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Uniqueness of Dirichlet series in the light of shared set and values

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Abstract. In this article, we have studied the uniqueness problem of Dirichlet series, which is convergent in a right half-plane and having analytic continuation in the complex plane as a meromorphic function sharing some sets and values. Our first result partially improve a result of [Ann. Univ. Sci. Budapest., Sect. Comput., **48**(2018), 117-128] by relaxing the sharing conditions. Most importantly, we have pointed out a number of big gaps in a recent paper [J. Contemp. Math. Anal., **56**(2021), 80-86], which makes the existence of the paper under question. Finally, under a different approach, we have provided the corrected form of the result of [J. Contemp. Math. Anal., **56**(2021), 80-86] as much as practicable.

1 Introduction and Main results

In 1737, Euler showed that the series $\sum \frac{1}{p}$ extended over all primes, diverges, which in turn actually proved the famous theorem on the existence of infinitely many primes. He deduced this from the fact that, for $\mathrm{Re}(s) > 1$, the zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ and $\zeta(s) \longrightarrow \infty$, when $s \longrightarrow 1$. Hundred years later, while studying a more general series $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$, where χ is a Dirichlet character and $s > 1$, Dirichlet proved his theorem on primes. This two types of series are actually a series of the form

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$$L(s,f)=\sum_{n\geq 1}\frac{f(n)}{n^s}, \tag{1}$$

where the coefficients are given by $f:\mathbb{N}\to\mathbb{C}$, is arithmetical function. This type of series are known as a Dirichlet series and these are very important in analytic number theory.

We recall briefly some classical facts about such series. There are a number of critical lines or abscissa connected to (1). We have the abscissa of absolute convergence σ_a and the abscissa of ordinary convergence σ_c. These numbers are such that the series converges in the prescribed sense to the right but not to the left of the abscissa. We also have the abscissa of uniform convergence σ_u, defined as the infimum of those σ for which the series converges uniformly in the half-plane $\mathrm{Re}(s)>\sigma$. We have trivially $-\infty\leq\sigma_c\leq\sigma_u\leq\sigma_a\leq+\infty$ and $\sigma_a-\sigma_c\leq 1$, if anyone of the abscissa is finite. A theorem of Bohr [2] says that

$$0\leq\sigma_a-\sigma_u\leq\frac{1}{2}. \tag{2}$$

This article deals with the uniqueness problem of Dirichlet series while two series share some values and set. In this paper, we will mainly study those Dirichlet series which are convergent in half plane and analytically continued as a meromorphic function. Uniqueness problem of Selberg class L-functions have recently been studied in various settings (see [10], [6], [11]). Recently Oswald-Steuding [12] considered a general class of entire functions re-presentable as Dirichlet series in some half plane Ω_a ($\{s=\sigma+it:\sigma>a\}$) and studied it's uniqueness properties.

Recall that two meromorphic functions f and g in $M(\mathbb{C})$ are said to share a value a $(\in\mathbb{C})$ CM (IM) if $f-a$ and $g-a$ have same zeros with same multiplicity (ignoring it's multiplicity).

Considering value sharing property of Dirichlet series in some right half plane, Oswald-Steuding obtained the following result.

Theorem A. [12] $L_j=L(s,f_j)$ $(j=1,2)$ *be non-constant entire function of finite order having convergent Dirichlet series representation in some right half plane. If* $L(s,f_1)$ *and* $L(s,f_2)$ *share two complex values CM, then they are identical.*

Next, Li [8] showed that *Theorem A* still holds when $L(s,f_j)$ $(j=1,2)$ have finitely many poles and obtained the following result.

Theorem B. [8] *Let* $L_j=L(s,f_j)$ $(j=1,2)$ *be two Dirichlet series convergent in a right half-plane and admit an analytic continuation in the complex plane* *as a meromorphic function of finite order having finitely many poles. If* L_1, L_2 *share two distinct complex values CM, then they are identical. The conclusion need not to hold if they have infinitely many poles.*

The above theorem does not hold for general Dirichlet series. Li in [8] showed $L_1(s) = 1 + e^s$ $L_2(s) = 1 + e^{-s}$, which are not ordinary Dirichlet series share 0, 1 CM, but they are not identical. Again considering Dirichlet series with infinitely many poles, Li proved the following theorem.

Theorem C. [8] $L_j = L(s, f_j)$ $(j = 1, 2)$ *be two Dirichlet series convergent in a right half-plane and admit an analytic continuation in the complex plane as a meromorphic function of finite order and* $f_1(1) = f_2(1)$. *If* $L(s, f_1)$ *and* $L(s, f_2)$ *share two complex values* a, b *CM and* $f_1(1) \neq a$, b *then they are identical.*

Natural question comes to one's mind whether it is possible to relax the strictly CM sharing condition in *Theorem A, B*. In our next theorem we will discuss this.

Let f and g be two non-constant meromorphic functions and consider a finite value $a \in \mathbb{C}$. In [9], Li defined that $f - a$ and $g - a$ have enough common zeros if $f - a$ and $g - a$ have same zeros with same multiplicities except an exceptional set G (say) of their zeros such that $n(r, G) = o(r)$ as $r \longrightarrow \infty$. Here by $n(r, G)$ we denote the counting function of G, i.e., the number of points in $G \cap \{|s| \leq r\}$ counted according it's multiplicity.

The order of G is denoted by $\rho(G)$ and is defined in a standard way as follows:

$$\rho(G) = \limsup_{r \longrightarrow \infty} \frac{\log n(r, G)}{\log r},$$

where $n(r, G)$ is an increasing function. According to (see p.17, [4]) we know that G is said to be of order k convergence type if $\int_{r_0}^{\infty} \frac{n(r,G)}{r^{k+1}}\, dr$ converges.

Now let us consider $L(s, f_p) = \sum_{n \geq 1} \frac{f_p(n)}{n^s}$ where $f_p : \mathbb{N} \longrightarrow \mathbb{C}$ is an arithmetical function with $f_p(n + x) = f_p(n)$ for some positive integer x. This series converges for $\sigma > 1$. Also by the periodicity of f_p we can write the series as

$$\begin{aligned} L(s, f_p) &= \sum_{n \geq 1} \frac{f_p(n)}{n^s} \\ &= \frac{1}{x^s} \left(f_p(1) \sum_{n \geq 0} \frac{1}{(n + 1/x)^s} + f_p(2) \sum_{n \geq 0} \frac{1}{(n + 2/x)^s} + \ldots + f_p(x) \sum_{n \geq 1} \frac{1}{n^s} \right) \quad (3) \\ &= \frac{1}{x^s} \sum_{a=1}^{x} f_p(a)\, \zeta\left(s, \frac{a}{x}\right). \end{aligned}$$

The analytic continuation of the Hurwitz zeta-function $(\zeta(s, a/x))$ leads immediately to an analytic continuation of $L(s, f_p)$. Hence, $L(s, f_p)$ is analytic throughout the whole complex plane and it can have a simple pole at $s = 1$, if $\sum_{a=1}^{x} f_p(a) \neq 0$.

Considering Dirichlet series with coefficients as periodic arithmetical functions and a meromorphic function g with finitely many poles, we have relaxed the CM sharing in *Theorems A, B* and obtained the following result.

Theorem 1 *Let $L = L(s, f_p)$ be a Dirichlet series where f_p is periodic arithmetical function. Also let g be a meromorphic function with finitely many poles and of order < 2 and $a,\ b \in \mathbb{C}$. If L and g share the value a CM except possibly a set G of order one convergence type and share b IM, then they are identical.*

Corollary 1 *Let $L = L(s, f_p)$ be an Dirichlet series and f_p is periodic arithmetical function. Also let g be a meromorphic function having finitely many poles and of order < 2 and $a,\ b \in \mathbb{C}$. If L, g share a CM and b IM, then they are identical.*

Corollary 2 *Let $L_j = L(s, f_j)$ $(j = 1, 2)$ be analytic functions of finite order having a convergent Dirichlet series representation of the form (1.1) in some right half-plane and f_j $(j = 1, 2)$ are periodic functions. Also let $a,\ b \in \mathbb{C}$. If L_1 and L_2 share a CM, b IM then they are identical.*

Before proceeding further, we require the following definitions.

Definition 1 *Let f and g be two non-constant meromorphic functions in $\mathcal{M}(\mathbb{C})$ and let S be a subset of $\mathbb{C}$. For some $a \in \mathbb{C} \cup \{\infty\}$, we define $E_f(S) = \cup_{a \in S}\{z : f(z) - a = 0\}$, where each point is counted according to its multiplicity. If we do not count the multiplicity then the set $\cup_{a \in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ then we say f and g share the set S CM. On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$ then we say f and g share the set S IM.*

Recently inspired by *Theorem A*, under IM sharing of some set of zeros of a uniqueness polynomial, Halder-Sahoo [3] obtained the following result.

Theorem D. [3] *Let L_j $j = 1, 2$, be two non constant entire functions having convergent Dirichlet series representations of the form (1) in certain right half-plane and one of them is of finite order. Let $S = \{a_1, a_2, \ldots, a_l\}$, where $a_1, a_2, \ldots, a_l$ are all distinct roots of the algebraic equation $P(w) = w^p + aw^q + b = 0$. Here, l is a positive integer satisfying $1 \leq l \leq p$, and p , q are* *relatively prime positive integers with* $p > 2$ *and* $p > q$*, and a and b are two finite nonzero constants. If* L_1 *and* L_2 *share* S *IM and they assume a common complex value* c $(\neq a_j, 1 \leq j \leq l)$ *for some* $s_0 \in \mathbb{C}$*, then* $L_1 = L_2$ *in some right half-plane.*

Remark 1 *The proof of Theorem D was mainly based on the Lemma 2.5 in* [3]*. But there are some logical errors in the proof of the lemma and this also caused a flaw in the proof of the Theorem D.*
(i) *First we would like to discuss the equation (2.6) and the argument just before (2.6) in* [3]*. We recall that, in* [3] $F(s; f_i) = \prod_{j=1}^{k}(L(s; f_i) - \gamma_j)^{l_j}$ $(i = 1, 2)$*. Since the set of Dirichlet series form a ring, it follows that each* $F(s; f_i)$ *is also a Dirichlet series and obviously it is zero free in some half plane. But that does not mean* $\lim_{\sigma \longrightarrow +\infty} F(s; f_i) =$ *some non zero constant, because as we know* $\lim_{\sigma \longrightarrow +\infty} L(s; f) = f(1) = 0 (\neq 0)$ *according as* $f(1) = 0 (\neq 0)$*. The authors claimed that "* $\lim_{\sigma \longrightarrow +\infty} F(s; f_1) = d_1$ *and* $\lim_{\sigma \longrightarrow +\infty} F(s; f_2) = d_2$ *for some non-zero constants* $d_1, d_2 \in \mathbb{C}$*, as* $F(s; f_1)$ *and* $F(s; f_2)$ *are non vanishing and convergent for all sufficiently large Re s". But it is easy to verify if* $f_i(1) = \gamma_j$ *for some* $1 \leq j \leq k$*, then we will have* $F(s; f_i) = \sum_{n \geq 2} \frac{\hat{f}_i(n)}{n^s}$ *and therefore* $\limsup_{r \longrightarrow +\infty} F(s; f_i) = 0$*. Next in* p. 83, l. 11*; using* Lemma 2.1 *authors get* $(L(s, f_2) - \gamma_j)^{-1} = L(s, g)$*, but the statement of Lemma 2.1 in* [3]*; which is actually Landu's theorem, is valid only when the constant term of the convergent Dirichlet series is non-zero. In* [3] (see p. 83, l. 18)*, using the same lemma the authors get* $\frac{F(s;f_1)}{F(s;f_2)} = L(s; \chi) = \sum_{n \geq m_1} \frac{\chi(n)}{n^s}$*; but clearly if* $f_2(1) = \gamma_j$*, for some* $1 \leq j \leq k$*, then we can not get any Dirichlet series as an inverse of* $(L(s; f_2) - \gamma_j)^{-1}$*, and so the construction of* $L(s; \chi)$ *is possible only if* $f_2(1) \neq \gamma_j$*. Also* $\lim_{\sigma \longrightarrow +\infty} \frac{F(s;f_1)}{F(s;f_2)} = 0$ *or* ∞ *according as* $f_1(1) = \gamma_j$ *or* $f_2(1) = \gamma_j$*; for some* $1 \leq j \leq k$*. Since there is no restriction on the choices of* γ_j*,* $j = 1, 2, \ldots, k$*; one can not always get* $\lim_{\sigma \longrightarrow +\infty} \frac{F(s;f_1)}{F(s;f_2)} =$ *non-zero constant. Hence the set considered in Theorem D should have been* $S = \{a_1, a_2, \ldots, a_l\}$ *with* $f_i(1) \notin S$*,* $i = 1, 2$ *and since the most important part of this lemma was based on the argument "* $\lim_{\sigma \longrightarrow +\infty} \frac{F(s;f_1)}{F(s;f_2)} = d_3$ $(\neq 0, \infty)$*", the proof of the theorem* [3] *is cease to be hold.*
(ii) *Next we want to point out another lacuna corresponding to Lemma 2.5 which actually makes a question about the existence of this lemma and consequently the same of the theorem. First we note that, as* $F(s; f_i)$*,* $i = 1, 2$ *are zero free in half plane, it follows that* $W(s) = \frac{F(s;f_1)}{F(s;f_2)}$ *is also zero and pole free in some half plane. Using this fact, with the assistance of the Hadamard factorization theorem, for sufficiently large* Re s*; the authors claimed that in some*

half plane, $W(s)$ *can be expressed as* $W(s) = \frac{F(s;f_1)}{F(s;f_2)} = e^{P_1(s)}$, *where* $P_1(s)$ *is a polynomial with* $\text{degree}(P_1) \leq \max\{\rho(F(s;f_1)), \rho(F(s;f_2))\}$. *But the Hadamard factorization theorem is only stands for the entire complex plane, not for some half plane. From* p. 142, [13], *we know that if a function* f *is holomorphicc and zero free in some simply connected domain* D, *then we can write it as* $f = e^g$ *for some* g, *holomorphic in* D *and therefore we can have some* $\hat{P}$, *holomorphic in the same half plane, so that we can write* $W(s) = e^{\hat{P}}$. *That is to say, in this case one can not assure* $\hat{P}$ *as a polynomial because it is not holomorphic in the entire plane.*

So in the proof of *Theorem D*, there are several gaps and as a result the existence of the whole paper is under question. Under these circumstances, it will be interesting to re-investigate the theorem. Here considering some arbitrary set $S \subset \mathbb{C}$, in the next theorem, we have obtained the best possible analogous corrected form of *Theorem D.*

Before going to the next theorem we recall the following definition.

Definition 2 *A polynomial* P *is called a uniqueness polynomial for meromorphic functions if for any two non-constant meromorphic functions* $f, g \in M(\mathbb{C})$, *the condition* $P(f) = P(g)$ *implies* $f \equiv g$.

Now let us consider a set $S = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{C}$ and denote the polynomial generated by the set S as

$$P(s) = s^n - \left(\sum a_i\right) s^{n-1} + \ldots + (-1)^{n-1} \sum (a_{i_1} a_{i_2} \ldots a_{i_{n-1}}) s + (-1)^n a_1 a_2 \ldots a_n.$$

We also have

$$P'(s) = n s^{n-1} - (n-1) \left(\sum a_i\right) s^{n-2} + \ldots + (-1)^{n-1} \sum (a_{i_1} a_{i_2} \ldots a_{i_{n-1}}).$$

Theorem 2 *Let* $L_j = L(s, f_j)$, $j = 1, 2$, *be entire functions of finite order having a convergent Dirichlet series representation of the form* (1) *in some right half-plane. Also let us consider a set* $S = \{a_i; i = 1, 2, \ldots, n\}$ *and* L_1, L_2 *share the set* S *CM except for a set* G *with* $n(r, G) = o(r)$. *Also let* $f_1(1) = f_2(1)$ *and* $P(f_1(1))$, $P'(f_1(1)) \neq 0$, *then we will have* $L_1 \equiv L_2$.

Corollary 3 *Let* $L_j = L(s, f_j)$, $j = 1, 2$, *be an entire function of finite order having a convergent Dirichlet series representation of the form* (1) *in some right half-plane. Also let us consider a set* $S = \{w : Q(w) = 0\}$, *where* Q *is a uniqueness polynomial and* $Q(f_j(1)) \neq 0$ *for* $j = 1, 2$ *and* L_1 *and* L_2 *share the set* S *CM except for a set* G *with* $n(r, G) = o(r)$. *If there exist a* $s_0 \in \mathbb{C}$ *where both* L_1, L_2 *take the same value* c $(Q(c) \neq 0)$, *then we will have* $L_1 \equiv L_2$.

As in *Theorem D*, it is easy to verify that the considered polynomial $w^n + aw^m + b$ is an uniqueness polynomial, so *Corollary 3* is also applicable to the set of zeros of the same polynomial.

In this paper we use Nevanlinna theory to prove our results. It is assumed that the readers are familiar with standard notations like the characteristic function $T(r,f)$, the proximity function $m(r,f)$, counting (reduced counting) function $N(r,f)$ ($\overline{N}(r,f)$) that are also explained in [15]. By $S(r,f)$ we mean any quantity that satisfies $S(r,f) = O(\log(rT(r,f)))$ when $r \longrightarrow \infty$, except possibly on a set of finite Lebesgue measure. When f has finite order, then $S(r,f) = O(\log r)$ for all r.

Let us take f a meromorphic function over $\mathbb{C}$, then the order of f is defined as

$$\rho(f) := \limsup_{r\longrightarrow\infty} \frac{\log T(r,f)}{\log r}.$$

In this paper we will need the following definition.

Definition 3 *Let f and g share a value a CM except for a set G. Let s_0 be a zero of $f-a$ of order p and a zero of $g-a$ of order q then by $\overline{N}(r,a;f \mid p=q)$ ($\overline{N}(r,a;f \mid p \neq q)$); we denote the reduced counting function of those common zeros of $f-a$ and $g-a$ where $p=q$ ($p \neq q$) where $p \geq 1,\ q \geq 0$.*

2 Lemma

Lemma 1 [5] *Let $g(s)$ a non-vanishing function represented by a convergent Dirichlet series $g(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ for some right half plane Ω_p and $f(1) \neq 0$ then its reciprocal also obeys a Dirichlet series representation, i.e., $1/g(s) = \sum_{n=1}^{\infty} \frac{f_o(n)}{n^s}$ in the same half-plane.*

Lemma 2 [1] *Let $f(s) = \sum_{n=1}^{\infty} \frac{f_o(n)}{n^s}$ in some Ω_0 and $g(s) = \sum_{n=1}^{\infty} \frac{g_o(n)}{n^s}$ in some Ω_1. Then in some right half-plane*

$$f(s)g(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s},$$

where $h = f_o * g_o$ Dirichlet convolution and $h(n) = \sum_{d|n} f_o(d) g_o(\frac{n}{d})$.

Lemma 3 [1] *Assume that $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges absolutely for $\sigma > \sigma_a$ and let $F(s)$ denote the sum function*

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \textit{for } \sigma > \sigma_a,$$

then $\lim_{\sigma \longrightarrow +\infty} F(s) = f(1)$.

Lemma 4 [9] *Let* $f(s) = \sum a_n e^{-\lambda_n s}$ *uniformly convergent in a half-plane* Ω_b *and admit an analytic continuation in* $\mathbb{C}$ *as a meromorphic function of finite order. Suppose that* $f(s)$ *tends to a finite non-zero limit as* $\sigma \longrightarrow \infty$, *then,*

$$\limsup_{r \longrightarrow +\infty} \frac{n(r, 0; f) + n(r, \infty; f)}{r} > 0.$$

Lemma 5 ([4], p.27]) *If* $D = \{a_n\}$ *is a sequence of non zero complex number such that* $\sum |a_n|^{-1}$ *converges, then the product* $E(s) = \prod(1 - \frac{s}{a_n})$ *is an entire function and satisfies*

$$\log |E(s)| \leq \int_0^{|s|} \frac{n(t, D)}{t} dt + |s| \int_{|s|}^{+\infty} \frac{n(t, D)}{t^2} dt.$$

3 Proofs of the theorems

Proof. [Proof of Theorem 2] It is given that $\prod_{i=1}^{n}(L_1 - a_i)$ and $\prod_{i=1}^{n}(L_2 - a_i)$ have enough common zeros. Let us denote the following function

$$F = \frac{\prod_{i=1}^{n}(L_1 - a_i)}{\prod_{i=1}^{n}(L_2 - a_i)}.$$

From the given condition we must have $n(r, 0; F) + n(r, \infty; F) \leq n(r, G) = o(r)$.

Now arranging the zeros c_k $(k = 1, 2 \ldots)$ and poles d_k $(k = 1, 2, \ldots)$ of F in an increasing order according to their moduli, we can make infinite Weierstrass product of it's zeros $\prod_1 = \prod E\left(\frac{s}{c_k}, r_1\right)$ and $\prod_2 = \prod E\left(\frac{s}{d_k}, r_2\right)$ of poles.

Then we can write F as

$$F = \frac{\prod_1}{\prod_2} s^k e^p, \tag{4}$$

where k is an integer and p is a polynomial.

Again $L_j = \sum_{n=1}^{\infty} \frac{f_j(n)}{n^s}$ is convergent in some half plane. Now consider $\epsilon : \mathbb{N} \longrightarrow \mathbb{C}$ by $\epsilon(n) = 1$ when $n = 1$ and 0 elsewhere. Now we can define $L(s; f_j) - a_i = L(s; f_j - a_i \epsilon)$ is convergent in some half plane. Since the collection of Dirichlet series form a ring then clearly $\prod_{i=1}^{n}(L_j - a_i) = \sum_{n=1}^{\infty} \frac{h_j(n)}{n^s}$ for $j = 1, 2$ is also a Dirichlet series under Dirichlet convolution '*' and convergent in the same half plane.

From *Lemma 2* clearly $h_j(1) \neq 0$ for $j = 1, 2$, since $f_j(1) \notin S$ for $j = 1, 2$. Then in the same half plane we can write it as $\frac{\prod_{i=1}^{n}(L_1-a_i)}{\prod_{i=1}^{n}(L_2-a_i)} = \frac{\sum_{n=1}^{\infty}\frac{h_1(n)}{n^s}}{\sum_{n=1}^{\infty}\frac{h_2(n)}{n^s}}$.

Again using *Lemma 1* we have the reciprocal of $\sum_{n=1}^{\infty}\frac{h_2(n)}{n^s}$ is also Dirichlet series convergent in the same half plane. Now using this fact and Dirichlet convolution we will get

$$\frac{\prod_{i=1}^{n}(L_1-a_i)}{\prod_{i=1}^{n}(L_2-a_i)} = \sum_{n=1}^{\infty}\frac{\hat{f}(n)}{n^s} = \sum_{n\geq m_a}^{\infty}\frac{\hat{f}(n)}{n^s},$$

in some half plane, where m_a is the least natural number with $\hat{f}(m_a) \neq 0$.

Now we can write it as

$$\sum_{n\geq m_a}^{\infty}\frac{\hat{f}(n)}{n^s} = \hat{f}(m_a)m_a^{-s}\sum_{n\geq m_a}\frac{\hat{f}(n)}{\hat{f}(m_a)}\left(\frac{n}{m_a}\right)^{-s} = \hat{f}(m_a)m_a^{-s}\hat{g}(s), \tag{5}$$

where $\hat{g}(s) = \sum_{n\geq m_a}\frac{\hat{f}(n)}{\hat{f}(m_a)}e^{-\alpha_n s}$ and $\{\alpha_n\}$ is a strictly increasing sequence $\longrightarrow \infty$ as $n \longrightarrow \infty$.

Since $\hat{g}(s) \longrightarrow 1$ as $\sigma \longrightarrow +\infty$ and clearly the Dirichlet series is convergent in some Ω_q and hence σ_c is finite. Now using the fact (2) we can say it converges uniformly in some Ω_{u_o}.

Now from *Lemma 4* we must have

$$\limsup_{r\longrightarrow+\infty}\frac{n(r,0;\hat{g})+n(r,\infty;\hat{g})}{r} > 0 \implies \limsup_{r\longrightarrow+\infty}\frac{n(r,0;F)+n(r,\infty;F)}{r} > 0,$$

a contradiction. Hence we must have $\hat{g} = 1$.

Clearly from (4) and (5) we have

$$F = \frac{\prod_{i=1}^{n}(L_1-a_i)}{\prod_{i=1}^{n}(L_2-a_i)} = \hat{f}(m_a)m_a^{-s}. \tag{6}$$

Again from *Lemma 3*, taking $\sigma \longrightarrow +\infty$ the left hand side of (6) tends to a finite value 1, and hence we must have $m_a = 1$ and then it becomes

$$\frac{\prod_{i=1}^{n}(L_1-a_i)}{\prod_{i=1}^{n}(L_2-a_i)} = 1. \text{ i.e., } \frac{P(L_1)}{P(L_2)} = 1. \tag{7}$$

Now

$P(L_1) = P(L_2)$

$$(L_1^n - L_2^n) - \sum a_i\left(L_1^{n-1} - L_2^{n-1}\right) + \ldots (-1)^{n-1}\sum(a_{i_1}a_{i_2}\ldots a_{i_{n-1}})(L_1 - L_2) = 0. \quad (8)$$

Taking $\sigma \longrightarrow +\infty$ from (8) and the fact $P'(f_1(1)) \neq 0$ we have $L_1 \equiv L_2$.

□

Proof. [Proof of Theorem 1] The proof of this theorem is mainly based on the idea of a paper of Li [7].

Here it is given that g is a meromorphic function of order < 2 and g, L share two values b IM and a CM except a set G of order one convergence type, i.e., $\int_{r_0}^{+\infty} \frac{n(t,G)}{t^2} < +\infty$ for some $r_0 \geq 0$. Now let $a_i, i = 1, 2, \ldots$ be the non-zero elements of the set G repeated according to its multiplicity then we have

$$\sum_{i=1}^{\infty} |a_i|^{-1} = \int_0^{\infty} \frac{d(n(t, G\backslash\{0\}))}{t} \leq \int_0^{+\infty} \frac{n(t, G\backslash\{0\})}{t^2} < +\infty.$$

Let us consider the following auxiliary function

$$H = \frac{g - a}{L - a}.$$

Clearly zeros and poles of H come from the set G as well as the poles of L and g respectively. Let us consider two sets G_1 and G_2 such that $G_1 \cup G_2 = G\backslash\{0\}$ and the elements of G_1 are zeros of H and the elements of G_2 are poles of H.

Next let us construct the functions $h_i(s) = \prod_{k=1}^{\infty}\left(1 - \frac{s}{a_k^i}\right)$, where a_k^i $(i = 1, 2)$ are points of G_i arranging as $|a_k^i| \leq |a_{k+1}^i|$, repeated according to their multiplicities.. Since G is of order one convergence type then clearly G_i $(i = 1, 2)$ are also order one convergence type, i.e., $\int_{r_0}^{+\infty} \frac{n(t,G_i)}{t^2} dr < +\infty$ $(i = 1, 2)$ for some $r_0 \geq 0$.

We have,

$$\begin{aligned}\sum |a_k^i|^{-1} &= \int_0^{+\infty} \frac{dn(t, G_i)}{t} \leq \lim_{t \longrightarrow \infty} \frac{n(t, G_i)}{t} + \int_0^{+\infty} \frac{n(t, G_i)}{t^2} dt \\ &= \int_0^{+\infty} \frac{n(t, G_i)}{t^2} dt < +\infty.\end{aligned}$$

Using *Lemma 5* we have,

$$\begin{aligned}\log |h_i(s)| &\leq \int_0^{|s|} \frac{n(t, G_i)}{t} dt + |s| \int_{|s|}^{\infty} \frac{n(t, G_i)}{t^2} dt \quad (9) \\ &= \int_0^{r_0} \frac{n(t, G_i)}{t} dt + \int_{r_0}^{|s|} \frac{n(t, G_i)}{t} dt + |s| \int_{|s|}^{\infty} \frac{n(t, G_i)}{t^2} dt.\end{aligned}$$

As we have $\int_{r_0}^{\infty} \frac{n(t,G_i)}{t^2} dt < +\infty$ for some $r_0 \geq 0$, it follows that we can have a positive ϵ such that $\int_{r}^{\infty} \frac{n(t,G_i)}{t^2} dt < \epsilon$ for some large $r \geq r_0$ and so we have

$$\int_{r}^{r_1} \frac{n(t, G_i)}{t^2} dt < \epsilon.$$

Again as $n(t, G_i)$ is an increasing function so we get

$$\begin{aligned} n(r, G_i) \int_{r}^{r_1} \frac{dt}{t^2} &\leq \int_{r}^{r_1} \frac{n(t, G_i)}{t^2} dt \\ \frac{n(r, G_i)}{Kr} \leq n(r, G_i) \int_{r}^{r_1} \frac{dt}{t^2} &\leq \int_{r}^{r_1} \frac{n(t, G_i)}{t^2} dt < \epsilon, \end{aligned}$$

hence for some large r we have $n(r, G_i) \leq Kr\epsilon$, where K is some constant.

From (9), we can have $\log |h_i(s)| \leq O(\epsilon|s|)$. As h_i's, $(i = 1, 2)$ are entire functions, it follows that $T(r, h_i) = m(r, h_i)$. Then,

$$\begin{aligned} T(r, h_i) = m(r, h_i) &= \frac{1}{2\pi} \int_{0}^{2\pi} \log^{+} |h_i(re^{i\theta})| d\theta \\ &= \frac{1}{2\pi} \int_{\theta \in \Theta} \log |h_i(re^{i\theta})| d\theta \leq O(r), \end{aligned} \tag{10}$$

where $\Theta = \{\theta : |h_i(re^{i\theta})| > 1\}$.

We can also write H as

$$H = \frac{g - a}{L - a} = \frac{h_1}{h_2} (s - 1)^k s^m Q e^{p(s)}, \tag{11}$$

for some polynomial $p(s)$ and a rational Q which actually comes from the poles of g.

Using the Second Fundamental Theorem we have,

$$\begin{aligned} T(r, L) &\leq \overline{N}(r, a; L) + \overline{N}(r, b; L) + \overline{N}(r, \infty; L) + S(r, L) \\ &\leq \overline{N}(r, a; g \mid p = q) + \overline{N}(r, b; g) + \sum_{i=1}^{2} \overline{N}(r, 0; h_i) + O(\log r) + S(r, L) \\ &\leq 2T(r, g) + O(r) + O(\log r) + S(r, L) < O(r^2) + S(r, L), \end{aligned}$$

as $r \longrightarrow \infty$, clearly here $\rho(L) < 2$.

Since g and L have order < 2 then from (11) we must have $p(s) = p_1 s + p_2$, a linear polynomial of degree at most one and so we have

$$H = \frac{h_1}{h_2} Q s^m (s-1)^k e^{p_1 s + p_2}.$$

Again,

$$\begin{aligned} \overline{N}(r, b; L) &= \overline{N}(r, b; g) \le \overline{N}(r, 1; H) \le T(r, H) \\ &\le T(r, h_1) + T(r, h_2) + T(r, e^{p_1 s + p_2}) + O(\log r) \le O(r). \end{aligned} \tag{12}$$

Now let us consider an auxiliary function

$$I = \left(\frac{g'}{(g-a)(g-b)} - \frac{L'}{(L-a)(L-b)} \right)(g-L). \tag{13}$$

Clearly here the poles of I come from the poles of g and L which are finitely many and from the set G. So in view of (10)-(12) we have, $T(r, I) = m(r, I) + N(r, I) \le O(T(r, H)) + O(\log r) + O(r) \le O(r)$.

Now,

$$\begin{aligned} \overline{N}(r, a; L) &= \overline{N}(r, a; L \mid p = q) + \overline{N}(r, a; L \mid p \ne q) \\ &\le \overline{N}(r, 0; I) + \sum_{i=1}^{2} \overline{N}(r, 0; h_i) \le O(r). \end{aligned} \tag{14}$$

Using (14), (12) and the Second Fundamental Theorem we have,

$$T(r, L) \le \overline{N}(r, a; L) + \overline{N}(r, b; L) + O(\log r) \le O(r), \tag{15}$$

but in view of p. 214, [14] we know, $O(r \log r) = N(r, 0; L) \le T(r, L) \le O(r)$, a contradiction.

Therefore in view of (12) and (14) we must have either $I \equiv 0$ or $H \equiv 1$. Now $H \equiv 1$ gives $f \equiv \mathcal{L}$. If $f \not\equiv \mathcal{L}$, then from $I \equiv 0$, on integration we have $c\frac{g-a}{g-b} = \frac{L-a}{L-b}$ and hence using the First Fundamental Theorem we have $T(r, g) = T(r, L) + O(1)$. Now if $c \ne 1$ then we get $\frac{(c+1)g-ca-b}{(c-1)g-ca+b} = \frac{2L-a-b}{b-a}$, since L can have at most one pole at $s = 1$ then we can write $(c-1)g - ca + b = (s-1)^m R e^q$ where R is a rational function, $m = 0$ or 1 and q is a polynomial of degree at most one. From this we will have $T(r, g) = O(r)$ and this implies $T(r, L) = O(r)$ which leads towards a contradiction. Hence $c = 1$ and therefore we get $g \equiv L$. □

Proof. [Proof of Corollary 2] It is given that L_1 and L_2 share a CM and b IM. Now let us consider the following auxiliary function $F = \frac{L_1-a}{L_2-a}$. Now we can have a polynomial $\widehat{p(s)}$ such that $F = \frac{L_1-a}{L_2-a} = e^{\widehat{p(s)}}$. Now from *Theorem 1.1* in [8] we can write F as $F = ce^{\alpha s}$ for some constant α and c. Now considering the same auxiliary function I and proceeding similarly as done in (12)-(15) we will have a contradiction. And finally we will get L_1, L_2 share a, b CM, then with the help of *Theorem B* we will get the result. □

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