of the balancing and Lucas-balancing numbers for $0 \leq n \leq 7$.

Table 1.

n	0	1	2	3	4	5	6	7
B_n	0	1	6	35	204	1189	6930	40391
C_n	1	3	17	99	577	3363	19601	114243

The Binet type formulas for the balancing and Lucas-balancing numbers have the forms

$$B_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$
$$C_n = \frac{\alpha^n + \beta^n}{2},$$

respectively, for $n \geq 0$, where $\alpha = 3 + 2\sqrt{2}$, $\beta = 3 - 2\sqrt{2}$.

The concept of balancing numbers has been extended and generalized by many authors, see [7, 8, 10]. In this paper, we focus our attention on k-Lucas balancing numbers and k-Lucas-balancing numbers and their applications in the theory of dual-complex numbers.

Based on the concept from [6], Özkoç in [7] introduced k-Lucas balancing numbers as follows.

For some positive integer $k \geq 1$ let C_n^k denote the nth k-Lucas balancing number which is the number defined by

$$C_n^k = 6kC_{n-1}^k - C_{n-2}^k$$

for $n \geq 2$, with $C_0^k = 1$, $C_1^k = 3$.

Theorem 1 ([7]) *The Binet type formula for* k*-Lucas balancing numbers is*

$$C_n^k = \frac{(3-\beta_k)\alpha_k^n - (3-\alpha_k)\beta_k^n}{2\sqrt{9k^2-1}}, \tag{2}$$

for $n \geq 0,\ k \geq 1,$ *where* $\alpha_k = 3k + \sqrt{9k^2-1},\ \beta_k = 3k - \sqrt{9k^2-1}.$

Another generalization of the Lucas-balancing numbers was presented in [12]. For integer $k \geq 1$ the sequence of k-Lucas-balancing numbers (written with two hyphens) is defined recursively by

$$C_{k,n} = 6kC_{k,n-1} - C_{k,n-2}$$

for $n \geq 2$, with $C_{k,0} = 1,\ C_{k,1} = 3k.$

Theorem 2 ([13]) *The Binet type formula for* k*-Lucas-balancing numbers is*

$$C_{k,n} = \frac{\alpha_k^n + \beta_k^n}{2} \tag{3}$$

for $n \geq 0,\ k \geq 1,$ *where* $\alpha_k = 3k + \sqrt{9k^2-1},\ \beta_k = 3k - \sqrt{9k^2-1}.$

Note that for $k = 1$ we have $C_n^1 = C_{1,n} = C_n$.

Complex and dual numbers are well known two dimensional number systems. Let $\mathbb{C}$ and $\mathbb{D}$ denote the set of complex numbers with imaginary unit i and the set of dual numbers with nilpotent unit ε, respectively. The set of dual-complex numbers is expressed in the form

$$\mathbb{DC} = \{w = z_1 + \varepsilon z_2 :\ z_1, z_2 \in \mathbb{C},\ \ \varepsilon^2 = 0,\ \ \varepsilon \neq 0\},$$

see [1]. Here if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then any dual-complex number can be written as

$$w = x_1 + iy_1 + \varepsilon x_2 + i\varepsilon y_2. \tag{4}$$

If $w_1 = z_1 + \varepsilon z_2$ and $w_2 = z_3 + \varepsilon z_4$ are any two dual-complex numbers then the equality, the addition, the subtraction, the multiplication by scalar and the multiplication are defined in the natural way:

$$\begin{aligned}
&w_1 = w_2 \ \text{ only if } \ z_1 = z_3,\ z_2 = z_4,\\
&w_1 \pm w_2 = (z_1 \pm z_3) + \varepsilon(z_2 \pm z_4),\\
&\text{for } \ s \in \mathbb{R} :\ sw_1 = sz_1 + \varepsilon sz_2,\\
&w_1 \cdot w_1 = z_1 z_3 + \varepsilon(z_1 z_4 + z_2 z_3).
\end{aligned}$$

If we write the dual-complex numbers using (4) then the multiplication of dual-complex numbers can be made analogously as multiplications of algebraic expressions using Table 2.

Table 2. The dual-complex numbers multiplication

$\cdot$	i	ε	$i\varepsilon$
i	-1	$i\varepsilon$	$-\varepsilon$
ε	$i\varepsilon$	0	0
$i\varepsilon$	$-\varepsilon$	0	0

Balancing and Lucas-balancing numbers are numbers defined by the linear recurrence relation and they are named as numbers of the Fibonacci type. These numbers have many applications in the theory of hypercomplex numbers, for details see [14]. Some interesting properties of dual-complex Fibonacci and dual-complex Lucas numbers we can find in [5]. The dual-complex Pell numbers (quaternions) were introduced quite recently in [3]. In [2], the author investigated one-parameter generalization of dual-complex Fibonacci numbers, called dual-complex k-Fibonacci numbers. Based on these ideas we define and study dual-complex Lucas-balancing numbers and their generalizations.

2 Main results

Let $n \geq 0$ be an integer. The nth dual-complex balancing number $\mathbb{DC}B_n$ and nth dual-complex Lucas-balancing number $\mathbb{DC}C_n$ are defined as

$$\mathbb{DC}B_n = B_n + iB_{n+1} + \varepsilon B_{n+2} + i\varepsilon B_{n+3},$$

$$\mathbb{DC}C_n = C_n + iC_{n+1} + \varepsilon C_{n+2} + i\varepsilon C_{n+3},$$

where B_n is the nth balancing number, C_n is the nth Lucas-balancing number and i, ε, $i\varepsilon$ are dual-complex units.

In the similar way we define the nth dual-complex k-Lucas balancing number $\mathbb{DC}C_n^k$ and the nth dual-complex k-Lucas-balancing number $\mathbb{DC}C_{k,n}$ as

$$\mathbb{DC}C_n^k = C_n^k + iC_{n+1}^k + \varepsilon C_{n+2}^k + i\varepsilon C_{n+3}^k,$$
$$\mathbb{DC}C_{k,n} = C_{k,n} + iC_{k,n+1} + \varepsilon C_{k,n+2} + i\varepsilon C_{k,n+3},$$

respectively.

For $k = 1$ we have $\mathbb{DC}C_n^1 = \mathbb{DC}C_{1,n} = \mathbb{DC}C_n$.

Theorem 3 (*Binet type formulas*) *Let* $n \geq 0$, $k \geq 1$ *be integers. Then*

$$\text{(i)}\quad \mathbb{DC}C_n^k = \frac{(3-\beta_k)\alpha_k^n\widehat{\alpha_k} - (3-\alpha_k)\beta_k^n\widehat{\beta_k}}{2\sqrt{9k^2-1}},$$

$$\text{(ii)}\quad \mathbb{DC}C_{k,n} = \frac{\alpha_k^n\widehat{\alpha_k} + \beta_k^n\widehat{\beta_k}}{2},$$

where

$$\alpha_k = 3k + \sqrt{9k^2-1}, \quad \beta_k = 3k - \sqrt{9k^2-1} \tag{5}$$

and

$$\widehat{\alpha_k} = 1 + i\alpha_k + \varepsilon\alpha_k^2 + i\varepsilon\alpha_k^3, \quad \widehat{\beta_k} = 1 + i\beta_k + \varepsilon\beta_k^2 + i\varepsilon\beta_k^3. \tag{6}$$

Proof. By formula (2) we get

$$\begin{aligned}\mathbb{DC}C_n^k &= C_n^k + iC_{n+1}^k + \varepsilon C_{n+2}^k + i\varepsilon C_{n+3}^k \\ &= \frac{(3-\beta_k)\alpha_k^n - (3-\alpha_k)\beta_k^n}{2\sqrt{9k^2-1}} + i\frac{(3-\beta_k)\alpha_k^{n+1} - (3-\alpha_k)\beta_k^{n+1}}{2\sqrt{9k^2-1}} \\ &+ \varepsilon\frac{(3-\beta_k)\alpha_k^{n+2} - (3-\alpha_k)\beta_k^{n+2}}{2\sqrt{9k^2-1}} + i\varepsilon\frac{(3-\beta_k)\alpha_k^{n+3} - (3-\alpha_k)\beta_k^{n+3}}{2\sqrt{9k^2-1}}\end{aligned}$$

and after calculation we obtain (i). By the same method, using (3), we can prove formula (ii). □

For $k = 1$ we obtain the Binet type formula for the dual-complex Lucas-balancing numbers.

Corollary 1 *Let* $n \geq 0$ *be an integer. Then*

$$\mathbb{DC}C_n = \frac{\alpha^n\hat{\alpha} + \beta^n\hat{\beta}}{2},$$

where

$$\begin{aligned}\alpha &= 3 + 2\sqrt{2}, \quad \beta = 3 - 2\sqrt{2}, \\ \hat{\alpha} &= 1 + i(3+\sqrt{8}) + \varepsilon(17+6\sqrt{8}) + i\varepsilon(99+35\sqrt{8}), \\ \hat{\beta} &= 1 + i(3-\sqrt{8}) + \varepsilon(17-6\sqrt{8}) + i\varepsilon(99-35\sqrt{8}).\end{aligned} \tag{7}$$

Moreover, by simple calculations, we get

$$\begin{aligned}
\alpha_k + \beta_k &= 6k,\\
\alpha_k - \beta_k &= 2\sqrt{9k^2-1},\\
\alpha_k\beta_k &= 1,\\
(3-\alpha_k)(3-\beta_k) &= 10-18k,\\
\alpha_k^3+\beta_k^3 &= (\alpha_k+\beta_k)^3 - 3\alpha_k\beta_k(\alpha_k+\beta_k) = 216k^3-18k
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\alpha_k}\widehat{\beta_k} &= \left(1+i\alpha_k+\varepsilon\alpha_k^2+i\varepsilon\alpha_k^3\right)\left(1+i\beta_k+\varepsilon\beta_k^2+i\varepsilon\beta_k^3\right)\\
&= i(\alpha_k+\beta_k)+i\varepsilon(\alpha_k^3+\beta_k^3+\alpha_k+\beta_k)\\
&= i(6k)+i\varepsilon(216k^3-12k).
\end{aligned}$$

In particular, for $k=1$, we have

$$\hat{\alpha}\hat{\beta} = 6i + 204i\varepsilon.$$

Now we will give some identities such as Catalan type, Cassini type and d'Ocagne type identities for the dual-complex k-Lucas balancing numbers and dual-complex k-Lucas-balancing numbers. These identities can be proved using the Binet type formulas for these numbers.

Theorem 4 (*Catalan type identity for dual-complex k-Lucas balancing numbers*) *Let $k \geq 1$, $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$\mathbb{DC}C_{n-r}^k \cdot \mathbb{DC}C_{n+r}^k - \left(\mathbb{DC}C_n^k\right)^2 = \\ = \frac{(3-\beta_k)(3-\alpha_k)}{4(9k^2-1)}\left(2-\left(\frac{\beta_k}{\alpha_k}\right)^r - \left(\frac{\alpha_k}{\beta_k}\right)^r\right)\widehat{\alpha_k}\widehat{\beta_k},$$

where α_k, β_k and $\widehat{\alpha_k}$, $\widehat{\beta_k}$ are given by (5) and (6), respectively.

Proof. By formula (i) of Theorem 3 we have

$$\begin{aligned}
&\mathbb{DC}C_{n-r}^k \cdot \mathbb{DC}C_{n+r}^k - \left(\mathbb{DC}C_n^k\right)^2\\
&= \frac{-(3-\alpha_k)(3-\beta_k)\alpha_k^{n-r}\beta_k^{n+r}\widehat{\alpha_k}\widehat{\beta_k} - (3-\alpha_k)(3-\beta_k)\alpha_k^{n+r}\beta_k^{n-r}\widehat{\alpha_k}\widehat{\beta_k}}{4(9k^2-1)}\\
&+ \frac{2(3-\alpha_k)(3-\beta_k)\alpha_k^n\beta_k^n\widehat{\alpha_k}\widehat{\beta_k}}{4(9k^2-1)}\\
&= \frac{(3-\alpha_k)(3-\beta_k)\alpha_k^n\beta_k^n\widehat{\alpha_k}\widehat{\beta_k}}{4(9k^2-1)}\left(2-\left(\frac{\beta_k}{\alpha_k}\right)^r - \left(\frac{\alpha_k}{\beta_k}\right)^r\right).
\end{aligned}$$

Using the fact that $\alpha_k\beta_k = 1$, we obtain the desired formula. □

Theorem 5 (*Catalan type identity for dual-complex* k*-Lucas-balancing numbers*) *Let* $k \geq 1$, $n \geq 0$, $r \geq 0$ *be integers such that* $n \geq r$. *Then*

$$\mathbb{DC}C_{k,n-r} \cdot \mathbb{DC}C_{k,n+r} - (\mathbb{DC}C_{k,n})^2 = \frac{1}{4}\left(\left(\frac{\beta_k}{\alpha_k}\right)^r + \left(\frac{\alpha_k}{\beta_k}\right)^r - 2\right)\widehat{\alpha_k}\widehat{\beta_k},$$

where α_k, β_k *and* $\widehat{\alpha_k}$, $\widehat{\beta_k}$ *are given by* (5) *and* (6), *respectively.*

Proof. By formula (ii) of Theorem 3 we have

$$\begin{aligned}
&\mathbb{DC}C_{k,n-r} \cdot \mathbb{DC}C_{k,n+r} - (\mathbb{DC}C_{k,n})^2 \\
&= \frac{\alpha_k^{n-r}\widehat{\alpha_k}\beta_k^{n+r}\widehat{\beta_k} + \beta_k^{n-r}\widehat{\beta_k}\alpha_k^{n+r}\widehat{\alpha_k} - 2\alpha_k^n\widehat{\alpha_k}\beta_k^n\widehat{\beta_k}}{4} \\
&= \frac{1}{4}\alpha_k^n\beta_k^n\widehat{\alpha_k}\widehat{\beta_k}\left(\left(\frac{\beta_k}{\alpha_k}\right)^r + \left(\frac{\alpha_k}{\beta_k}\right)^r - 2\right).
\end{aligned}$$

Using the fact that $\alpha_k\beta_k = 1$, we obtain the desired formula. □

Note that for $r = 1$ we obtain Cassini type identities for the dual-complex k-Lucas balancing numbers and the dual-complex k-Lucas-balancing numbers.

Corollary 2 (*Cassini type identity for dual-complex* k*-Lucas balancing numbers*) *Let* $k \geq 1$, $n \geq 1$ *be integers. Then*

$$\mathbb{DC}C_{n-1}^k \cdot \mathbb{DC}C_{n+1}^k - \left(\mathbb{DC}C_n^k\right)^2 = (18k - 10)\,\widehat{\alpha_k}\widehat{\beta_k},$$

where α_k, β_k *and* $\widehat{\alpha_k}$, $\widehat{\beta_k}$ *are given by* (5) *and* (6), *respectively.*

Corollary 3 (*Cassini type identity for dual-complex* k*-Lucas-balancing numbers*) *Let* $k \geq 1$, $n \geq 1$ *be integers. Then*

$$\mathbb{DC}C_{k,n-1} \cdot \mathbb{DC}C_{k,n+1} - (\mathbb{DC}C_{k,n})^2 = \left(9k^2 - 1\right)\widehat{\alpha_k}\widehat{\beta_k},$$

where α_k, β_k *and* $\widehat{\alpha_k}$, $\widehat{\beta_k}$ *are given by* (5) *and* (6), *respectively.*

Theorem 6 (*d'Ocagne type identity for dual-complex* k*-Lucas balancing numbers*) *Let* $k \geq 1$, $m \geq 0$, $n \geq 0$ *be integers such that* $m \geq n$. *Then*

$$\begin{aligned}
&\mathbb{DC}C_m^k \cdot \mathbb{DC}C_{n+1}^k - \mathbb{DC}C_{m+1}^k \cdot \mathbb{DC}C_n^k = \\
&= \frac{(3-\alpha_k)(3-\beta_k)\alpha_k^n\beta_k^n\left(\alpha_k^{m-n} - \beta_k^{m-n}\right)}{2\sqrt{9k^2-1}}\widehat{\alpha_k}\widehat{\beta_k},
\end{aligned}$$

where α_k, β_k *and* $\widehat{\alpha_k}$, $\widehat{\beta_k}$ *are given by* (5) *and* (6), *respectively.*

Proof. By formula (i) of Theorem 3 we have

$$\begin{aligned}
&\mathbb{DC}C^k_m \cdot \mathbb{DC}C^k_{n+1} - \mathbb{DC}C^k_{m+1} \cdot \mathbb{DC}C^k_n \\
&= \frac{-(3-\beta_k)\alpha_k^m\widehat{\alpha_k}(3-\alpha_k)\beta_k^{n+1}\widehat{\beta_k} - (3-\alpha_k)\beta_k^m\widehat{\beta_k}(3-\beta_k)\alpha_k^{n+1}\widehat{\alpha_k}}{4(9k^2-1)} \\
&+ \frac{(3-\beta_k)\alpha_k^{m+1}\widehat{\alpha_k}(3-\alpha_k)\beta_k^n\widehat{\beta_k} + (3-\alpha_k)\beta_k^{m+1}\widehat{\beta_k}(3-\beta_k)\alpha_k^n\widehat{\alpha_k}}{4(9k^2-1)} \\
&= \frac{(3-\alpha_k)(3-\beta_k)\alpha_k^n\beta_k^n\left(\alpha_k^{m-n+1} + \beta_k^{m-n+1} - \alpha_k\beta_k^{m-n} - \alpha_k^{m-n}\beta_k\right)}{4(9k^2-1)}\widehat{\alpha_k}\widehat{\beta_k} \\
&= \frac{(3-\alpha_k)(3-\beta_k)\alpha_k^n\beta_k^n\left(\alpha_k^{m-n} - \beta_k^{m-n}\right)(\alpha_k - \beta_k)}{4(9k^2-1)}\widehat{\alpha_k}\widehat{\beta_k} \\
&= \frac{(3-\alpha_k)(3-\beta_k)\alpha_k^n\beta_k^n\left(\alpha_k^{m-n} - \beta_k^{m-n}\right)}{2\sqrt{9k^2-1}}\widehat{\alpha_k}\widehat{\beta_k},
\end{aligned}$$

which ends the proof. □

Theorem 7 (*d'Ocagne type identity for dual-complex* k*-Lucas-balancing numbers*) *Let* $k \geq 1$, $m \geq 0$, $n \geq 0$ *be integers such that* $m \geq n$. *Then*

$$\begin{aligned}
&\mathbb{DC}C_{k,m} \cdot \mathbb{DC}C_{k,n+1} - \mathbb{DC}C_{k,m+1} \cdot \mathbb{DC}C_{k,n} = \\
&= \frac{1}{4}\left(\alpha_k^{m-n} - \beta_k^{m-n}\right)(\beta_k - \alpha_k)\widehat{\alpha_k}\widehat{\beta_k},
\end{aligned}$$

where α_k, β_k *and* $\widehat{\alpha_k}$, $\widehat{\beta_k}$ *are given by* (5) *and* (6), *respectively.*

Proof. By formula (ii) of Theorem 3 we have

$$\begin{aligned}
&\mathbb{DC}C_{k,m} \cdot \mathbb{DC}C_{k,n+1} - \mathbb{DC}C_{k,m+1} \cdot \mathbb{DC}C_{k,n} \\
&= \frac{\alpha_k^m\widehat{\alpha_k}\beta_k^{n+1}\widehat{\beta_k} + \beta_k^m\widehat{\beta_k}\alpha_k^{n+1}\widehat{\alpha_k} - \alpha_k^{m+1}\widehat{\alpha_k}\beta_k^n\widehat{\beta_k} - \beta_k^{m+1}\widehat{\beta_k}\alpha_k^n\widehat{\alpha_k}}{4} \\
&= \frac{1}{4}\alpha_k^n\beta_k^n\left(\alpha_k^{m-n} - \beta_k^{m-n}\right)(\beta_k - \alpha_k)\widehat{\alpha_k}\widehat{\beta_k} \\
&= \frac{1}{4}\left(\alpha_k^{m-n} - \beta_k^{m-n}\right)(\beta_k - \alpha_k)\widehat{\alpha_k}\widehat{\beta_k},
\end{aligned}$$

which ends the proof. □

For $k = 1$ we obtain the Catalan, Cassini and d'Ocagne identities for the dual-complex Lucas-balancing numbers.

Corollary 4 (*Catalan type identity for dual-complex Lucas-balancing numbers*) *Let* $n \geq 0$, $r \geq 0$ *be integers such that* $n \geq r$. *Then*

$$\mathbb{DC}C_{n-r} \cdot \mathbb{DC}C_{n+r} - (\mathbb{DC}C_n)^2 = \\ = -\frac{1}{4}\left(2 - \left(\frac{\beta}{\alpha}\right)^r - \left(\frac{\alpha}{\beta}\right)^r\right)\hat{\alpha}\hat{\beta},$$

where α, β, $\hat{\alpha}$ *and* $\hat{\beta}$ *are given by* (7).

Corollary 5 (*Cassini type identity for dual-complex Lucas-balancing numbers*) *Let* $n \geq 1$ *be an integer. Then*

$$\mathbb{DC}C_{n-1} \cdot \mathbb{DC}C_{n+1} - (\mathbb{DC}C_n)^2 = 8\hat{\alpha}\hat{\beta},$$

where $\hat{\alpha}$ *and* $\hat{\beta}$ *are given by* (7).

Corollary 6 (*d'Ocagne type identity for dual-complex Lucas-balancing numbers*) *Let* $m \geq 0$, $n \geq 0$ *be integers such that* $m \geq n$. *Then*

$$\mathbb{DC}C_m \cdot \mathbb{DC}C_{n+1} - \mathbb{DC}C_{m+1} \cdot \mathbb{DC}C_n = -\sqrt{2}\left(\alpha^{m-n} - \beta^{m-n}\right)\hat{\alpha_k}\hat{\beta_k},$$

where α, β, $\hat{\alpha}$ *and* $\hat{\beta}$ *are given by* (7).

3 Concluding Remarks

Cobalancing numbers were defined and introduced in [10] by modification of formula (1). The authors called positive integer number n a cobalancing number with cobalancer r if

$$1 + 2 + \ldots + n = (n+1) + (n+2) + \ldots + (n+r).$$

Let b_n denote the nth cobalancing number. The nth Lucas-cobalancing number c_n is defined with $c_n = \sqrt{8b_n^2 + 8b_n + 1}$, see [7, 8].

In [11], we can find some relations of balancing and cobalancing numbers with Pell numbers. Related to these dependences it seems to be interesting to define dual-complex cobalancing numbers, dual-complex Lucas-cobalancing numbers and next to find relations of dual-complex balancing and cobalancing numbers with dual-complex Pell numbers (quaternions). For dual-complex Pell numbers details, see [3].

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