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Gauss Lucas theorem and Bernstein-type inequalities for polynomials

Liyaqat Ali
P. G. Department of Mathematics,
Kashmir University, Hazratbal,
Srinagar-190006, India

N. A. Rather
P. G. Department of Mathematics,
Kashmir University, Hazratbal,
Srinagar-190006, India
email: dr.narather@gmail.com

Suhail Gulzar
Department of Mathematics,
Government College for Engineering &
Technology Ganderbal, J& K
email: sgmattoo@gmail.com

Abstract. According to Gauss-Lucas theorem, every convex set containing all the zeros of a polynomial also contains all its critical points. This result is of central importance in the geometry of critical points in the analytic theory of polynomials. In this paper, an extension of Gauss-Lucas theorem is obtained and as an application some generalizations of Bernstein-type polynomial inequalities are also established.

1 Extension of Gauss-Lucas theorem

Let g be a real differential function, then Rolle's theorem guarantees the existence of at least one critical point (zero of its derivative g') between any two real zeros of g. While as, in case of analytic functions of a complex variable, Rolle's theorem does not hold in general. This fact can be realized from the function $g(z) = e^{iz} - 1$ which has zeros at $z = 0$ and $z = 2\pi$, however, its derivative g' has no zeros whatsoever.

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If the idea of a critical point lying between two points on the real line is replaced in the complex plane by the concept of a critical point located in some region containing the zero of the function then following result (see [3, pp. 22], [4, pp. 179]) called as Gauss-Lucas theorem gives a relative location of critical points with respect to the zero for polynomials.

Theorem 1 *The critical points of a non-constant polynomial* f *lie in the convex hull* $\mathcal{H}$ *of the zeros of* f.

Let a circle $\mathcal{C}$ encloses all the zeros of f then by theorem 1, $\mathcal{H} \subset \mathcal{C}$. On the other hand, through each pair of vertices of the polygon $\mathcal{H}$ a family of circles C_δ can be drawn which contains $\mathcal{H}$ and consequently all the zeros and critical points of f. The region $\Gamma = \cap C_\delta$ would also contain all zeros of f and f'. Hence, all the critical points of f must lie in the region common to all possible Γ's, which is equal to $\mathcal{H}$. Thus, an equivalent form of Theorem 1 can be stated as follows.

Theorem 2 *A circle* $\mathcal{C}$ *containing all the zeros of a non-constant polynomial* f *also encloses all the zeros of its derivative* f'.

In literature, there exist different variants of Gauss-Lucas theorem (for references see [3, pp. 22], [4, pp. 180], [5, pp. 71]). In this paper, we first present the following extension of Gauss-Lucas theorem.

Theorem 3 *Let all the zeros of an* n*th degree polynomial* $f(z)$ *lie in* $|z| \leq r$, *then for every* $\alpha \in \mathbb{C}$ *with* $\Re(\alpha) \leq \frac{n}{2}$, *the zeros of* $zf'(z) - \alpha f(z)$ *also lie in* $|z| \leq r$.

Proof. Let $P(z) = zf'(z) - \alpha f(z)$ and $w \in \mathbb{C}$ with $|w| > r$. Suppose $z_1, z_2, \dots, z_n$ be the zeros of $f(z)$, then $|z_\nu| \leq r$ and $|w| - |z_\nu| > 0$ for $\nu = 1, 2, \dots, n$. Now,

$$\begin{aligned}\frac{P(w)}{f(w)} &= \frac{wf'(w)}{f(w)} - \alpha = \sum_{\nu=1}^{n} \frac{w}{w - z_\nu} - \alpha \\ &= \frac{1}{2} \sum_{\nu=1}^{n} \frac{(w - z_\nu) + (w + z_\nu)}{w - z_\nu} - \alpha \\ &= \frac{n}{2} - \alpha + \frac{1}{2} \sum_{\nu=1}^{n} \frac{(w + z_\nu)(\bar{w} - \bar{z_\nu})}{|w - z_\nu|^2}.\end{aligned}$$

This implies

$$\Re\left(\frac{P(w)}{f(w)}\right) = \frac{n}{2} - \Re(\alpha) + \frac{1}{2} \sum_{\nu=1}^{n} \frac{|w|^2 - |z_\nu|^2}{|w - z_\nu|^2}$$

$$\geq \frac{n}{2} - \frac{n}{2} + \frac{1}{2} \sum_{\nu=1}^{n} \frac{|w|^2 - |z_\nu|^2}{|w - z_\nu|^2} > 0.$$

This further implies that $P(w) \neq 0$. Hence, $P(z)$ cannot have a zero in $|z| > r$. Therefore, we conclude that all the zeros of the polynomial $zf'(z) - \alpha f(z)$ lie in $|z| \leq r$. □

Note that the Theorem 2 follows by taking $\alpha = 0$ in Theorem 3.

2 Bernstein-type inequalities

The zero-preserving property of the derivative, which emerge out of Gauss-Lucas theorem, plays an important role in Bernstein-type inequalities for polynomials. A simple proof of the following result using Gauss-Lucas theorem can be found in a comprehensive book of Rahman & Schmeisser [5, pp. 510].

Theorem 4 *Let a polynomial* $F(z)$ *of degree* n *has all its zeros in* $|z| \leq 1$ *and* $G(z)$ *be a polynomial of degree at most* n *such that* $|G(z)| \leq |F(z)|$ *for* $|z| = 1$, *then*

$$|G'(z)| \leq |F'(z)| \qquad \text{for} \quad |z| \geq 1. \tag{1}$$

The equality holds outside the closed unit disk if and only if $G(z) \equiv e^{i\delta} F(z)$ *for some* $\delta \in \mathbb{R}$.

By taking $F(z) = Mz^n$, where $M = \max_{|z|=1} |G(z)|$, following sharp estimate for the derivative over closed unit disc, called as Bernstein's inequality [1], follows immediately.

Theorem 5 *Let* $G(z)$ *be a polynomial of degree at most* n, *then*

$$\max_{|z|=1} |G'(z)| \leq n \max_{|z|=1} |G(z)|. \tag{2}$$

The equality is attained in (2) if and only if $G(z) = az^n$, $a \neq 0$. Therefore, for the polynomials having zeros away from origin, there is a scope for an improvement in (2). In this regard, Erdös conjectured that if a polynomial $G(z)$ of degree n has no zero in $|z| < 1$, then

$$\max_{|z|=1} |G'(z)| \leq \frac{n}{2} \max_{|z|=1} |G(z)|. \tag{3}$$

This conjecture was later proved by Lax [2].

As an application of Theorem 3, here we next present the following extension of Theorem 4. The proof of this theorem is similar to that of theorem 4 given in [5].

Theorem 6 *Let a polynomial* $F(z)$ *of degree* n *has all its zeros in* $|z| \le 1$ *and* $G(z)$ *be a polynomial of degree at most* n *such that* $|G(z)| \le |F(z)|$ *for* $|z| = 1$, *then for every* $\alpha, \beta \in \mathbb{C}$ *with* $\Re(\alpha) \le n/2$, $\Re(\beta) \le n/2$ *and* $|z| \ge 1$, *we have*

$$|zG'(z) - \alpha G(z)| \le |zF'(z) - \alpha F(z)| \tag{4}$$

and

$$\begin{aligned} &|z^2G''(z) + (1-\alpha-\beta)zG'(z) + \alpha\beta G(z)| \\ &\qquad \le |z^2F''(z) + (1-\alpha-\beta)zF'(z) + \alpha\beta F(z)|. \end{aligned} \tag{5}$$

The bound is sharp and equality holds for some point z in $|z| > 1$ *if and only if* $G(z) = e^{i\delta}F(z)$ *for some* $\delta \in \mathbb{R}$.

Proof. Since the result holds trivially true, if $G(z) = e^{i\delta}F(z)$ for some $\delta \in \mathbb{R}$. Therefore, let $G(z) \ne e^{i\delta}F(z)$. Consider the function $\phi(z) = G^*(z)/F^*(z)$ where $F^*(z) = z^n\overline{F(1/\overline{z})}$ and $G^*(z) = z^n\overline{G(1/\overline{z})}$. Since $F(z)$ has its all zeros in $|z| \le 1$, then $F^*(z)$ has no zero in $|z| < 1$. This implies that the rational function $\phi(z)$ is analytic for $|z| \le 1$. Also, $|G(z)| = |G^*(z)|$ and $|F(z)| = |F^*(z)|$ for $|z| = 1$, therefore, $|\phi(z)| \le 1$ for $|z| = 1$. By invoking maximum modulus theorem, we obtain;

$$|\phi(z)| < 1 \qquad \text{for} \quad |z| < 1.$$

On replacing z by $1/z$ in the above inequality, we get $|G(z)| < |F(z)|$ for $|z| > 1$. It follows by Rouche's theorem that for any $\lambda \in \mathbb{C}$ with $|\lambda| \ge 1$, the polynomial $G(z) - \lambda F(z)$ of degree n has all its zeros in $|z| \le 1$. Applying Theorem 3 to the polynomial $P(z) = G(z) - \lambda F(z)$, for $\alpha \in \mathbb{C}$ with $\Re(\alpha) \le n/2$, we obtain that the polynomial

$$\begin{aligned} zP'(z) - \alpha P(z) &= z(G'(z) - \lambda F'(z)) - \alpha(G(z) - \lambda F(z)) \\ &= (zG'(z) - \alpha G(z)) - \lambda(zF'(z) - \alpha F(z)) \end{aligned}$$

has all zeros in $|z| \le 1$. This implies

$$|zG'(z) - \alpha G(z)| \le |zF'(z) - \alpha F(z)| \qquad \text{for} \quad |z| > 1. \tag{6}$$

If inequality (6) were not true, then there is some point $w \in \mathbb{C}$ with $|w| > 1$ such that $|wG'(w) - \alpha G(w)| > |wF'(w) - \alpha F(w)|$. By Theorem 3, $wF'(w) - \alpha F(w) \neq 0$. Now, choose $\lambda = -\frac{wG'(w)-\alpha G(w)}{wF'(w)-\alpha F(w)}$ and note that λ is a well defined complex number with modulus greater than 1. So, with this choice of λ, one can easily observe that w is a zero of $zP'(z) - \alpha P(z)$ of modulus greater than one. This is a contradiction, since all the zeros of this polynomial lie in $|z| \leq 1$. Hence, the inequality (6) is true, by continuity (6) also holds for $|z| = 1$. This proves the inequality (4).

Finally, if we take $H(z) = zG'(z) - \alpha G(z)$ and $K(z) = zF'(z) - \alpha F(z)$ with $\Re(\alpha) \leq n/2$, then by inequality (4), $|H(z)| \leq |K(z)|$ for $|z| = 1$. Therefore, by using inequality (4) again, for $\beta \in \mathbb{C}$ with $\Re(\beta) \leq n/2$, we get, $|zH'(z) - \beta H(z)| \leq |zK'(z) - \beta K(z)|$ for $|z| \geq 1$, which is equivalent to (5). This completes the proof of this theorem. □

For $\alpha = 0$ the inequality (4) reduces to (1).

The following result can be deduced from Theorem 6 by taking $F(z) = Mz^n$ where $M = \max_{|z|=1} |G(z)|$.

Corollary 1 *Let $G(z)$ be a polynomial of degree n and $\alpha, \beta \in \mathbb{C}$ with $\Re(\alpha) \leq n/2$, $\Re(\beta) \leq n/2$, then for $|z| \geq 1$,*

$$|zG'(z) - \alpha G(z)| \leq |n - \alpha||z|^n \max_{|z|=1} |G(z)| \tag{7}$$

$$\begin{aligned}|z^2 G''(z) + (1 - \alpha - \beta) zG'(z) + \alpha\beta G(z)| \\ \leq |n(n - \alpha - \beta) + \alpha\beta||z|^n \max_{|z|=1} |G(z)|.\end{aligned} \tag{8}$$

Equality in (7) and (8) hold for $G(z) = az^n$ where $a \neq 0$.

The next corollary follows by taking $\alpha = \beta = n/2$ in (7) and (8).

Corollary 2 *Let $G(z)$ be a polynomial of degree n, then for $|z| \geq 1$,*

$$\left| zG'(z) - \frac{n}{2} G(z) \right| \leq \frac{n}{2} |z|^n \max_{|z|=1} |G(z)| \tag{9}$$

$$\left| z^2 G''(z) + (1 - n) zG'(z) + \frac{n^2}{4} G(z) \right| \leq \frac{n^2}{4} |z|^n \max_{|z|=1} |G(z)|. \tag{10}$$

The inequalities (9) and (10) are sharp and equality holds for $G(z) = az^n$, $a \neq 0$

Next, if we take $\alpha = 1$ in (7) and $\beta = 0$ in (8), we obtain the following:

Corollary 3 *Let* $G(z)$ *be a polynomial of degree* $n \geq 2$ *and* $\alpha \in \mathbb{C}$ *with* $\Re(\alpha) \leq n/2$, *then for* $|z| \geq 1$,

$$|zG'(z) - G(z)| \leq (n-1)|z|^n \max_{|z|=1} |G(z)|$$

$$|zG''(z) + (1-\alpha)G'(z)| \leq n|n-\alpha||z|^{n-1} \max_{|z|=1} |G(z)|.$$

The results are best possible and $G(z) = az^n$, $a \neq 0$ *is the extremal polynomial for both the inequalities.*

The Theorem 6 and preceding corollaries can be improved for the class of polynomials having no zero in $|z| < 1$. For that, we require following lemmas.

3 Lemmas

If we are given an nth degree polynomial $f(z)$ which does not vanish for $|z| < 1$, then all the zeros of $q(z) = z^n\overline{f(1/\overline{z})}$ lie in $|z| \leq 1$ and $|f(z)| = |q(z)|$ for $|z| = 1$. Applying Theorem 6 by taking $G(z) = f(z)$ and $F(z) = q(z)$, we get:

Lemma 1 *Let a polynomial* $f(z)$ *of degree* n *has no zero in* $|z| < 1$ *and* $q(z) = z^n\overline{f(1/\overline{z})}$, *then for every* $\alpha, \beta \in \mathbb{C}$ *with* $\Re(\alpha) \leq n/2$, $\Re(\beta) \leq n/2$ *and* $|z| \geq 1$, *we have*

$$|zf'(z) - \alpha f(z)| \leq |zq'(z) - \alpha q(z)| \tag{11}$$

and

$$|z^2f''(z) + (1-\alpha-\beta)zf'(z) + \alpha\beta f(z)| \leq |z^2q''(z) + (1-\alpha-\beta)zq'(z) + \alpha\beta q(z)|. \tag{12}$$

Lemma 2 *Let* $f(z)$ *be a polynomial of degree* n *and* $q(z) = z^n\overline{f(1/\overline{z})}$, *then for every* $\alpha, \beta \in \mathbb{C}$ *with* $\Re(\alpha) \leq n/2$, $\Re(\beta) \leq n/2$ *and* $|z| \geq 1$, *we have*

$$|zf'(z) - \alpha f(z)| + |zq'(z) - \alpha q(z)| \leq (|n-\alpha| + |\alpha|)|z|^n \max_{|z|=1} |f(z)|,$$

and

$$|z^2f''(z) + (1-\alpha-\beta)zf'(z) + \alpha\beta f(z)| + |z^2q''(z) + (1-\alpha-\beta)zq'(z) + \alpha\beta q(z)| \leq |n(n-\alpha-\beta) + \alpha\beta| + |\alpha\beta|)|z|^n \max_{|z|=1} |f(z)|.$$

Proof. Let $M = \max_{|z|=1} |f(z)|$ then by Rouche's theorem, for every $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$, the polynomial $f(z) + \lambda M$ does not vanish in $|z| < 1$. Applying Lemma 1 to $f(z) + \lambda M$, then for $\alpha, \beta \in \mathbb{C}$ with $\Re(\alpha) \leq n/2$, $\Re(\beta) \leq n/2$ and $|z| \geq 1$, we have

$$|(zf'(z) - \alpha f(z)) + \lambda\alpha M| \leq |(zq'(z) - \alpha q(z)) + \bar{\lambda}(n-\alpha)Mz^n| \tag{13}$$

and

$$\begin{aligned}|z^2f''(z) + (1-\alpha-\beta)zf'(z) + \alpha\beta f(z) + \alpha\beta\lambda M|& \\ \leq |z^2q''(z) + (1-\alpha-\beta)zq'(z) + \alpha\beta q(z)& \\ + \bar{\lambda}\{n(n-\alpha-\beta) + \alpha\beta\}Mz^n|,&\end{aligned} \tag{14}$$

where $q(z) = z^n\overline{f(1/\bar{z})}$. In view of Corollary 1, we can choose argument of λ in (13) and separately in (14) such that

$$|(zq'(z) - \alpha q(z)) + \bar{\lambda}(n-\alpha)Mz^n| = |\bar{\lambda}||n-\alpha||z|^nM - |zq'(z) - \alpha q(z)|$$

and

$$\begin{aligned}&|z^2q''(z) + (1-\alpha-\beta)zq'(z) + \alpha\beta q(z)) + \bar{\lambda}\{n(n-\alpha-\beta) + \alpha\beta\}Mz^n| \\ &\quad = |\bar{\lambda}||n(n-\alpha-\beta) + \alpha\beta||z|^nM - |z^2q''(z) + (1-\alpha-\beta)zq'(z) + \alpha\beta q(z)|\end{aligned}$$

for $|z| \geq 1$. Using these inequalities in (13) and (14) and taking $|\lambda| = 1$, we conclude for $|z| \geq 1$,

$$|zf'(z) - \alpha f(z)| + |zq'(z) - \alpha q(z)| \leq (|n-\alpha| + |\alpha|)M|z|^n,$$

and

$$\begin{aligned}|z^2f''(z) + (1-\alpha-\beta)zf'(z) + \alpha\beta f(z)|& \\ + |z^2q''(z) + (1-\alpha-\beta)zq'(z) + \alpha\beta q(z)|& \\ \leq |n(n-\alpha-\beta) + \alpha\beta| + |\alpha\beta|)M|z|^n.&\end{aligned}$$

This completes the proof. □

4 Extension of Erdös-Lax theorem

Finally, we prove the following extension of inequality (3) for the class of polynomials having no zero in $|z| < 1$.

Theorem 7 *Let* $G(z)$ *be a polynomial of degree* n *and has no zero in* $|z| < 1$, *then for every* $\alpha, \beta \in \mathbb{C}$ *with* $\Re(\alpha) \leq n/2$, $\Re(\beta) \leq n/2$ *and* $|z| \geq 1$, *we have*

$$|zG'(z) - \alpha G(z)| \leq \frac{|n-\alpha| + |\alpha|}{2} |z|^n \max_{|z|=1} |G(z)| \tag{15}$$

and

$$|z^2 G''(z) + (1-\alpha-\beta)zG'(z) + \alpha\beta G(z)| \leq \frac{|n(n-\alpha-\beta) + \alpha\beta| + |\alpha\beta|}{2} |z|^n \max_{|z|=1} |G(z)|. \tag{16}$$

Equality in (15) *and* (16) *hold for* $G(z) = az^n + b$ *where* $|a| = |b| \neq 0$.

Proof. The proof follows by combining the lemmas 1 and 2. □

Note that inequality (3) follows from (15) by taking $\alpha = 0$.

If we take $\alpha = 1$ in (15) and $\beta = 0$ in (16), we obtain the following:

Corollary 4 *Let a polynomial* $G(z)$ *of degree* $n \geq 2$ *does not vanish for* $|z| < 1$ *and* $\alpha \in \mathbb{C}$ *with* $\Re(\alpha) \leq n/2$, *then for* $|z| \geq 1$,

$$|zG'(z) - G(z)| \leq \frac{n}{2} |z|^n \max_{|z|=1} |G(z)| \tag{17}$$

$$|zG''(z) + (1-\alpha)zG'(z)| \leq \frac{n|n-\alpha|}{2} |z|^{n-1} \max_{|z|=1} |G(z)|. \tag{18}$$

These inequalities are sharp.

Remark 1 *A polynomial* $f(z)$ *of degree* n *is said to be self-inversive if* $f(z) = \sigma q(z)$, *where* $q(z) = z^n \overline{f(1/\overline{z})}$ *and* $|\sigma| = 1$. *It is not difficult to prove that the Theorem 7 also holds for self-inversive polynomials as well.*

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