Acta Univ. Sapientiae, Mathematica, **16**, 1 (2024) 152–168

DOI: 10.47745/ausm-2024-0012

Uniqueness of an entire function sharing a small function with its linear differential polynomial with non-constant coefficients

Imrul Kaish
Department of Mathematics and Statistics, Aliah University, Kolkata, West Bengal 700160, India
email: imrulksh3@gmail.com

Md Majibur Rahaman
Department of Mathematics and Statistics, Aliah University, Kolkata, West Bengal 700160, India
email: majiburjrf107@gmail.com

Abstract. The uniqueness problems of entire functions sharing at least two values with their derivatives or linear differential polynomials have been studied and many results on this topic have been obtained. In our paper, we study the uniqueness of an entire function when it shares a small function with its first derivative and two linear differential polynomials of different orders. Here we consider the differential polynomial with non-constant coefficients. In particular, the result of the paper improves the results due to P. Li [7], I. Kaish and Md. M. Rahaman [4].

1 Introduction, definitions and results

Let us consider a non-constant meromorphic function f in the open complex plane $\mathbb{C}$. For a meromorphic function $a = a(z)$ defined in $\mathbb{C}$, we denote by $E(a; f)$ the set of zeros of $f - a$, counted with multiplicities and by $\overline{E}(a; f)$, the set of distinct zeros of $f - a$.

The investigation of uniqueness of an entire function sharing two values has been introduced by L. A. Rubel and C. C. Yang [9] in 1977 by the following result.

2010 Mathematics Subject Classification: 30D35
Key words and phrases: entire function, small function, differential polynomial, sharing

Theorem A [9] Let f be a non-constant entire function satisfying $E(a;f) = E(a;f^{(1)})$ and $E(b;f) = E(b;f^{(1)})$, for distinct finite complex numbers a and b. Then $f \equiv f^{(1)}$.

If for two meromorphic functions f and g, $E(a;f) = E(a;g)$ then we say that f and g share a CM and if $\overline{E}(a;f) = \overline{E}(a;g)$ then we say that f and g share a IM. In Theorem A, f and $f^{(1)}$ share a and b CM.

In 1979 considering IM sharing, E. Mues and N. Steinmetz [8] proved the following result.

Theorem B [8] Let f be a non-constant entire function satisfying $\overline{E}(a;f) = \overline{E}(a;f^{(1)})$ and $\overline{E}(b;f) = \overline{E}(b;f^{(1)})$. Then $f = f^{(1)}$.

From the following example we see that the two values cannot be replaced by a single value.

Example 1 *Let* $f(z) = \exp(e^z)\int_0^z \exp(-e^t)(1-e^t)dt$. *Then* $f^{(1)}-1 = e^z(f-1)$ *and so* $E(1;f) = E(1;f^{(1)})$ *but* $f \neq f^{(1)}$.

Considering a single shared value G. Jank, E. Mues and L. Volkmann [3] established the following result.

Theorem C [3] Let f be a non-constant entire function satisfying $\overline{E}(a;f) = \overline{E}(a;f^{(1)}) \subset \overline{E}(a;f^{(2)})$, for a non-zero constant a. Then $f = f^{(1)}$.

J. Chang and F. Fang [1] extended Theorem C by considering shared fixed points. Their result may be stated as follows.

Theorem D [1] Let f be a non-constant entire function satisfying $\overline{E}(z;f) = \overline{E}(z;f^{(1)}) \subset \overline{E}(z;f^{(2)})$, then $f = f^{(1)}$.

In 2009, I. Lahiri and G.K. Ghosh [5] extended Theorem D and proved the following theorem.

Theorem E [5] Let f be a non-constant entire function and $a(z) = \alpha z + \beta$, where $\alpha(\neq 0)$, β are constants. If $E(a;f) \subset E(a;f^{(1)})$ and $E(a;f^{(1)}) \subset E(a;f^{(2)})$, then either $f = A\exp(z)$ or $f = \alpha z + \beta + (\alpha z + \beta - 2\alpha)\exp\{\frac{\alpha z+\beta-2\alpha}{\alpha}\}$, where A is a non-zero constant.

For further discussion we need the following notation.

Let f be a non-constant meromorphic function and A be a set of complex numbers. For any meromorphic function $a = a(z)$, the integrated counting function $N_A(r, a; f)$ of zeros of $f - a$ which lie in $A \cap \{z : |z| \leq r\}$ is defined as

$$N_A(r, a; f) = \int_0^r \frac{n_A(t, a; f) - n_A(0, a; f)}{t} dt + n_A(0, a; f) \log r,$$

where $n_A(t, a; f)$ is the number of zeros of $f - a$, counted according to their multiplicities in $A \cap \{z : |z| \leq r\}$ and $n_A(0, a; f)$ be the multiplicity of the zeros of $f - a$ at origin. $T(r, f)$ be the characteristic function of f and $S(r, f)$ is any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. A meromorphic function $a = a(z)$ defined in $\mathbb{C}$ is called a small function of f if $T(r, a) = S(r, f)$. For standard definitions and notations we refer the reader to [2] and [10].

For two subsets A and B of $\mathbb{C}$, we denote by $A \triangle B$ the symmetric difference of A and B i.e. $A \triangle B = (A - B) \cup (B - A)$.

I. Lahiri and I. Kaish [6] extended Theorem E in the following way.

Theorem F [6] Let f be a non-constant entire function and $a = a(z)$ be a polynomial. Suppose that $A = \overline{E}(a; f) \triangle \overline{E}(a; f^{(1)})$ and $B = \overline{E}(a; f^{(1)}) \setminus \{\overline{E}(a; f^{(n)}) \cap \overline{E}(a; f^{(n+1)})\}$. If

(i) $deg(a) \neq deg(f)$,

(ii) $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{logT(r, f)\}$ and $N_B(r, a; f^{(1)}) = S(r, f)$,

(iii) each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity, then $f = \lambda e^z$, where $\lambda(\neq 0)$ is a constant.

Suppose that f be a non-constant entire function and $a_1, a_2, ..., a_n (\neq 0)$ are complex numbers. Then

$$L = L(f) = a_1 f^{(1)} + a_2 f^{(2)} + ... + a_n f^{(n)}, \tag{1}$$

is called a linear differential polynomial generated by f.

In 1999 P. Li [7] extended Theorem C by considering a linear differential polynomial and they prove the following theorem.

Theorem G [7] Let f be a non-constant entire function and L be defined by (1). Suppose that a be a non-zero finite value. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; L) \cap \overline{E}(a; L^{(1)})$, then $f = f^{(1)} = L$.

In 2018 I. Kaish and Md.M. Rahaman [4] improved Theorem F and Theorem G in the following way.

Theorem H [4] Let f be a non-constant entire function and $L = a_2f^{(2)} + a_3f^{(3)} + ... + a_nf^{(n)}$, where $a_2, a_3, ..., a_n(\neq 0)$ are constants , and $n(\geq 2)$ be an integer. Also let $a(z) \neq 0$ be a polynomial with $deg(a) \neq deg(f)$. Suppose that $A = \overline{E}(a; f) \triangle \overline{E}(a; f^{(1)})$ and $B = \overline{E}(a; f^{(1)}) \setminus \{\overline{E}(a; L) \cap \overline{E}(a; L^{(1)})\}$. If

(1) $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{logT(r, f)\}$,

(2) $N_B(r, a; f^{(1)}) = S(r, f)$, and

(3) each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity,

then $f = L = \lambda e^z$, where $\lambda(\neq 0)$ is a constant .

In this paper we consider a linear differential polynomial of an entire function f whose coefficients are small functions of f and we improve Theorem G and Theorem H by considering small function sharing by an entire function and its differential polynomials of various orders. The following theorem is our main result in the paper.

Theorem 1 *Let f be a non-constant entire function and $a = a(z)(\neq 0, \infty)$ be a small function of f with $a \not\equiv a^{(1)}$. Suppose that $A = \overline{E}(a; f) \triangle \overline{E}(a; f^{(1)})$, $B = \overline{E}(a; f^{(1)}) \setminus \{\overline{E}(a; L^{(p)}) \cap \overline{E}(a; L^{(q)})\}$, and $L = a_1(z)f^{(1)}(z) + a_2(z)f^{(2)}(z) + ... + a_n(z)f^{(n)}(z)$, where $a_1(z), a_2(z), ..., a_n(z)(\neq 0)$ are small functions of f and n, p, q are positive integers, $q > p \geq 0$. If*

(i) $E_{1)}(a; f) \subset \overline{E}(a; f^{(1)})$,

(ii) $N_A(r, a; f) + N_{A\cup B}(r, a; f^{(1)}) = S(r, f)$, *and*

(iii) *each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity,*

then $f = L = \delta e^z$, where $\delta(\neq 0)$ is a constant.

Putting $A = B = \emptyset$, we get the following corollary.

Corollary 1 *Let f be a non-constant entire function and $a = a(z)(\neq 0, \infty)$ be a small function of f with $a \not\equiv a^{(1)}$. If $E(a; f) = E(a; f^{(1)})$ and $\overline{E}(a; f^{(1)}) \subset \{\overline{E}(a; L^{(p)}) \cap \overline{E}(a; L^{(q)})\}$, where L is defined as in Theorem 1, then $f = L = \delta e^z$, where $\delta(\neq 0)$ is a constant.*

In Corollary 1, if we consider that a is a constant and L be a linear differential polynomial with constant coefficient then it is a particular case of Theorem G. Also in corollary if we consider that a is a polynomial with $\deg(a) \neq \deg(f)$, L is a linear differential polynomial with constant coefficient and $a_1 = 0$, $p = 0$, $q = 1$ then it is Theorem H.

We assume the following:

1. The degree of a transcendental entire function is infinity.
2. The order of a differential polynomial of f is the order of the highest ordered derivative of f presented in the polynomial.

2 Lemmas

In this section we give some necessary lemmas.

Lemma 1 [1] *Let f be a meromorphic function and k be a positive integer. Suppose that f is a solution of the following differential equation : $a_0 w^{(k)} + a_1 w^{(k-1)} + ... + a_k w = 0$, where $a_0 (\neq 0), a_1, a_2, ..., a_k$ are constants. Then $T(r, f) = O(r)$. Furthermore, if f is transcendental, then $r = O(T(r, f))$.*

Lemma 2 [1] *Let f be a meromorphic function and n be a positive integer. If there exists meromorphic functions $a_0 (\not\equiv 0), a_1, a_2, ..., a_n$ such that*

$$a_0 f^n + a_1 f^{n-1} + ... + a_{n-1} f + a_n \equiv 0,$$

then

$$m(r, f) \quad \leq \quad nT(r, a_0) + \sum_{j=1}^{n} m(r, a_j) + (n-1) \log 2.$$

Lemma 3 ([2], p. 68). *Let f be a transcendental meromorphic function and $f^n P(z) = Q(z)$, where $P(z), Q(z)$ are differential polynomials generated by f and the degree of Q is at most n. Then $m(r, P) = S(r, f)$.*

Lemma 4 ([2], p. 69). *Let f be a non-constant meromorphic function and*

$$g(z) = f^n(z) + P_{n-1}(f),$$

where $P_{n-1}(f)$ is a differential polynomial generated by f and of degree at most $n-1$.

If $N(r, \infty; f) + N(r, 0; g) = S(r, f)$, then $g(z) = h^n(z)$, where $h(z) = f(z) + \frac{a(z)}{n}$ and $h^{n-1}(z)a(z)$ is obtained by substituting $h(z)$ for $f(z)$, $h^{(1)}(z)$ for $f^{(1)}(z)$ etc. in the terms of degree $n-1$ in $P_{n-1}(f)$.

Lemma 5 ([2], p. 57). *Suppose that* g *be a non-constant meromorphic function and* $\psi = \sum_{\nu=0}^{l} a_\nu g^{(\nu)}$, *where* $a_\nu' s$ *are meromorphic functions satisfying* $T(r, a_\nu) = S(r, g)$ *for* $\nu = 1, 2, \ldots, l$. *If* ψ *is non-constant, then*

$$T(r,g) \quad \leq \quad \overline{N}(r,\infty;g) + N(r,0;g) + \overline{N}(r,1;\psi) + S(r,g).$$

Lemma 6 *Let* f *be a non-constant meromorphic function and* $a = a(z)$ *be a small function of* f *with* $a \not\equiv a^{(1)}$. *Then*

$$T(r,f) \quad \leq \quad \overline{N}(r,\infty;f) + N(r,a;f) + \overline{N}(r,a;f^{(1)}) + S(r,f).$$

Proof. Lemma follows from Lemma 5 for $g = f - a$ and $\psi = \frac{g^{(1)}}{a - a^{(1)}}$. □

Lemma 7 *Let* f *be a non-constant entire function and* $a = a(z)(\neq 0, \infty)$ *be a small function of* f *with* $a \not\equiv a^{(1)}$. *If*

(i) $N_A(r,a;f) + N_A(r,a;f^{(1)}) = S(r,f)$, *where* $A = \overline{E}(a;f) \triangle \overline{E}(a;f^{(1)})$,

(ii) *each common zero of* $f - a$ *and* $f^{(1)} - a$ *has the same multiplicity,*

then $T(r,f) \leq 2\overline{N}(r,a;f) + S(r,f)$.

Proof. Let z_0 be a zero of $f - a$ and $f^{(1)} - a$ with multiplicity $q(\geq 2)$. Then z_0 is a zero of $f^{(1)} - a^{(1)}$ with multiplicity $q - 1$. Hence z_0 is a zero of $a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a)$ with multiplicity $q - 1$.

Then we have

$$\begin{aligned} N_{(2}(r,a;f) \quad &\leq \quad 2N(r,0;a - a^{(1)}) + N_A(r,a;f) \\ &= \quad S(r,f). \end{aligned} \tag{2}$$

Again

$$\begin{aligned} \overline{N}(r,a;f^{(1)}) \quad &\leq \quad \overline{N}(r,a;f) + N_A(r,a;f^{(1)}) + S(r,f) \\ &= \quad \overline{N}(r,a;f) + S(r,f). \end{aligned} \tag{3}$$

Now using (2) and (3) and from Lemma 6, we get

$$T(r,f) \leq 2\overline{N}(r,a;f) + S(r,f).$$

□

Lemma 8 ([2], p.47). *Let* f *be a non-constant meromorphic function and* a_1, a_2, a_3 *be three distinct meromorphic functions satisfying* $T(r, a_\nu) = S(r, f)$ *for* $\nu = 1, 2, 3$. *Then*

$$T(r, f) \leq \sum_{\nu=1}^{3} \overline{N}(r, a_\nu; f) + S(r, f).$$

Lemma 9 ([10], p.92). *Suppose that* $f_1, f_2, \ldots, f_n (n \geq 3)$ *are meromorphic functions which are not constants except for* f_n. *Furthermore, let* $\sum_{j=1}^{n} f_j \equiv 1$.

If $f_n \not\equiv 0$ *and* $\sum_{j=1}^{n} N(r, 0; f_j) + (n-1) \sum_{j=1}^{n} \overline{N}(r, \infty; f_j) < \{\lambda + o(1)\} T(r, f_k)$, *where* $r \in I$, $k = 1, 2, \ldots, n-1$ *and* $\lambda < 1$, *then* $f_n \equiv 1$.

Lemma 10 *Let* f *be a non-constant entire function and* $a = a(z)(\not\equiv 0, \infty)$ *be a small function of* f *with* $a \not\equiv a^{(1)}$. *Suppose that* $A = \overline{E}(a; f) \triangle \overline{E}(a; f^{(1)})$, $B = \overline{E}(a; f^{(1)}) \setminus \{\overline{E}(a; L^{(p)}) \cap \overline{E}(a; L^{(q)})\}$, *where* L *is defined in Theorem 1 and* $q > p \geq 0$. *If*

(i) $E_{1)}(a; f) \subset \overline{E}(a; f^{(1)})$,

(ii) $N_A(r, a; f) + N_{A \cup B}(r, a; f^{(1)}) = S(r, f)$, *and*

(iii) *each common zero of* $f - a$ *and* $f^{(1)} - a$ *has the same multiplicity,*

then the function $h = \frac{f^{(1)} - a}{f - a}$ *is a small function of* f.

Proof. Let $F = f - a$. Then from

$$h = \frac{f^{(1)} - a}{f - a}, \tag{4}$$

we get

$$\begin{aligned} F^{(1)} = f^{(1)} - a^{(1)} &= f^{(1)} - a + (a - a^{(1)}) \\ &= hF + (a - a^{(1)}) \\ &= b_1 F + c_1, \end{aligned} \tag{5}$$

where $b_1 = h$, $c_1 = a - a^{(1)} = b$ (say).

Differentiating (5) and then using (5), we get

$$\begin{aligned} F^{(2)} &= b_1 F^{(1)} + b_1^{(1)} F + c_1^{(1)} \\ &= b_1 (b_1 F + c_1) + b_1^{(1)} F + c_1^{(1)} \end{aligned}$$

$$= (b_1 b_1 + b_1^{(1)})F + b_1 c_1 + c_1^{(1)}$$
$$= b_2 F + c_2,$$

where $b_2 = b_1 b_1 + b_1^{(1)}$ and $c_2 = b_1 c_1 + c_1^{(1)}$.

Similarly,

$$F^{(k)} = b_k F + c_k, \tag{6}$$

where $b_{k+1} = b_1 b_k + b_k^{(1)}$ and $c_{k+1} = c_1 b_k + c_k^{(1)}$.

If h is a constant then $T(r, h) = S(r, f)$ i.e. h is a small function of f. So we suppose h is non-constant.

Clearly from the hypothesis, we can obtain

$$\begin{aligned} N(r, 0; h) + N(r, \infty; h) &\le N_A(r, a; f) + N_A(r, a; f^{(1)}) \\ &= S(r, f). \end{aligned} \tag{7}$$

Now putting $k = 1$ in $b_{k+1} = b_1 b_k + b_k^{(1)}$, we get

$$b_2 = b_1 b_1 + b_1^{(1)} = h^2 + h^{(1)} = h^2 + h d_1,$$

where $d_1 = \frac{h^{(1)}}{h}$.

Again putting $k = 2$ in $b_{k+1} = b_1 b_k + b_k^{(1)}$, we have

$$\begin{aligned} b_3 &= b_1 b_2 + b_2^{(1)} \\ &= h^3 + 3 d_1 h^2 + d_2 h, \end{aligned}$$

where $d_2 = d_1^{(1)} + d_1^2$.

Similarly,

$$b_4 = h^4 + 6 d_1 h^3 + (d_2 + 6 d_1^2 + 3 d_1^{(1)}) h^2 + (d_2^{(1)} + d_1 d_2) h.$$

Therefore in general, we get for $k \ge 2$

$$b_k = h^k + \sum_{j=1}^{k-1} \alpha_j h^j, \tag{8}$$

where $T(r, \alpha_j) = O(\overline{N}(r, 0; h) + \overline{N}(r, \infty; h)) + S(r, h) = S(r, f)$, for $j = 1, 2, \ldots,$ $k - 1$.

Again putting $k = 1$ in $c_{k+1} = c_1 b_k + c_k^{(1)}$, we have

$$c_2 = c_1 b_1 + c_1^{(1)} = h b + b^{(1)}.$$

Also putting $k = 2$ in $c_{k+1} = c_1 b_k + c_k^{(1)}$, we can obtain

$$c_3 = bh^2 + (b^{(1)} + 2bd_1)h + b^{(2)}.$$

Similarly,

$$c_4 = bh^3 + (5hd_1 + b^{(1)})h^2 + (3b^{(1)}d_1 + 4bd_1^{(1)} + b^2 + d_2 b)h + b^{(3)}.$$

Therefore in general, we get for $k \geq 2$

$$c_k = \sum_{j=1}^{k-1} \beta_j h^j + b^{(k-1)}, \tag{9}$$

where $T(r, \beta_j) = O(\overline{N}(r, 0; h) + \overline{N}(r, \infty; h)) + S(r, h) = S(r, f)$, for $j = 1, 2, \ldots, k-1$.

Case 1. In this case we suppose that either $n \geq 2$ or $n = 1$, $a_1 \neq 1$ and $p \geq 0$ or $n = 1$, $a_1 = 1$ and $p > 0$.

We put

$$\Psi = \frac{(a - L^{(p)}(a))(f^{(1)} - a^{(1)}) - (a - a^{(1)})(L^{(p)}(f) - L^{(p)}(a))}{f - a} \tag{10}$$

Then by lemma of the logarithmic derivative, we get $m(r, \Psi) = S(r, f)$.

Also from the hypothesis

$$\begin{aligned} N(r, \Psi) &\leq N_{(2}(r, a; f) + N_A(r, a; f) + N_B(r, a; f^{(1)}) + N(r, \infty; a_k) \\ &= S(r, f). \end{aligned}$$

Therefore $T(r, \Psi) = S(r, f)$.

Now from (11), we have

$$\begin{aligned} \Psi F &= (a - L^{(p)}(a))F^{(1)} - bL^{(p)}(F) \\ &= (a - L^{(p)}(a))(hF + b) - b\sum_{k=1}^{n} a_k F^{(k+p)}, \quad \text{using (5)} \\ &= (a - L^{(p)}(a))(hF + b) - b\sum_{k=1}^{n} a_k [b_{k+p}F + c_{k+p}], \quad \text{using (6)} \\ &= (a - L^{(p)}(a))(hF + b) - b\sum_{k=1}^{n} a_k \left\{ h^{k+p} + \sum_{j=1}^{k+p-1} \alpha_j h^j \right\} F \end{aligned}$$

$$-b\sum_{k=1}^{n}a_k\left\{\sum_{j=1}^{k+p-1}\beta_j h^j + b^{(k+p-1)}\right\},\quad \text{using (8)},\ \text{using (9)}$$

Or,

$$\begin{aligned}&\left[\Psi - h(a-L^{(p)}(a)) + b\sum_{k=1}^{n}a_k\left\{h^{k+p}+\sum_{j=1}^{k+p-1}\alpha_j h^j\right\}\right]F\\ &+\left[b\sum_{k=1}^{n}a_k\left\{\sum_{j=1}^{k+p-1}\beta_j h^j + b^{(k+p-1)}\right\} - b(a-L^{(p)}(a))\right]=0.\end{aligned} \tag{11}$$

Or,

$$\Delta_1 F + \Delta_2 = 0, \tag{12}$$

where

$$\Delta_1 = \Psi - h(a-L^{(p)}(a)) + b\sum_{k=1}^{n}a_k\left\{h^{k+p}+\sum_{j=1}^{k+p-1}\alpha_j h^j\right\}$$

and

$$\Delta_2 = b\sum_{k=1}^{n}a_k\left\{\sum_{j=1}^{k+p-1}\beta_j h^j + b^{(k+p-1)}\right\} - b(a-L^{(p)}(a)).$$

If $\Delta_1 \equiv 0$, then by Lemma 2 we get $m(r,h) = S(r,f)$ and from (7), $T(r,h) = S(r,f)$.

Therefore we suppose $\Delta_1 \not\equiv 0$.

From (12) we get

$$F = -\frac{\Delta_2}{\Delta_1}. \tag{13}$$

From First Fundamental theorem and the properties of characteristic function, we can obtain

$$T(r,F) = O(T(r,h)) + S(r,f)$$

i.e.

$$\begin{aligned}T(r,f) &= T(r,F) + S(r,f)\\ &= O(T(r,h)) + S(r,f).\end{aligned} \tag{14}$$

Here Δ_1 is a polynomial of h of degree $n+p$ and Δ_2 is a polynomial of h of degree $n+p-1$. Also the coefficients of both the polynomials are small functions of h .

Without loss of generality we may suppose F is irreducible if not cancelling the common factor it can be made irreducible.

Since $N(r,F) = S(r,f)$, from (13) and (14), we get

$$N(r,0;\Delta_1) = S(r,h).$$

Also from (7) and (14), we have

$$N(r,\infty;h) = S(r,f) = S(r,h).$$

Then by Lemma 4, we get

$$\Delta_1 = \left(h + \frac{c}{n+p}\right)^{n+p}, \tag{15}$$

where c is the coefficient of h^{n+p-1} in Δ_1.

If $c \neq 0$ then from Lemma 8, we can obtain

$$\begin{aligned} T(r,h) &\leq \overline{N}(r,0;h) + \overline{N}(r,\infty;h) + \overline{N}\left(r,-\frac{c}{n+p};h\right) + S(r,h) \\ &= \overline{N}(r,0;\Delta_1) + S(r,h) \\ &= S(r,h), \end{aligned}$$

a contradiction.

Therefore $c = 0$ and from (15), $\Delta_1 = h^{n+p}$ and from (13), $F = -\frac{\Delta_2}{h^{n+p}}$.

Differentiating, we have

$$\begin{aligned} F^{(1)} &= -\frac{h^{n+p}\Delta_2^{(1)} - (n+p)h^{n+p-1}\Delta_2}{(h^{n+p})^2}h^{(1)} \\ &= d_1\frac{(n+p)\Delta_2 - h\Delta_2^{(1)}}{h^{n+p}}, \end{aligned}$$

where $d_1 = \frac{h^{(1)}}{h}$.

From the properties of characteristic function, we get

$$T(r,F^{(1)}) = (n+p)T(r,h) + S(r,h). \tag{16}$$

Again

$$F^{(1)} = hF + b = -\frac{\Delta_2}{h^{n+p-1}} + b,$$

Therefore

$$T(r, F^{(1)}) = (n + p - 1)T(r, h) + S(r, h). \tag{17}$$

From (16) and (17) we get $T(r, h) = S(r, h)$, which is a contradiction. Therefore

$$T(r, h) = S(r, f).$$

Case 2. In this case we suppose $n = 1$, $a_1 = 1$ and $p = 0$. Then $L^{(p)} = L = f^{(1)}$.
We put

$$\Psi_1 = \frac{(a - L^{(q)}(a))(f^{(1)} - a^{(1)}) - (a - a^{(1)})(L^{(q)}(f) - L^{(q)}(a))}{f - a}. \tag{18}$$

From the hypothesis

$$\begin{aligned} N(r, \Psi_1) &\leq N_A(r, a; f) + N_B(r, a; f^{(1)}) + N_{(2}(r, a; f) + N(r, \infty; a_k) \\ &= S(r, f). \end{aligned}$$

Also $m(r, \Psi_1) = S(r, f)$.
Therefore $T(r, \Psi_1) = S(r, f)$.
Now following the similar arguments of case-1 and using (18), we can prove

$$T(r, h) = S(r, f).$$

This proves the lemma. □

3 Proof of the main theorem

Proof.
To prove the theorem let us consider h as defined in Lemma 10.
That is,

$$h = \frac{f^{(1)} - a}{f - a}. \tag{19}$$

By Lemma 10, $T(r, h) = S(r, f)$.
Now from (19), we have

$$\begin{aligned} f^{(1)} &= hf + a(1 - h) \\ &= \xi_1 f + \eta_1, \end{aligned} \tag{20}$$

where $\xi_1 = h$ and $\eta_1 = a(1 - h)$.

Differentiating (20) and then using it, we get

$$\begin{aligned} f^{(2)} &= \xi_1^{(1)} f + \xi_1 f^{(1)} + \eta_1^{(1)} \\ &= \xi_1^{(1)} f + \xi_1 (\xi_1 f + \eta_1) + \eta_1^{(1)} \\ &= (\xi_1^{(1)} + \xi_1 \xi_1) f + \xi_1 \eta_1 + \eta_1^{(1)} \\ &= \xi_2 f + \eta_2, \end{aligned} \tag{21}$$

where $\xi_2 = \xi_1^{(1)} + \xi_1\xi_1$ and $\eta_2 = \eta_1^{(1)} + \xi_1\eta_1$.

Similarly

$$f^{(k)} = \xi_k f + \eta_k, \tag{22}$$

where $\xi_{k+1} = \xi_k^{(1)} + \xi_1\xi_k$ and $\eta_k = \eta_k^{(1)} + \eta_1\xi_k$.

Since $T(r, h) = S(r, f)$, we see that

$$T(r, \xi_k) + T(r, \eta_k) = S(r, f), \tag{23}$$

for $k = 1, 2, \ldots$.

Now

$$\begin{aligned} L^{(p)} &= \sum_{k=1}^{n} a_k f^{(k+p)} \\ &= \sum_{k=1}^{n} a_k (\xi_{k+p} f + \eta_{k+p}) \\ &= \left(\sum_{k=1}^{n} a_k \xi_{k+p} \right) f + \left(\sum_{k=1}^{n} a_k \eta_{k+p} \right) \\ &= \mu_1 f + \nu_1, \end{aligned} \tag{24}$$

where

$$\mu_1 = \sum_{k=1}^{n} a_k \xi_{k+p} \quad \text{and} \quad \nu_1 = \sum_{k=1}^{n} a_k \eta_{k+p}.$$

Since each a_k is a small function of f and from (23), $T(r, \mu_1) + T(r, \nu_1) = S(r, f)$.

Similarly

$$L^{(q)} = \mu_2 f + \nu_2, \tag{25}$$

where

$$\mu_2 = \sum_{k=1}^{n} a_k \xi_{k+q} \quad \text{and} \quad \nu_2 = \sum_{k=1}^{n} a_k \eta_{k+q}.$$

Also $T(r, \mu_2) + T(r, \nu_2) = S(r, f)$.

Let z_1 be a zero of $f - a$ such that $z_1 \notin A \cup B$. Then $f(z_1) = f^{(1)}(z_1) = L^{(p)}(z_1) = L^{(q)}(z_1) = a(z_1)$.

From (24) and (25), we get

$$\mu_1(z_1)a(z_1) + \nu_1(z_1) - a(z_1) = 0$$

and

$$\mu_2(z_1)a(z_1) + \nu_2(z_1) - a(z_1) = 0.$$

If $\mu_1(z)a(z) + \nu_1(z) - a(z) \not\equiv 0$, then

$$\begin{aligned}\overline{N}(r, a; f) &\leq N_A(r, a; f) + N(r, 0; \mu_1 a + \nu_1 - a) + S(r, f)\\ &= S(r, f).\end{aligned}$$

From Lemma 7, $T(r, f) = S(r, f)$, a contradiction. Therefore

$$\mu_1(z)a(z) + \nu_1(z) \equiv a(z). \tag{26}$$

Again if $\mu_2(z)a(z) + \nu_2(z) - a(z) \not\equiv 0$, then

$$\begin{aligned}\overline{N}(r, a; f) &\leq N_A(r, a; f) + N(r, 0; \mu_2 a + \nu_2 - a) + S(r, f)\\ &= S(r, f).\end{aligned}$$

From Lemma 7, $T(r, f) = S(r, f)$, a contradiction. Therefore

$$\mu_2(z)a(z) + \nu_2(z) \equiv a(z). \tag{27}$$

From (26) and (27), we see that $\mu_1(z) \equiv \mu_2(z) \equiv 1$ and $\nu_1(z) \equiv \nu_2(z) \equiv 0$.

Therefore from (24) and (25), we have $L^{(p)} \equiv L^{(q)} \equiv f$.

Let $q - p = r$. Then

$$L^{(p+r)} \equiv L^{(q)} \quad \text{or} \quad f^{(r)} \equiv f. \tag{28}$$

Solving (28), we get

$$f = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} + ... + p_t e^{\alpha_t z}, \tag{29}$$

where α_1, α_2,...,α_t are distinct roots of $z^r - 1 = 0$ and $p_1, p_2, ..., p_t$ are constants or polynomials.

Differentiating (29), we have

$$f^{(1)} = (p_1\alpha_1 + p_1^{(1)})e^{\alpha_1 z} + (p_2\alpha_2 + p_2^{(1)})e^{\alpha_2 z} + ... + (p_t\alpha_t + p_t^{(1)})e^{\alpha_t z}. \quad (30)$$

Now from (19), (29) and (30), we can obtain

$$hf - f^{(1)} = a(h-1).$$

Or,

$$\sum_{j=1}^{t}(hp_j - p_j\alpha_j - p_j^{(1)})e^{\alpha_j z} = a(h-1). \quad (31)$$

If $h \not\equiv 1$, then from (31) we get

$$\sum_{j=1}^{t}\frac{(hp_j - p_j\alpha_j - p_j^{(1)})}{a(h-1)}e^{\alpha_j z} \equiv 1. \quad (32)$$

Also we note that $T(r,f) = O(T(r,e^{\alpha_j z}))$ for $j = 1, 2, ..., t$.

If the left hand side of (32) contains more than two terms, then from Lemma 9 we get

$$\frac{(hp_j - p_j\alpha_j - p_j^{(1)})}{a(h-1)}e^{\alpha_j z} \equiv 1, \quad (33)$$

for one value of $j \in \{1, 2, ..., t\}$.

From (33), we see that

$$T(r, e^{\alpha_j z}) = S(r,f) = S(r, e^{\alpha_j z}),$$

a contradiction.

Now we suppose that the left hand side of (32) contains exactly two terms. Suppose (32) is of the form

$$\frac{(hp_l - p_l\alpha_l - p_l^{(1)})}{a(h-1)}e^{\alpha_l z} + \frac{(hp_m - p_m\alpha_m - p_m^{(1)})}{a(h-1)})e^{\alpha_m z} \equiv 1, \quad (34)$$

where $1 \leq l, m \leq t$.

From Lemma 8, we have

$$T(r, e^{\alpha_l z}) \leq \overline{N}(r, 0; e^{\alpha_l z}) + \overline{N}(r, \infty; e^{\alpha_l z})$$

$$
\begin{aligned}
&+\overline{N}\left(r, \frac{a(h-1)}{(hp_l - p_l\alpha_l - p_l^{(1)})}; e^{\alpha_l z}\right) + S(r, e^{\alpha_l z})\Bigg) \\
= \;& \overline{N}(r, 0; e^{\alpha_m z}) + S(r, e^{\alpha_l z}) \\
= \;& S(r, e^{\alpha_l z}),
\end{aligned}
$$

which is a contradiction.

Finally we suppose that the left hand side of (32) contains exactly one term, say, of the form

$$\frac{(hp_l - p_l\alpha_l - p_l^{(1)})}{a(h-1)} e^{\alpha_l z} \equiv 1.$$

This implies $T(r, e^{\alpha_l z}) = S(r, e^{\alpha_l z})$, a contradiction.

Therefore $h \equiv 1$. i.e., $f^{(1)} \equiv f$.
This implies $f = \delta e^z$, where $\delta(\neq 0)$ is a constant. Now

$$
\begin{aligned}
L^{(p)} &= \sum_{k=1}^{n} a_k f^{(p+k)} \\
&= \left(\sum_{k=1}^{n} a_k\right) \delta e^z.
\end{aligned}
$$

Since $L^{(p)} \equiv f$ i.e.,

$$\left(\sum_{k=1}^{n} a_k\right) \delta e^z \equiv \delta e^z, \quad \sum_{k=1}^{n} a_k \equiv 1.$$

Therefore

$$
\begin{aligned}
L &= \sum_{k=1}^{n} a_k f^{(k)} \\
&= \left(\sum_{k=1}^{n} a_k\right) \delta e^z = \delta e^z.
\end{aligned}
$$

Hence $f = L = \delta e^z$, where $\delta(\neq 0)$ is a constant. This completes the proof. □

References

[1] J. Chang and M. Fang, Uniqneness of entire function and fixed points, *Kodai Math. J.*, Vol. **25** (2002), pp. 309–320.

[2] W. K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).

[3] G. Jank, E. Mues and L. Volkmann, Meromorphe Funktionen, die mit ihrer ersten und zweiten Ableitung einen endlichen wert teilen, *Complex Var. Theory Appl.*, Vol.**6** (1986), pp.51–71.

[4] I. Kaish and Md. M. Rahaman, An entire function sharing polynomial with its linear differential polynomial, *Math. Bohem.* **143** (2018), No.2, pp. 213–224.

[5] I. Lahiri and G. K. Ghosh, Uniqueness of entire functions and their derivatives, *Ann. polon. Math.* **96** (2009), pp. 239–246.

[6] I. Lahiri, I. Kaish, An entire function sharing a polynomial with its derivatives, *Boll. Unione Mat. Ital.* **10** (2017), pp. 229–240.

[7] P. Li, Entire functions that share one value with their linear differential polynomials, *Kodai Math. J.*, **22**(1999), pp. 446–457.

[8] E. Mues and N. Steinmetz, Meromorphe functionen, die mit ihrer ableitung werte teilen, *Manuscripta Math.*, **29** (1979), pp. 195–206.

[9] L. A. Rubel and C. C. Yang, Values shared by an entire function and its derivative, in "Complex Analysis, Kentucky, 1976", *Lecture Notes in Math.*, Vol. 599, Springer (1977), pp. 101–103.

[10] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Science Press and Kluwer Academic Publishers (2003)

Received: June 04, 2023