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Composition of continued fractions convergents to $\sqrt[3]{2}$

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Abstract. Applying geometrical construction in the 3-dim space, we compose all good convergents of $\sqrt[3]{2}$. The problem tackled in this paper is the nature of the continued fraction expansion of $\sqrt[3]{2}$: are the partial quotients bounded or not.

1 Introduction

The present paper uses some notations and results of [5] and [3].

We investigate $\sqrt[3]{2}$ and its adjunction ring. It is a common belief that the partial quotients in C.F.E. of $\sqrt[3]{2}$ that begins with

[1,3,1,5,1,1,4,1,1,8,1,14,1,10,2,1,4,12,2,3,2,1,3,4,1,1,2,14,3,12,1,15,3,1,4,534,1,...]

are not bounded, as supported by extensive computations, but there is no proof [4].

In the adjunction ring, we have the unit $\rho = 1 + \sqrt[3]{2} + \sqrt[3]{4}$ and its inverse $\sigma = -1 + \sqrt[3]{2}$. Multiplicative norm is defined in $\mathbb{Z}[\sqrt[3]{2}]$. Let $a = x + y\sqrt[3]{2} + z\sqrt[3]{4}$, its norm is $N(a) = x^3 + 2y^3 + 4z^3 - 6xyz$.

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2 Ambient vector space $\mathcal{V}$ and its geometry

Now, let $\mathcal{V} = \mathbb{R}^3$ be the 3-dimensional space endowed with the usual scalar product $\langle \mathbf{a}, \mathbf{b} \rangle$ and cross product $\mathbf{a} \times \mathbf{b}$. We define a linear mapping

$$\eta\colon \mathbb{Z}[\sqrt[3]{2}] \to \mathcal{V}$$

by $\eta(x + y\sqrt[3]{2} + z\sqrt[3]{4}) = (x, y, z)$, the resulting image consisting of all vectors with integer entries, multiplication inherited from $\mathbb{Z}[\sqrt[3]{2}]$.

Multiplication with σ will prove very important and we observe

$$\eta(\sigma \cdot a) = S\eta(a)$$

where S is the matrix

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

If $\mathbf{s}_j = \eta(\sigma^j)$, then we have $\mathbf{s}_{j+1} = S\mathbf{s}_j$,

$$\mathbf{s}_0 = (1, 0, 0), \quad \mathbf{s}_1 = (-1, 1, 0), \quad \mathbf{s}_2 = (1, -2, 1), \quad \mathbf{s}_3 = (1, 3, -3).$$

With the aid of diagonalization we can write

$$\mathbf{s}_j = \sigma^j \mathbf{h} + \rho^{\frac{j}{2}}(\mathbf{g}\cos(j\theta) + \mathbf{k}\sin(j\theta)) \tag{1}$$

where $\mathbf{h}$ and $\mathbf{g} \pm i\mathbf{k}$ are eigenvectors of matrix S

$$\begin{aligned} \mathbf{g} &= \frac{1}{6}(4, -\sqrt[3]{4}, -\sqrt[3]{2}), \\ \mathbf{k} &= \frac{\sqrt{3}}{6}(0, \sqrt[3]{4}, -\sqrt[3]{2}), \\ \mathbf{h} &= \frac{1}{6}(2, \sqrt[3]{4}, \sqrt[3]{2}) \end{aligned}$$

and the rotation angle is

$$\theta = \pi - \arctan \frac{\sqrt{3}\sqrt[3]{2}}{2 + \sqrt[3]{2}} \doteq 146.2^0.$$

Remark: Formula (1) can be extended for noninteger $t \in \mathbb{R}$

$$\mathbf{s}_t = \sigma^t \mathbf{h} + \rho^{\frac{t}{2}}(\mathbf{g}\cos(t\theta) + \mathbf{k}\sin(t\theta)) \tag{2}$$

Plane P, spanned by $\mathbf{g}, \mathbf{k}$, is the eigenplane, invariant for S, and together with the line of $\mathbf{h}$ forms the locus of zero norm.

The basic vectors $\mathbf{s}_j$ with increasing positive j are approaching the invariant plane and are for negative j almost collinear to eigenvector $\mathbf{h}$.

For each real N we consider the funnel

$$F_N = \{(x, y, z) \in \mathcal{V}; x^3 + 2y^3 + 4z^3 - 6xyz = N\},$$

i.e. points of norm $= N$. The positive funnels lie "above" the invariant plane $P : x + y\sqrt[3]{2} + z\sqrt[3]{4} = 0$, the negative ones "below". Figure 1 shows the funnel F_1 containing all the above units $\mathbf{s_j}$. The funnel flattens towards the invariant plane P spanned by vectors $\mathbf{g}$, $\mathbf{k}$, and embraces the line of $\mathbf{h}$.

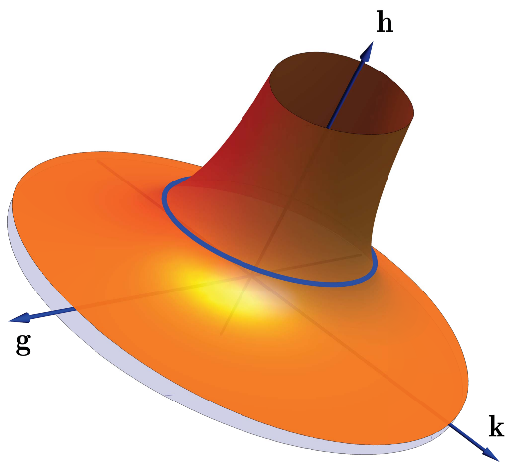

Figure 1: Funel F_1 with collar $\mathbf{c}_\phi$ and vectors $\mathbf{g}$, $\mathbf{k}$, $\mathbf{h}$.

3 Shortest vector algorithm

Definition 1 *We denote by M_j the lattice of integral vectors, orthogonal to $\mathbf{s}_j$.*

$$M_j = \{(x, y, z) \in \mathbb{Z}^3; \langle (x, y, z), \mathbf{s}_j \rangle = 0\}.$$

Using (1) we get a result on orthogonality

Lemma 1

$$T\mathbf{s}_{-j+1} \times T\mathbf{s}_{-j} = \mathbf{s}_j$$

where transposition T *is the linear transformation*

$$T(x, y, z) = (z, y, x).$$

Thus, vectors orthogonal to $\mathbf{s}_j$ are $T\mathbf{s}_{-j}$ and $T\mathbf{s}_{-j+1}$ and they form a basis for lattice M_j.

Lemma 2

$$M_j = \{mT\mathbf{s}_{-j+1} + nT\mathbf{s}_{-j}; m, n \in \mathbb{Z}\}.$$

Proof. Let (x, y, z) be point from M_j. Then $(x, y, z) = \alpha T\mathbf{s}_{-j+1} + \beta T\mathbf{s}_{-j}$ for some real α and β. Applying transformations T and S^j to this equation, we get

$$S^j(z, y, x) = \alpha \mathbf{s}_1 + \beta \mathbf{s}_0 = \alpha(-1, 1, 0) + \beta(1, 0, 0)$$

□

To prove our theorem, the length of vectors that form a basis of the lattice M_j is crucial to get good estimates. Therefore, we need the shortest basis vectors $\mathbf{u}_j$, $\mathbf{v}_j$ of lattice M_j. In [1] we find the construction called the *shortest vector algorithm* SVA, which gives the shortest lattice vectors $\mathbf{u}_j$, $\mathbf{v}_j$ and cross product preserved by construction

$$\mathbf{u}_j \times \mathbf{v}_j = \mathbf{s}_j. \quad (3)$$

Computations of the shortest vectors can be done inductively, because vectors $\left(S^T\right)^{-1}\mathbf{u}_j$, $\left(S^T\right)^{-1}\mathbf{v}_j$ form the basis of lattice M_{j+1}. This essentially reduces the SVA algorithm.

4 Multiplications in $\mathcal{V}$

We shall endow the 3-dim vector space $\mathcal{V}$ with some additional structures. We already know the usual scalar and vector products. The multiplication can also be inherited from the immersion of $\mathbb{Z}[\sqrt[3]{2}]$.

Definition 2 $(x, y, z) \otimes (a, b, c) = (ax + 2cy + 2bz, bx + ay + 2cz, cx + by + az)$.

If we allow for any real entries, the multiplication retains its favorable properties of commutativity, associativity and distibutivity.

Function $\gamma\colon \mathcal{V} \to \mathbb{R}$, $\gamma(x, y, z) = x + \sqrt[3]{2}y + \sqrt[3]{4}z$ is multiplicative with respect to the $\otimes$ product.

5 Collar and collar coordinates

First we shall define the collar in F_1, which is a topological circle of points $\mathbf{c}_\phi$ near the origin

$$\mathbf{c}_\phi = \mathbf{h} + \mathbf{g}\cos\phi + \mathbf{k}\sin\phi.$$

We shall prove a uniqueness theorem.

Theorem 1 *For every point* $(x, y, z) \in \mathcal{V}$*, which does not lie on the invariant plane or the invariant line, we have a unique representation*

$$(x, y, z) = \sqrt[3]{N}\mathbf{c}_\phi \otimes \mathbf{s}_t$$

for some $\phi \in [0, 2\pi)$*,* $t \in \mathbb{R}$ *and* N *is the norm of the given point.*

Proof. Since multiplication with $\sqrt[3]{N}$ moves points from F_1 to F_N, we can suppose $(x, y, z) \in F_1$ and try to solve the equation

$$(x, y, z) = \mathbf{c}_\phi \otimes \mathbf{s}_t \tag{4}$$

uniquely for $\phi \in [0, 2\pi)$, $t \in \mathbb{R}$.

Function γ is positive on F_1 and $\gamma(\mathbf{c}_\phi \otimes \mathbf{s}_t) = \sigma^t$, so $t = \log_\sigma \gamma(x, y, z)$ is defined. Point $T_0 = (x, y, z) \otimes \mathbf{s}_{-t}$ lies on F_1 and has development

$$T_0 = \mathbf{h} + \alpha\mathbf{g} + \beta\mathbf{k},$$

with $\alpha^2 + \beta^2 = 1$, and (4) holds for some $\phi \in [0, 2\pi)$.

Uniqueness is the consequence of identity

$$\mathbf{c}_\phi \otimes \mathbf{s}_t = \sigma^t\mathbf{h} + \rho^{t/2}(\mathbf{g}\cos(\phi + t\theta) + \mathbf{k}\sin(\phi + t\theta)).$$

□

Corollary 1 *Every point* $(x, y, z) \in \mathcal{V}$ *has a unique representation*

$$(x, y, z) = \sqrt[3]{N}\mathbf{c}_\phi \otimes \mathbf{s}_j \otimes \mathbf{s}_\kappa$$

where j *is integer,* $\kappa \in [-0.723, 0.277)$*,* $\phi \in [0, 2\pi)$ *and* N *is the norm of the point.*

In continuation of the article, *Mathematica* [6] is used to get some crucial numerical not sharp estimates of smooth elementary functions on compact interval or rectangle.

6 Some technical lemmas

Lemma 3

$$\rho^{\frac{\kappa}{4}} |\mathbf{c}_\phi \otimes \mathbf{s}_\kappa| \leq 1.152$$

for all $\phi \in [0, 2\pi]$ *and* $\kappa \in [-0.723, 0.277]$.

The chosen interval of unit length gives optimal inequality.

Lemma 4

$$0.5773 < |\mathbf{g} \cos \phi + \mathbf{k} \sin \phi| < 0.7534$$

for all $\phi \in [0, 2\pi]$.

Lemma 5

$$|\mathbf{s}_j| \geq \rho^{\frac{j}{2}} 0.576$$

for $j \geq 4$.

Proof. Estimate is the consequence of (1), Lemma 4 and

$$\begin{aligned} |\mathbf{s}_j| &= \rho^{\frac{j}{2}} \left| \sigma^{\frac{3j}{2}} \mathbf{h} + \mathbf{g} \cos(j\theta) + \mathbf{k} \sin(j\theta) \right| \\ &\geq \rho^{\frac{j}{2}} \left(|\mathbf{g} \cos(j\theta) + \mathbf{k} \sin(j\theta)| - \sigma^{\frac{3j}{2}} |\mathbf{h}| \right). \end{aligned}$$

□

Lemma 6

$$K = 1 + \frac{\delta}{\sqrt[3]{2} q_n^2} + \frac{\delta^2}{3 \sqrt[3]{4} q_n^4} < 1.0032$$

for $|\delta| < 0.196$ *and* $q_n \geq 7$.

Lemma 7

$$|\mathbf{u}_j| < 0.9328 \rho^{\frac{j}{4}}$$

for $j \geq 5$.

Proof. Because the angle $\varphi = \sphericalangle(\mathbf{u}_j, \mathbf{v}_j) \in [\pi/3, \pi/2]$, [1], we use (1), (3) and Lemma 4

$$\begin{aligned} |\mathbf{u}_j|^2 \frac{\sqrt{3}}{2} &\leq |\mathbf{u}_j| \, |\mathbf{v}_j| \sin \varphi = \left| \sigma^j \mathbf{h} + \rho^{\frac{j}{2}} (\mathbf{g} \cos(j\theta) + \mathbf{k} \sin(j\theta)) \right| \\ &\leq \sigma^j |\mathbf{h}| + \rho^{\frac{j}{2}} 0.7534 \leq \rho^{\frac{j}{2}} 0.7535 \end{aligned}$$

and Lemma folows. □

From this lemma and (1) we see that the length of vector $\mathbf{u}_j$ is of the order of the fourth root of the length of basis vectors $T\mathbf{s}_{-j+1}$, $T\mathbf{s}_{-j}$ of M_j.

Lemma 8 *On the unit sphere* $|\gamma(x, y, z)| \leq 1 + \sqrt[3]{2}$.

Lemma 9 *On the unit sphere* $\sqrt{N\gamma}(x, y, z) < 2.627$.

7 Representation with the shortest vector

Take now a n-th C.F. convergent $\frac{p_n}{q_n}$. As usual, we say

$$\frac{p_n}{q_n} - \sqrt[3]{2} = \frac{\delta}{q_n^2}.$$

We express the norm of the vector $(p_n, -q_n, 0)$ as

$$N = N(p_n, -q_n, 0) = p_n^3 - 2q_n^3 = \left(q_n\sqrt[3]{2} + \frac{\delta}{q_n}\right)^3 - 2q_n^3 = 3\sqrt[3]{4}q_n\delta K \quad (5)$$

where K is from Lemma 6.

Apply the collar representation

$$(p_n, -q_n, 0) = \sqrt[3]{N}\mathbf{c}_\phi \otimes \mathbf{s}_j \otimes \mathbf{s}_\kappa \quad (6)$$

and we shall first express q_n computing γ of the above equation:

$$\gamma(p_n, -q_n, 0) = p_n - \sqrt[3]{2}q_n = \sqrt[3]{N}\gamma(\mathbf{c}_\phi)\gamma(\mathbf{s}_j)\gamma(\mathbf{s}_\kappa),$$
$$\frac{\delta}{q_n} = \left(3\sqrt[3]{4}q_n\delta K\right)^{\frac{1}{3}} 1\sigma^j\sigma^\kappa.$$

We get

$$q_n = \sqrt{|\delta|}\left(3\sqrt[3]{4}\right)^{-\frac{1}{4}} K^{-\frac{1}{4}}\rho^{\frac{3j}{4}}\rho^{\frac{3\kappa}{4}}. \quad (7)$$

Now, let in the representation (6) be $\mathbf{a} = \sqrt[3]{N}\mathbf{c}_\phi \otimes \mathbf{s}_\kappa$ and calculate its length

$$|\mathbf{a}| = \left(\sqrt[3]{|N|}\rho^{-\frac{\kappa}{4}}\right)\left(\rho^{\frac{\kappa}{4}}|\mathbf{c}_\phi \otimes \mathbf{s}_\kappa|\right). \quad (8)$$

We transform the first factor in (8) using (5) and (7)

$$\sqrt[3]{|N|}\rho^{-\frac{\kappa}{4}} = \left(3\sqrt[3]{4}\right)^{\frac{1}{4}}\sqrt{|\delta|}K^{\frac{1}{4}}\rho^{\frac{j}{4}}. \quad (9)$$

On the other hand, since $\mathbf{u}_j \times \mathbf{v}_j = \mathbf{s}_j$ by (3) and using Lemma 5 we get

$$|\mathbf{v}_j||\mathbf{v}_j| \geq |\mathbf{u}_j||\mathbf{v}_j| > 0.576\rho^{\frac{j}{2}},$$

$$|\mathbf{v}_j| > 0.758\rho^{\frac{j}{4}}.$$

In the representation $(p_n, -q_n, 0) = \mathbf{a} \otimes \mathbf{s}_j$, the last component of $\otimes$ product is scalar product $\langle T\mathbf{a}, \mathbf{s}_j \rangle$, thus $T\mathbf{a} \perp \mathbf{s}_j$, i.e. $T\mathbf{a} \in M_j$. As $\mathbf{v}_j$ is the second shortest basis vector, we have

Lemma 10 *From* $|\mathbf{a}| < 0.758\rho^{\frac{j}{4}}$, *follows* $T\mathbf{a} = \pm\mathbf{u}_j$.

We are now prepared to formulate and prove

Theorem 2 *Let a C.F. convergent* $\frac{p_n}{q_n}$ *have* $|\delta| < 0.196$. *Then, for some* $j \in \mathbb{N}$, *there exists a representation*

$$(p_n, -q_n, 0) = T\mathbf{u}_j \otimes \mathbf{s}_j.$$

Proof. In equation (8) we estimate factors one by one using (9), Lemma 3 and Lemma 6

$$\begin{aligned} |\mathbf{a}| &< \left(3\sqrt[3]{4}\right)^{\frac{1}{4}} \sqrt{|\delta|} K^{\frac{1}{4}} \rho^{\frac{j}{4}} \left(\rho^{\frac{\kappa}{4}} |\mathbf{c}_\phi \otimes \mathbf{s}_\kappa|\right) \\ &< 1.478 \cdot 0.443 \cdot 1.0008 \cdot \rho^{\frac{j}{4}} \cdot 1.152 \\ &< 0.755\rho^{\frac{j}{4}}. \end{aligned}$$

This yields the condition of Lemma 10 and thus proves the theorem.

Conditions used in lemmas are satisfied for all convergents, which have $|\delta| < 0.196$, except convergent $\frac{5}{4}$, which has $j = 3$ and the theorem is true by inspection. □

From the first line of the proof we get an estimate of $\mathbf{u}_j$ in terms oh δ.

Corollary 2

$$|\mathbf{u}_j|^2 < 2.91|\delta|\rho^{\frac{j}{2}}.$$

If $\frac{p}{q}$ is convergent and B next partial quotient, then we have [2]

$$\frac{1}{q_n(B+2)} < \left|p_n - q_n\sqrt[3]{2}\right| < \frac{1}{qB}.$$

From this it follows that integer part of $\frac{1}{|\delta|}$ is B or $B+1$. Our Theorem covers all partial quotiens with B greater than 5. This may prove useful in search of big partial quotients.

Let as before, u_j be the shortes lattice vector of M_j. Then we have $T\mathbf{u}_j \otimes \mathbf{s}_j$ of the form $(p, -q, 0)$, $\frac{p}{q}$ not necessarily a C.F. convergent. Still it is a good approximation as the Theorem 3, some sort of converse of the Theorem 2 shows.

Theorem 3 *Let* j *be at least 5 and* $(p, -q, 0) = T\mathbf{u}_j \otimes \mathbf{s}_j$*. Then it holds*

$$|p - q\sqrt[3]{2}| < 2.11\sigma^{\frac{3j}{4}} \tag{10}$$

and for $\delta = q(p - q\sqrt[3]{2})$

$$|\delta| < 1.054. \tag{11}$$

Proof. We use Lemmas 7 and 8

$$|p - q\sqrt[3]{2}| = |\gamma(p, -q, 0)| = |\gamma(T\mathbf{u}_j)\gamma(\mathbf{s}_j)| = |T\mathbf{u}_j| \left|\gamma\left(\frac{T\mathbf{u}_j}{|T\mathbf{u}_j|}\right)\right| \sigma^j$$
$$< 0.9328\rho^{\frac{j}{4}}(1 + \sqrt[3]{2})\sigma^j < 2.11\sigma^{\frac{3j}{4}}$$

and (10) is proved.

From inequality (10) we have for some constant $|c| < 2.11$

$$p = q\sqrt[3]{2} + c\sigma^{\frac{3j}{4}}.$$

Function $R = \sqrt{N/\gamma}$ is defined outside the invariant plane $\gamma = 0$, where it is $\otimes$ multiplicative.

$$R^2(p, -q, 0) = \frac{p^3 - 2q^3}{p - q\sqrt[3]{2}} = p^2 + pq\sqrt[3]{2} + q^2\sqrt[3]{4}$$
$$= 3\sqrt[3]{4}q^2\left(1 + \frac{c}{q\sqrt[3]{2}}\sigma^{\frac{3j}{4}} + \frac{c^2}{3q^2\sqrt[3]{4}}\sigma^{\frac{3j}{2}}\right)$$
$$= 3\sqrt[3]{4}q^2\hat{K}^2$$

and $|q| = \frac{|R|}{\sqrt{3}\sqrt[3]{2}\hat{K}}$.

We have

$$R(p, -q, 0) = R(T\mathbf{u}_j)R(\mathbf{s}_j) = |T\mathbf{u}_j|R(\mathbf{b})\rho^{\frac{j}{2}},$$
$$\gamma(p, -q, 0) = \gamma(T\mathbf{u}_j)\gamma(\mathbf{s}_j) = |T\mathbf{u}_j|\gamma(\mathbf{b})\sigma^j,$$

where vector $\mathbf{b}$ is from the unit sphere. Using these equalities, Lemmas 7 and 9, we estimate

$$|\delta| = |q||p - q\sqrt[3]{2}| = \frac{|R(p, -q, 0)|}{\sqrt{3}\sqrt[3]{2}\hat{K}}|\gamma(p, -q, 0)| = \frac{|T\mathbf{u}_j|^2}{\sqrt{3}\sqrt[3]{2}\hat{K}}\sqrt{N(\mathbf{b})\gamma(\mathbf{b})}\sigma^{\frac{j}{2}}$$
$$< \frac{0.9328^2 2.627}{\sqrt{3}\sqrt[3]{2}\hat{K}} < \frac{1.048}{\hat{K}} < \frac{1.048}{\sqrt{0.9892}} < 1.054.$$

We have used inequality

$$\hat{K}^2 > 1 - \frac{2.11}{\sqrt[3]{2} \cdot 1} \sigma^{\frac{3 \cdot 5}{4}} - \frac{2.11^2}{3 \cdot 1^2 \sqrt[3]{4}} \sigma^{\frac{3 \cdot 5}{2}} > 0.9892.$$

□

Thus $\frac{p}{q}$ is a good rational approximation to $\sqrt[3]{2}$. If $|\delta| < 0.5$, then $\frac{p}{q}$ is C.F. convergent.

From the proof we get an estimate of δ in terms of $\mathbf{u}_j$.

Corollary 3

$$|\delta| < 1.22 |\mathbf{u}_j|^2 \sigma^{\frac{j}{2}}.$$

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