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Asymptotic properties of a nonparametric conditional distribution function estimator in the local linear estimation for functional data via a functional single-index model

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Abstract. This paper deals with the conditional distribution function estimator of a real response variable given a functional random variable (i.e takes values in an infinite dimensional space). Specifically, we focus on the functional index model, this approach represents a good compromise between nonparametric and parametric models. Then we give under general conditions and when the variables are independent, the quadratic error and asymptotic normality of estimator by local linear method, based on the single-index structure. Moreover, as an application, the asymptotic $(1-\gamma)$ confidence interval of the conditional distribution function is given for $0 < \gamma < 1$.

Keywords: Mean squared error, single functional index , conditional distribution function, nonparametric estimation, local linear estimation, Asymptotic normality, functional data.

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1 Introduction

The estimation of the conditional cumulative distribution function has great importance. In fact, it is involved in many applications, such as reliability, survival analysis (see Zamanzade and all. [31], Tabti and Ait Saadi [28]), ... Moreover, there are several prediction tools in the nonparametric statistics branch, for instance the conditional mode, the conditional median or the conditional quantiles, which are based on the preliminary estimation of this nonparametric model. In the nonfunctional case, the local polynomial fitting has been the subject of considerable studies and key references on this topic are Fan and Yao [14], Fan [16], Fan and Gijbels [15] and references therein. However, only few results are available for the local linear modeling in the functional statistics setup. Indeed, the first results, in this direction, were established in Baillo and Grané [6]. These papers focus on the local linear estimation of the regression operator when the explanatory variable takes values in a Hilbert space. The general case, where the regressors do not belong to a Hilbert space but just to a semi-metric space, has been considered in Barrientos-Marin et al. [7] and El Methni and Rachdi [13]. In these works, authors obtained the almost-complete convergence (a.co.), with rates, of the proposed estimator. Other alternative versions of the local linear modeling for functional data were investigated (see Boj et al.[8]; Baillo and Grané [6]; El Methni and Rachdi [13]), for the regression operator and Demongeot et al.[10]; Demongeot et al.[12], Xiong and al. [30], for the conditional density function, Demongeot et al.[11] for the conditional distribution function). in the case of spatial data Laksaci et al.[23] they established pointwise almost complete convergence with rate.
Furthermore, the functional index model plays a major role in statistics. The interest of this approach comes from its use to reduce the dimension of the data by projection in fractal space. The literature on this topic is closely limited, the first work which was interested in the single-index model on the nonparametric estimation is Ferraty et al.[17] they stated for i.i.d. variables and obtained the almost complete convergence under some conditions. Based on the cross-validation procedure, Ait Saidi et al. [2] proposed an estimator of this parameter, where the functional single-index is unknown. See Ait Saidi et al. [1] for the dependant case. Attaoui et al.[4] considered the nonparametric estimation of the conditional density in the single functional model. They established its pointwise and uniform almost complete convergence (a.co.) rates. In the same topic, Attaoui et al.[5] proved the asymptotic results of a nonparametric conditional cumulative distribution estimator for time series data. Ait Saadi and Mecheri [3], established the pointwise and the uniform almost

complete convergence (with the rate) of the kernel estimate of of the conditional cumulative distribution function of a scalar response variable Y given a Hilbertian random variable X when the observations are linked via a single-index structure. Ferraty and al. [21] proposed an estimator based on the idea of functional derivative estimation of a single index parameter. Hamdaoui and al. [22] established The asymptotic normality of the conditional distribution kernel estimator.

Tabti and al. [28] obtained the almost complete convergence and the uniform almost complete convergence of a kernel estimator of the hazard function with quasi-association condition when the observations are linked with functional single-index structure. In this paper, we focus on the local linear estimation with the single-index structure to compute under some conditions, the quadratic error of the conditional distribution function estimator. In practice, this study has great importance, because, it permits to construct a prediction method based on the maximum risk estimation with a single functional index.

In Section 2, We introduce the estimator of our model in the single-functional index. In Section 3 we introduce assumptions and asymptotic properties are given.

Finally, Section 5 is devoted to the proofs of the results.

2 The model

Let $\{(X_i, Y_i), \ 1 \le i \le n\}$ be n random variables, independent and identically distributed as the random pair (X, Y) with values in $\mathcal{H} \times \mathbb{R}$, where $\mathcal{H}$ is a separable real Hilbert space with the norm $\| \,.\, \|$ generated by an inner product $< .,. >$. We consider the semi-metric d_θ associated to the single index $\theta \in \mathcal{H}$ defined by $\forall x_1, x_2 \in \mathcal{H} : d_\theta(x_1, x_2) := |< x_1 - x_2, \theta >|$. Assume that the explanation of Y given X is done through a fixed functional index θ in $\mathcal{H}$. In the sense that, there exists a θ in $\mathcal{H}$ (unique up to a scale normalization factor) such that: $\mathbb{E}[Y|X] = \mathbb{E}[Y| < \theta, X >]$. The conditional probability distribution of Y given $X = x$ denoted by $F_\theta(.|x)$ exists and is given by $\forall y \in \mathbb{R}, \ F_\theta(y|x) := F(y| < x, \theta >)$. In the following, we denote by $F(\theta, ., x)$, the conditional distribution function of Y given $< x, \theta >$ and we define the local linear estimator for single-index structure $\widehat{F}(\theta, ., x)$ of $F(\theta, ., x)$ by:

$$\widehat{F}(\theta, y, x) = \frac{\sum\limits_{1 \le i,j \le n} W_{ij}(\theta, x) H({h_H}^{-1}(y - Y_j))}{\sum\limits_{1 \le i,j \le n} W_{ij}(\theta, x)} = \frac{\sum\limits_{1 \le j \le n} \Omega_j K_j H_j}{\sum\limits_{1 \le j \le n} \Omega_j K_j},$$

with

$$W_{ij}(\theta,x)=\beta_\theta(X_i,x)\Big(\beta_\theta(X_i,x)-\beta_\theta(X_j,x)\Big)K(h_K^{-1}d_\theta(x,X_i))K(h_K^{-1}d_\theta(x,X_j)),$$

and $\Omega_j K_j = \sum_{i=1}^{n} W_{ij}$ with $\beta_\theta(X_i,x)$ is a known bi-functional operator from $\mathcal{H}^2$ into $\mathbb{R}$ where K is a kernel, H is a cumulative distribution function and $h_K := h_{n,K}$ (resp $h_H := h_{n,H}$) is a sequence that decrease to zero as n goes to infinity.

3 Assumptions and Mains results

All along the paper, we will denote by C, C' and $C_{\theta,x}$ some strictly positive generic constants and by $K_i(\theta,x) := K(h_K^{-1}d_\theta(x,X_i))$, $\forall x\in\mathcal{H}, i=1,\dots,n$, $H_j := H({h_H}^{-1}(y-Y_j)), \forall y\in\mathbb{R}, j=1,\dots,n.$, $\beta_{\theta,i} := \beta_\theta(X_i,x)$, $W_{ij}(\theta,x) := W_{\theta,ij}$ and we will use the notation $B_\theta(x,h_K) := \{x_1\in\mathcal{H}:\ 0<|<x-x_1,\theta>|<h_K\}$, the ball centered at x with radius h_K. Moreover, for find the results in our paper we denote: for any $l\in\{0,2\}$ $\psi_l(.,y) := \dfrac{\partial^l F(.,y,.)}{\partial y^l}$,
$\Phi_l(s)=\mathbb{E}[\psi_l(X,y)-\psi_l(x,y)|\beta_\theta(x,X)=s]$,
and $\phi_{\theta,x}(r_1,r_2)=\mathbb{P}(r_1\le d_\theta(x,X)\le r_2)$.
In order to study our asymptotic results we need the following assumptions:

(H1) **(i)** $\mathbb{P}(X\in B_\theta(x,h_K)) =: \phi_{\theta,x}(h_K)>0$,

(ii) assume that there exists a function $\chi_{\theta,x}(\cdot)$ such that

$$\forall s\in[-1,1]\ \lim_{h_K\to 0}\frac{\phi_{\theta,x}(sh_K,h_K)}{\phi_{\theta,x}(h_K)}=\chi_{\theta,x}(s).$$

(iii) For any $l\in\{0,2\}$, the quantities $\Phi_l'(0)$ and $\Phi_l^{(2)}(0)$ exist, where Φ_l' (resp.$\Phi_l^{(2)}$) denotes the first (resp. the second) derivative of Φ_l

(H2) The conditional distribution function $F(\theta,y,x)$ satisfies that there exist some positive constants b_1 and b_2, such that for all (x_1,x_2,y_1,y_2)

$$|F(\theta,y_1,x)-F(\theta,y_2,x)|\le C(|d_\theta(x_1,x_2)|^{b_1}+|y_1-y_2|^{b_2})$$

(H3) The bi-functional $\beta_\theta(.,.)$ satisfies:

(i) $\forall x'\in\ \mathcal{F},\ C_1\,d_\theta(x,x')\le|\beta_\theta(x,x')|\le C_2\,d_\theta(x,x'),$ where $C_1,C_2>0,$

(ii) $\sup_{u\in B(x,r)} |\beta_\theta(u,x) - d_\theta(x,u)| = o(r)$,

(iii) $h_K \int_{B(x,h_K)} \beta_\theta(u,x) dP(u) = o\left(\int_{B(x,h_K)} \beta_\theta^2(u,x)\, dP(u)\right)$

Where $B_\theta(x,r) = \{x' \in \mathcal{H}/|d_\theta(x,x') \leq r\}$ and $dP(x)$ is the cumulative distribution of X.

(H4) **(i)** The kernel K is a positive function, which is supported within $[-1,1]$, and $K(1) > 0$.

(ii) The kernel K is a differentiable function and its derivative K' satisfies

$$K^2(1) - \int_{-1}^{1} (K^2(u))' \chi_{\theta,x}(u) du > 0$$

(H5) The kernel H is a differentiable function and bounded, such that:

$$\int H^{(1)}(t)dt = 1\ ,\ \int |t|^{b_2} H^{(1)}(t)dt < \infty \text{ and } \int H^2(t)dt < \infty.$$

(H6) The bandwidths h_K,h_H satisfies:

(i) $\lim_{n\to\infty} h_K = 0$, $\lim_{n\to\infty} h_H = 0$ and $\lim_{n\to\infty} \frac{\log\log n}{n\phi_{\theta,x}(h_K)}$,

(ii) $\exists \eta_0 \in \mathbb{N}$, $\forall \eta > \eta_0$, $\frac{1}{\phi_{\theta,x}(h_K)} \int_{-1}^{1} \phi_{\theta,x}(th_K, h_K) \frac{d}{dt}(t^2K(t))dt > C_3 > 0$

Comments on assumptions: The first part of assumption (H1) characterizes the concentration property of the probability measure of the functional variable X, which permits to control the effect of the topological structure in the asymptotic results (see Ferraty et al. [19]), the second part of assumption is known as (for small h) the concentration assumption acting on the distribution of X in infinite dimensional spaces.The function χ_x plays a determinant role. It is possible to specify this function in the above examples by
1. $\chi_0(u) = \delta_1(u)$; where $\delta_1(.)$ is Dirac function,
2. $\chi_0(u) = 1_{]0;1]}(u)$.
The third part of (H1) characterizes the functional space of our model, it is obvious that this condition is closely related to the existence of the functions, ψ_l and Φ_l, (see Ferraty et al.[20], for more discussions on the link between their derivatives). Moreover, this condition is used in order to keep the usual form of the quadratic error (see Vieu, 1991 [29]). However, if we replace the

third part of assumption (H1), by the following Lipschitz condition (where $\mathcal{N}_z$ denotes a neighborhood of z):
$\forall(y_1, y_2) \in \mathcal{N}_y \times \mathcal{N}_y$ and $\forall(x_1, x_2) \in \mathcal{N}_x \times \mathcal{N}_x$

$$|F(\theta, y_1, x) - F(\theta, y_2, x)| \leq C(|d_\theta(x_1, x_2)|^{b_1} + |y_1 - y_2|^{b_2})$$

which is less restrictive than assumption (H2), then Theorem 3.1's final result becomes as follows:

$$\mathbb{E}\left[\widehat{F}(\theta, y, x) - F(\theta, y, x)\right]^2 = o(h_H^4 + h_K^2) + o\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right).$$

Such expression of the rate of convergence of our estimator is inaccurate and cannot be useful to determine the smoothing parameters. In other words, the third part of assumption (H1) on the differentiability of the conditional density permits to determine the unknown constants in the mean squared error (MSE). Thus, the third part of assumption (H1) may be considered as a good compromise permitting to obtain an asymptotically exact expression of the convergence rate of $\hat{F}(\theta, x, y)$. while the assumption (H2) is a regularity condition which characterizes the functional space, of our model, and is needed to evaluate the bias term in the asymptotic results. Then, assumption (H3) has been introduced and commented, first, in Barrientos et al. [7] and it plays an important role in our methodology, particularly when we will compute exact constant terms involved in the asymptotic result. The second part of the condition (H3) is verified, for instance, if $d_\theta(\cdot, \cdot) = \beta_\theta(\cdot, \cdot)$, moreover if

$$\lim_{d_\theta(x,u)\to 0} \left|\frac{\beta_\theta(u, x)}{d_\theta(x, u)} - 1\right| = 0.$$

Moreover, assumption (H6) is classicaly used and is standard in the context of the quadratic error determination in functional statistics and is common in the setting of functional local linear fitting (see for instance Laksaci et al. [23] and Rachdi et al. [26]). The rest of the hypotheses are imposed for a sake of brevity of our results's proofs. Moreover, one could find in Ferraty and Vieu [18] some examples of kernels K and H satisfying assumptions (H4) and (H5). The small ball probability effects are really inherent to our infinite dimensional context, as exemples, we can cite diffusion processes and Gaussian processes (see F.Ferraty, A.Laksaci and P. Vieu [19]).

3.1 Mean square convergence

In this part, we are going to show the asymptotic results of quadratic-mean convergence

Theorem 1 *Under assumptions (H1)-(H6), we obtain:*

$$\mathbb{E}\left[\widehat{F}(\theta,y,x)-F(\theta,y,x)\right]^2 = B_H^2(\theta,x,y)h_H^4 + B_K^2(\theta,x,y)h_K^4 + \frac{V_{HK}(\theta,x,y)}{n\phi_{\theta,x}(h_K)} + o(h_H^4) + o(h_K^4) + o\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right),$$

where

$$B_H(\theta,x,y) = \frac{1}{2}\frac{\partial^2 F(\theta,y,x)}{\partial y^2}\int t^2 H^{(1)}(t)dt, \quad B_K(\theta,x,y) = \frac{1}{2}\Phi_0^{(2)}(0)\frac{M_0}{M_1} + o(h_K^2),$$

and

$$V_{HK}(\theta,x,y) = \frac{M_2}{M_1^2}F(\theta,y,x)(1-F(\theta,y,x)),$$

with

$$M_0 = K(1) - \int_{-1}^{1} s^2 K'(s)\chi_{\theta,x}(s)ds \quad \textit{and}$$

$$M_j = K^j(1) - \int_{-1}^{1}(K^j)'(s)\chi_{\theta,x}(s)ds \ \textit{for} \ \ j=1,2.$$

we set

$$\widehat{F}(\theta,y,x) = \frac{\widehat{F}_N(\theta,y,x)}{\widehat{F}_D(\theta,x)}.$$

where

$$\widehat{F}_N(\theta,y,x) = \frac{1}{n(n-1)\mathbb{E}[W_{12}(\theta,x)]}\sum_{1\le i\ne j\le n} W_{ij}(\theta,x)H(h_H^{-1}(y-Y_j)),$$

and

$$\widehat{F}_D(\theta,x) = \frac{1}{n(n-1)\mathbb{E}[W_{12}(\theta,x)]}\sum_{1\le i\ne j\le n} W_{ij}(\theta,x),$$

The following lemmas will be useful for proof of Theorem 1.

Lemma 1 *Under the assumptions of Theorem 1, we obtain:*

$$\mathbb{E}\left[\widehat{F}_N(\theta,y,x)\right] - F(\theta,y,x) = B_H(\theta,x,y)h_H^2 + B_K(\theta,x,y)h_K^2 + o(h_H^2) + o(h_K^2).$$

Lemma 2 *Under the assumptions of Theorem 1, we obtain:*

$$\mathrm{Var}\left[\widehat{F}_N(\theta, y, x)\right] = \frac{V_{HK}(\theta, x, y)}{n\phi_{\theta,x}(h_K)} + o\left(\frac{1}{n\phi_{\theta,x}(h_k)}\right).$$

Lemma 3 *Under the assumptions of Theorem 1, we get:*

$$\mathrm{Cov}(\widehat{F}_N(\theta, y, x), \widehat{F}_D(\theta, x)) = O\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right).$$

Lemma 4 *Under the assumptions of Theorem 1, we get:*

$$\mathrm{Var}\left[\widehat{F}_D(\theta, x)\right] = O\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right).$$

Comments

Since the mean squart error depend on the bias and variance, The idea of the proof of both variance term, and bias term is to treat separately the numerator and the denominator of the estimator. Lemma 3.2 is auxiliary result wish allow us to determine the bias of the estimator, wile Lemma 3.3-3.5, allow us to determine the variance of our estimator, by means the variance decomposition of Sarda and Vieu [27] and Lecoutre [24], see also Ferraty and al. [20]. As all asymptotic result in functional statistic, the dispersion term is related to the "dimensionality" of the functional variable in sense that the variance term depends on the function $\phi_x(h_K)$ which is closely linked on bi-functional operator δ and the latter can be related to the topological structure on the functional space $\mathcal{H}$.
Another way to highlight the interest of our asymptotic result is to show how the exact calculation of the leading terms in the quadratic error leads to the build of confidence intervals. Indeed, it is well known that the computation of the bias and the variance terms is commonly a preliminary result permitting to obtain the asymptotic normality result of the estimator.

3.2 Asymptotic normality

This section contains results on the asymptotic normality of $\widehat{F}(\theta, y, x)$. Before announcing our main results, we introduce the quantity $N(a, b)$, which will appear in the bias and variance dominant terms:

$$N(a, b) = K^a(1) - \int_{-1}^{1} (u^b K^a(u))' \chi_x(u) du \text{ for all } a > 0 \text{ and } b = 2, 4$$

Then, we have the following theorem:

Theorem 2 *Under assumptions (H1)-(H6), we obtain:*

$$\sqrt{n\phi_{\theta,x}(h_K)}(\widehat{F}(\theta,y,x) - F(\theta,y,x) - \mathbb{B}_n(\theta,x,y)) \xrightarrow{D} \mathcal{N}(0, V_{HK}(\theta,x,y)) \quad (1)$$

where,

$$V_{HK}(\theta,x,y) = \frac{M_2}{M_1^2}F(\theta,y,x)(1 - F(\theta,y,x)) \quad (2)$$

and

$$\mathbb{B}_n(\theta,x,y) = \frac{\mathbb{E}(\widehat{F}_N(\theta,y,x)(y))}{\mathbb{E}(\widehat{F}_D(\theta,x))} - F(\theta,y,x) \quad (3)$$

with $\xrightarrow{D}$ *denoting the convergence in distribution.*

Proof of Theorem 2.
Inspired by the decomposition given in Masry [25], we set.

$$\begin{aligned}&\widehat{F}(\theta,y,x) - F(\theta,y,x) - \mathbb{B}_n(\theta,x,y)\\ &\qquad = \frac{\widehat{F}_N(\theta,y,x) - F(\theta,y,x)\widehat{F}_D(\theta,x) - \widehat{F}_D(\theta,x)\mathbb{B}_n(\theta,x,y)}{\widehat{F}_D(\theta,x)}\end{aligned}$$

If we denote by

$$\begin{aligned}Q_n(\theta,x,y) = \widehat{F}_N(\theta,y,x) - F(\theta,y,x)\widehat{F}_D(\theta,x) - \mathbb{E}(\widehat{F}_N(\theta,y,x)\\ - F(\theta,y,x)\widehat{F}_D(\theta,x) = \widehat{F}_N(\theta,y,x)\\ - F(\theta,y,x)\widehat{F}_D(\theta,x) - \mathbb{B}_n(\theta,x,y)\end{aligned} \quad (4)$$

since

$$\widehat{F}_N(\theta,y,x) - F(\theta,y,x)\widehat{F}_D(\theta,x) = Q_n(\theta,x,y) + \mathbb{B}_n(\theta,x,y)$$

then the proof of this theorem will be completed from the following expression

$$\begin{aligned}&\widehat{F}(\theta,y,x) - F(\theta,y,x) - \mathbb{B}_n(\theta,x,y)\\ &\quad = \frac{Q_n(\theta,x,y) - \mathbb{B}_n(\theta,x,y)(\widehat{F}_D(\theta,x) - \mathbb{E}(\widehat{F}_D(\theta,x)))}{\widehat{F}_D(\theta,x)}\end{aligned} \quad (5)$$

and the following auxiliary results which play a main role and for which proofs are given in the appendix.

Lemma 5 *Under assumptions (H1)-(H5), we have*

$$\widehat{F}_D(\theta,x) \xrightarrow{P} \mathbb{E}(\widehat{F}_D(\theta,x)) = 1$$

where $\xrightarrow{P}$ *denotes the convergence in probability.*

Lemma 6 *Under assumptions (H2), (H4) and (H5), as* $n \to \infty$*, we have*

$$\mathbb{E}\left(K_1^2 var\left(H\left(\frac{y-Y_1}{h}\right)|X_1\right)\right) \to \mathbb{E}(K_1^2)F(\theta,y,x)(1-F(\theta,y,x))$$

So, Lemma 5, implies that $\widehat{F}_D(\theta,x) \to 1$. Moreover, $\mathbb{B}_n(\theta,x,y) = o(1)$ as $n \to \infty$ because of the continuity of $F(\theta,.,x)$. Then, we obtain that

$$\widehat{F}(\theta,y,x) - F(\theta,y,x) - \mathbb{B}_n(\theta,x,y) = \frac{Q_n(\theta,x,y)}{\widehat{F}_D(\theta,x)}(1+o_p(1))$$

Lemma 7 *Under assumptions (H1)-(H5), we have*

$$\sqrt{n\phi_{\theta,x}(h_K)}Q_n(\theta,x,y) \xrightarrow{D} \mathcal{N}(0, V_{HK}(\theta,x,y)), \tag{6}$$

where $V_{HK}(\theta,x,y)$ is defined by (2).

If we take advantage of the following assumptions,
(H7) $\lim_{n\to+\infty} \sqrt{nh_H\phi_{\theta,x}(h_K)}\mathbb{B}_n(\theta,x,y) = 0$, we can cancel the bias term and obtain the following corollary.

Corollary 1 *Under the assumptions of Theorem 2, we get*

$$\sqrt{\frac{nh_H\widehat{\phi}_{\theta,x}(h_K)}{V_{HK}(\theta,x,y)}}(\widehat{F}(\theta,y,x) - F(\theta,y,x)) \to \mathcal{N}(0,1)$$

Indeed: by the additional assumption (H7), we firstly obtain,

$$\sqrt{n\phi_{\theta,x}(h_K)}(\widehat{F}(\theta,y,x) - F(\theta,y,x)) \xrightarrow{D} \mathcal{N}(0, V_{HK}(\theta,x,y)),$$

to avoid estimating the constants in this last expression, one may consider the simple uniform kernel ($M_1 = M_2 = 1$) and get the above corollary (Corollary 3.11). So the practical utilization of our result in confidence intervals construction requires only the estimation of the function $\phi_{\theta,x}(t)$. This last can be empirically estimated by:

$$\widehat{\phi}_{\theta,x}(t) = \frac{\sharp\{i : |d(X_i,x)| \leq t\}}{n},$$

where $\sharp(A)$ denote the cardinality of the set A.
Finally, for $\gamma \in (0,1)$, we obtain the following $(1-\gamma)$ confidence interval for $F(\theta, x, y)$:

$$\widehat{F}(\theta, y, x) \underset{-}{+} t_{1-\frac{\gamma}{2}} \times \frac{\widehat{\sigma}(\theta, x)}{\sqrt{n\widehat{\phi}_{\theta,x}(h_K)}},$$

where $t_{1-\frac{\gamma}{2}}$ is the quantile of standard normal distribution, and $\widehat{\sigma}^2(x, y)$ denote the estimators of $V_{HK}(\theta, x, y)$.

Discussion on the importance of our model and on impacts of our results

It is well known that, the conditional distribution function (cdf) has the advantage of completely characterizing the conditional law of the considered random variables. In fact, the determination of the cdf allows to obtain the conditional density, the conditional hazard and the conditional quantile functions. Thus, even if the estimation of the conditional distribution has an interest in its own right, it is moreover of great aid in estimating various conditional models. On the other hand, the asymptotic results, obtained here, would have a great impact on the theoretical as well as on the practical aspects. The determination of the bias and of the variance terms of the estimator is a basic ingredient to obtain its asymptotic normality. This question is a natural way to extend results of this work. Notice also that this asymptotic property is very interesting to make statistical tests. The convergence in mean square wich study the L^2-consistency of $\widehat{F}(\theta, x, y)$ is one of the most useful/practical accuracy measures in the nonparametric smoothing estimation.

Remark 1 *The generalisation to multi-index model as mentioned by the reviewer, is an interesting subject, and a good prospect, to do that we consider* θ_D *as a matrix* $D \times D$ *of vectors* $(\theta_{jD})_{j=1,D}$ *of* $\mathcal{H}$*, where the direction* D *can be chosen by cross validation, and the inner product can be defined by* $|< \theta_D, x >| = \sum_j \theta_{jD} x_j$.

Remark 2 *Being independent refers to how the process of collectiong the sample was performed and it assures the representation fairness of the sampling. Dependent samples introduce bias into the results. From computational point of view independency significantly simplifies operation. If the random variables are not independents, the complexity of the problem explodes and we can not be able to use several results that need the random variables to be independent, in* *this case we can use mixing coeifficient to measure the dependency, this work is one of our goals to prepare another paper for submission.*

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4 Appendix

Proof of Theorem 1. We know the theorem is a consequence of a separate computes two quantities (bias and variance) of $\widehat{F}(\theta, y, x)$, we have

$$\mathbb{E}\left[\widehat{F}(\theta, y, x) - F(\theta, y, x)\right]^2 = \left[\mathbb{E}\left(\widehat{F}(\theta, y, x)\right) - F(\theta, y, x)\right]^2 + Var\left[\widehat{F}(\theta, y, x)\right]$$

By classical calculations, we obtain

$$\begin{aligned}\widehat{F}(\theta, y, x) - F(\theta, y, x) &= \left(\widehat{F}_N(\theta, y, x) - f(\theta, y, x)\right) - \widehat{F}_N(\theta, y, x)\left(\widehat{F}_D(\theta, x) - 1\right) \\ &\quad - \mathbb{E}[\widehat{F}_N(\theta, y, x)]\left(\widehat{F}_D(\theta, x) - 1\right) \\ &\quad - \mathbb{E}[\widehat{F}_N(\theta, y, x)]\left(\widehat{F}_D(\theta, x) - 1\right) + \left(\widehat{F}_D(\theta, x) - 1\right)^2 \widehat{F}(\theta, y, x).\end{aligned}$$

which implies that:

$$\begin{aligned}\mathbb{E}\left[\widehat{F}(\theta, y, x)\right] - F(\theta, y, x) &= \left(\mathbb{E}[\widehat{F}_N(\theta, y, x)] - F(\theta, y, x)\right) \\ &\quad - Cov\left(\widehat{F}_N(\theta, y, x), \widehat{F}_D(\theta, x)\right) \\ &\quad + \mathbb{E}\left[\left(\widehat{F}_D(\theta, x) - \mathbb{E}[\widehat{F}_D(\theta, x)]\right)^2 \widehat{F}(\theta, y, x)\right].\end{aligned}$$

Hence:

$$\begin{aligned}\mathbb{E}\left[\widehat{F}(\theta, y, x)\right] - F(\theta, y, x) &= \left(\mathbb{E}[\widehat{F}_N(\theta, y, x)] - F(\theta, y, x)\right) \\ &\quad - Cov\left(\widehat{F}_N(\theta, y, x), \widehat{F}_D(\theta, x)\right) \\ &\quad + Var\left[\widehat{F}_D(\theta, x)\right] O(h_H^{-1}).\end{aligned}$$

Now, by similar technics as those Sarda and Vieu [27] and by Bosq and Lecoutre [9], the variance term is

$$\begin{aligned} Var\left[\widehat{F}(\theta,y,x)\right] &= Var\left[\widehat{F}_N(\theta,y,x)\right] - 2\mathbb{E}[\widehat{F}_N(\theta,y,x)]Cov\left(\widehat{F}_N(\theta,y,x),\widehat{F}_D(\theta,x)\right) \\ &\quad + \left(\mathbb{E}[\widehat{F}_N(\theta,y,x)]\right)^2 Var\left(\widehat{F}_D(\theta,x)\right) + o\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right). \end{aligned}$$

Proof of Lemma 1. We have:

$$\begin{aligned} \mathbb{E}[\widehat{F}_N(\theta,y,x)] &= \mathbb{E}\left[\frac{1}{n(n-1)\mathbb{E}[W_{12}(\theta,x)]}\sum_{1\le i\ne j\le n} W_{ij}(\theta,x)H(h_H^{-1}(y-Y_j))\right] \\ &= \frac{1}{\mathbb{E}[W_{\theta,12}]}\mathbb{E}\left[W_{\theta,12}\mathbb{E}[H_2|X_2]\right]. \end{aligned}$$

We use an integration by part to show that:

$$\mathbb{E}[H_2|X_2] = h_H^{-1}\int_{\mathbb{R}} H^{(1)}(h_H^{-1}(y-z))F(\theta,z,x)dz$$

Now the change of variable $t=\frac{y-z}{h_H}$ allows to write:

$$|\mathbb{E}[H_2|X_2] + \int_{\mathbb{R}} H^{(1)}(t)|F(\theta,y-th_H,x)$$

By using a Taylor's expansion and under assumption (H5), we have

$$\mathbb{E}[H_2|X_2] = F(\theta,y,X_2) + \frac{h_H^2}{2}\left(\int t^2H^{(1)}(t)dt\right)\frac{\partial^2F(\theta,y,X_2)}{\partial y^2} + o(h_H^2).$$

Now, we can re-written as:

$$\mathbb{E}[H_2|X_2] = \psi_0(X_2,y) + \frac{h_H^2}{2}\left(\int t^2H^{(1)}(t)dt\right)\psi_2(X_2,y) + o(h_H^2).$$

Thus, we obtain

$$\begin{aligned} \mathbb{E}\left[\widehat{F}_N(\theta,y,x)\right] &= \frac{1}{E[W_{\theta,12}]}\mathbb{E}\left[W_{\theta,12}\psi_0(X_2,y)\right] \\ &\quad + \frac{1}{E[W_{\theta,12}]}\left(\int t^2H^{(1)}(t)dt\right)\mathbb{E}\left[W_{\theta,12}\psi_2(X_2,y)\right] + o(h_H^2). \end{aligned}$$

Accordingly to Ferraty et al. [20], for $l \in \{0, 2\}$, we show that

$$\begin{aligned}\mathbb{E}[W_{\theta,12}\psi_l(X_2, y)] &= \psi_l(x, y)\mathbb{E}[W_{\theta,12}] + \mathbb{E}[W_{\theta,12}(\psi_l(X_2, y) - \psi_l(x, y))] \\ &= \psi_l(x, y)\mathbb{E}[W_{\theta,12}] + \mathbb{E}[W_{\theta,12}\mathbb{E}[\psi_l(X_2, y) - \psi_l(x, y)|\beta_\theta(X_2, x)]] \\ &= \psi_l(x, y)\mathbb{E}[W_{\theta,12}] + \mathbb{E}[W_{\theta,12}\Phi_l(\beta_\theta(X_2, x))].\end{aligned}$$

Since $\mathbb{E}[\beta_{\theta,2}W_{\theta,12}] = 0$ and $\Phi_l(0) = 0$, for $l \in 0, 2$, we obtain

$$\mathbb{E}[W_{\theta,12}\Phi_l(\beta_\theta(X_2, x))] = \frac{1}{2}\Phi_l^{(2)}(0)\mathbb{E}[\beta_\theta^2(X_2, x)W_{\theta,12}] + o(\mathbb{E}\left[\beta_\theta(X_2, x)W_{\theta,12}\right]).$$

Then,

$$\begin{aligned}\mathbb{E}\left[\widehat{F}_N(\theta, y, x)\right] &= F(\theta, y, x) + \frac{h_H^2}{2}\frac{\partial^2 F(\theta, y, x)}{\partial y^2}\int t^2 H^{(1)}(t)dt \\ &+ o\left(h_H^2\frac{\mathbb{E}\left[\beta_\theta^2(X_2, x)W_{\theta,12}\right]}{\mathbb{E}[W_{\theta,12}]}\right) + \frac{1}{2}\Phi_l^{(2)}(0)\frac{\mathbb{E}\left[\beta_\theta^2(X_2, x)W_{\theta,12}\right]}{\mathbb{E}[W_{\theta,12}]} \\ &+ o\left(\frac{\mathbb{E}\left[\beta_\theta^2(X_2, x)W_{\theta,12}\right]}{\mathbb{E}[W_{\theta,12}]}\right).\end{aligned}$$

Therefore, it remains to determine the quantities $E\left[\beta_\theta^2(X_2, x)W_{\theta,12}\right]$ and $E[W_{\theta,12}]$. According to the definition of $W_{\theta,12}$, the behaviours of the two quantities $E\left[\beta_\theta^2(X_2, x)W_{\theta,12}\right]$ and $E[W_{\theta,12}]$ are based on the asymptotic evaluation of $E[K_1^a\beta_1^b]$. To do that, we treat firstly, the case $b = 1$. For this case,we use the assumptions (H3) and (H4) to get

$$h_K\mathbb{E}[K_1^a\beta_{\theta,1}] = o\left(\int_{B(x,h_K)}\beta_\theta^2(u, x)dP(u)\right) = o(h_K^2\phi_{\theta,x}(h_K)).$$

So, we obtain that,

$$\mathbb{E}[K_1^a\beta_{\theta,1}] = o(h_K\phi_{\theta,x}(h_K)). \tag{7}$$

Morever, for all $b > 1$, and after simplifications of the expressions, permits to write that

$$\mathbb{E}[K_1^a\beta_{\theta,1}^b] = \mathbb{E}[K_1^a d_\theta^b(x, X)] + o(h_K^b\phi_{\theta,x}(h_K)).$$

Concerning the first term, we write

$$\begin{aligned}
h_K^{-b}\mathbb{E}[K_1^a d_\theta^b] &= \int \nu^b K^a(\nu) dP^{h_K^{-1} d_\theta(x,X)}(\nu) \\
&= \int_{-1}^{1} \left[K^a(1) - \int_{\nu}^{1} \left((s^b K^a(s))' \right) du \right] dP^{h_K^{-1} d_\theta(x,X)}(\nu) \\
&= \left(K(1)\phi_{\theta,x}(h_K) - \int_{-1}^{1} (s^b K^a(s))^{(1)} \phi_{\theta,x}(sh_K, h_K) ds \right) \\
&= \phi_{\theta,x}(h_K) \left(K(1) - \int_{-1}^{1} (s^b K^a(s))' \frac{\phi_{\theta,x}(sh_K, h_K)}{\phi_{\theta,x}(h_K)} ds \right).
\end{aligned}$$

Finally, under assumptions (H1), we get

$$\mathbb{E}[K_1^a \beta_{\theta,1}^b] = h_K^b \phi_{\theta,x}(h_K) \left(K(1) - \int_{-1}^{1} (s^b K^a(u))' \chi_{\theta,x}(s) ds \right) + o(h_K^b \phi_{\theta,x}(h_K)). \tag{8}$$

On other hand, by following the same steps in Ferraty and al. [20], we have

$$\mathbb{E}[W_{\theta,12}] = O(h_K^2 \phi_{\theta,x}^2(h_K)), \tag{9}$$

and

$$\mathbb{E}(K_{\theta,1}^j) = M_j \phi_{\theta,x}(h_K) \text{ for } j = 1, 2 \tag{10}$$

So,

$$\frac{\mathbb{E}[\beta_\theta^2(X_2, x) W_{\theta,12}]}{\mathbb{E}[W_{\theta,12}]} = h_K^2 \left(\frac{K(1) - \int_{-1}^{1} (s^2 K(s))' \chi_{\theta,x}(s) ds}{K(1) - \int_{-1}^{1} (K'(u) \chi_{\theta,x}(s) ds} \right) + o(h_K^2).$$

Hence,

$$\begin{aligned}
\mathbb{E}\left[\widehat{F}_N(\theta, y, x)\right] &= F(\theta, y, x) + \frac{h_H^2}{2} \frac{\partial^2 F(\theta, y, x)}{\partial y^2} \int t^2 H^{(1)}(t) dt + o(h_H^2) \\
&\quad + h_K^2 \Phi_0^{(2)}(0) \frac{\left(K(1) - \int_{-1}^{1} (s^2 K(s))' \chi_{\theta,x}(s) ds \right)}{2 \left(K(1) - \int_{-1}^{1} K'(s) \chi_{\theta,x}(s) ds \right)} + o(h_K^2).
\end{aligned}$$

Proof of Lemma 2. We know

$$Var\left(\widehat{F}_N(\theta, y, x)\right) = \frac{1}{(n(n-1)(\mathbb{E}[W_{\theta,12}]))^2} Var\left(\sum_{1 \le i \ne j \le n} W_{\theta,ij} H_j \right)$$

$$= \frac{1}{(n(n-1)(\mathbb{E}[W_{\theta,12}]))^2}\Big[n(n-1)\mathbb{E}[W^2_{\theta,12}H_2^2] + n(n-1)\mathbb{E}[W_{\theta,12}W_{\theta,21}H_2H_1] \\ +n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,13}H_2H_3] + n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,23}H_2H_3] \\ +n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,31}H_2H_1] + n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,32}H_2^2] \\ -n(n-1)(4n-6)(\mathbb{E}[W_{\theta,12}H_2])^2\Big]. \tag{11}$$

By direct calculations, we get

$$\begin{cases} \mathbb{E}[W^2_{\theta,12}H_2^2] = O(h_K^4\phi^2_{\theta,x}(h_K)), \quad \mathbb{E}[W_{\theta,12}W_{\theta,21}H_2H_1] = O(h_K^4\phi^2_{\theta,x}(h_K)), \\ \mathbb{E}[W_{\theta,12}W_{\theta,13}H_2H_3] = (F(\theta,y,x))^2\mathbb{E}[\beta^4_{1,\theta}K^2_{\theta,1}](\mathbb{E}[K_{\theta,1}])^2 + o(h_K^4\phi^3_{\theta,x}(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,23}H_2H_3] = (F(\theta,y,x))^2\mathbb{E}[\beta^2_{1,\theta}K_{\theta,1}]\mathbb{E}[\beta^2_{1,\theta}K^2_{\theta,1}]\mathbb{E}[K_{\theta,1}] + o(h_K^4\phi^3_{\theta,x}(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,31}H_2H_3] = (F(\theta,y,x))^2\mathbb{E}[\beta^2_{1,\theta}K_{\theta,1}]\mathbb{E}[\beta^2_{1,\theta}K^2_{\theta,1}]\mathbb{E}[K_{\theta,1}] + o(h_K^4\phi^3_{\theta,x}(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,32}H_2^2] = F(\theta,y,x)\mathbb{E}^2[\beta_1^2K_1]\mathbb{E}[K_1^2] + o(h_K^4\phi^3_{\theta,x}(h_K)). \\ \mathbb{E}[W_{\theta,12}H_1] = O(h_K^2\phi^2_{\theta,x}(h_K)) \end{cases}$$

By equation (7), equation (8), (9) and (10)

$$Var\left(\widehat{F}_N(\theta,y,x)\right) = \frac{F(\theta,y,x)(1-F(\theta,y,x))}{n\phi_{\theta,x}(h_K)}\left[\frac{\left(K^2(1) - \int_{-1}^1 (K^2(s))'\chi_{\theta,x}(s)ds\right)}{\left(K(1) - \int_{-1}^1 (K(s))'\chi_{\theta,x}(s)ds\right)^2}\right] \\ +o\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right) = \frac{M_2}{M_1^2 n\phi_{\theta,x}(h_K)}F(\theta,y,x)(1-F(\theta,y,x)) + o\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right).$$

Proof of Lemma 3. The proof of this Lemma it's similar to Lemma 2 proof, it permits to write (with $I = \{(i,j) : 1 \le i \neq j \le n\}$)

$$Cov\left(\widehat{F}_N(\theta,y,x), \widehat{F}_D(\theta,x)\right) = \frac{1}{(n(n-1)\mathbb{E}[W_{\theta,12}])^2}Cov\Big(\sum_{i,j\in I} W_{\theta,ij}H_j, \sum_{k,l\in I} W_{\theta,kl}\Big) \\ = \frac{1}{(n(n-1)\mathbb{E}[W_{\theta,12}])^2}\Big[n(n-1)\mathbb{E}[W^2_{\theta 12}H_2] + n(n-1)\mathbb{E}[W_{\theta,12}W_{\theta,21}H_2] \\ +n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,13}H_2] + n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,23}H_2] \\ +n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,31}H_2] + n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,32}H_2] \\ -n(n-1)(4n-6)(\mathbb{E}[W_{\theta,12}H_2]\mathbb{E}[W_{\theta,12}]\Big].$$

By direct calculations, we get

$$\begin{cases} \mathbb{E}[W_{\theta,12}^2 H_2] = O(h_K^4 \phi_{\theta,x}^2(h_K)), \quad \mathbb{E}[W_{\theta,12}W_{\theta,21}H_2] = O(h_K^4 \phi_{\theta,x}^2(h_K)), \\ \mathbb{E}[W_{\theta,12}W_{\theta,13}H_2] = (F(\theta,y,x))\mathbb{E}[\beta_{1,\theta}^4 K_{\theta,1}^2](\mathbb{E}[K_{\theta,1}])^2 + o(h_K^4 \phi_{\theta,x}^3(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,23}H_2] = (F(\theta,y,x))\mathbb{E}[\beta_{1,\theta}^2 K_{\theta,1}]\mathbb{E}[\beta_{1,\theta}^2 K_{\theta,1}^2]\mathbb{E}[K_{\theta,1}] + o(h_K^4 \phi_{\theta,x}^3(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,31}H_2] = (F(\theta,y,x))\mathbb{E}[\beta_{1,\theta}^2 K_{\theta,1}]\mathbb{E}[\beta_{1,\theta}^2 K_{\theta,1}^2]\mathbb{E}[K_{\theta,1}] + o(h_K^4 \phi_{\theta,x}^3(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,32}H_2] = F(\theta,y,x)\mathbb{E}^2[\beta_{\theta,1}^2 K_1]\mathbb{E}[K_{\theta,1}^2] + o(h_K^4 \phi_{\theta,x}^3(h_K)). \\ \mathbb{E}[W_{\theta,12}H_1] = O(h_K^2 \phi_{\theta,x}^2(h_K)) \end{cases}$$

By equation (7), equation (8), (9) and (10), we obtain

$$\mathrm{Cov}\left(\widehat{F}_N(\theta,y,x), \widehat{F}_D(\theta,x)\right) = O\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right).$$

Proof of Lemma 4. We have that

$$Var(\widehat{f}_D(\theta,x)) = \frac{1}{(n(n-1)\mathbb{E}[W_{\theta,12}])^2} Var\left(\sum_{1\le i\ne j\le n} W_{\theta,ij}\right).$$

That is

$$\begin{aligned} Var\left(\widehat{F}_D(\theta,x)\right) &= \frac{1}{(n(n-1)(\mathbb{E}[W_{\theta,12}]))^2}\Big[n(n-1)\mathbb{E}[W_{\theta,12}^2] + n(n-1)\mathbb{E}[W_{\theta,12}W_{\theta,21}] \\ &\quad + n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,13}] + n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,23}] \\ &\quad + n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,31}] + n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,32}] \\ &\quad - n(n-1)(4n-6)(\mathbb{E}[W_{\theta,12}])^2\Big]. \end{aligned} \tag{12}$$

and similarly to the previous cases

$$\begin{cases} \mathbb{E}[W_{\theta,12}^2] = O(h_K^4 \phi_{\theta,x}^2(h_K)), \quad \mathbb{E}[W_{\theta,12}W_{\theta,21}] = O(h_K^4 \phi_{\theta,x}^2(h_K)), \\ \mathbb{E}[W_{\theta,12}W_{\theta,13}] = \mathbb{E}[\beta_{1,\theta}^4 K_{\theta,1}^2](\mathbb{E}[K_{\theta,1}])^2 + o(h_K^4 \phi_{\theta,x}^3(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,23}] = \mathbb{E}[\beta_{1,\theta}^2 K_{\theta,1}]\mathbb{E}[\beta_{1,\theta}^2 K_{\theta,1}^2]\mathbb{E}[K_{\theta,1}] + o(h_K^4 \phi_{\theta,x}^3(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,31}] = \mathbb{E}[\beta_{1,\theta}^2 K_{\theta,1}]\mathbb{E}[\beta_{1,\theta}^2 K_{\theta,1}^2]\mathbb{E}[K_{\theta,1}] + o(h_K^4 \phi_{\theta,x}^3(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,32}] = \mathbb{E}^2[\beta_{\theta,1}^2 K_1]\mathbb{E}[K_{\theta,1}^2] + o(h_K^4 \phi_{\theta,x}^3(h_K)). \\ \mathbb{E}[W_{\theta,12}] = O(h_K^2 \phi_{\theta,x}^2(h_K)) \end{cases}$$

By the same arguments used in the previous lemmas, we can write:

$$\begin{aligned} Var\left(\widehat{F}_D(\theta,x)\right) &= \frac{M_2\phi_{\theta,x}(h_K)}{n(M_1\phi_{\theta,x}(h_K))^2} + o\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right) \\ &= O\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right). \end{aligned}$$

Proof of Lemma 5. By applying the Bienaym'e-Tchebychev's inequality, as $n \to +\infty$,, we obtain, for all $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}(|\widehat{F}_D(\theta,x)) - \mathbb{E}(\widehat{F}_D(\theta,x)|) \geq \varepsilon) &< \frac{var(\widehat{F}_D(\theta,x))}{\varepsilon^2}\\ &< \frac{1}{\varepsilon^2}O\Big(\frac{1}{n\phi_{\theta,x}(h_K)}\Big)\\ &= \frac{o(1)}{\varepsilon^2}\\ &\to 0\end{aligned}$$

Proof of Lemma 6. We have,

$$\begin{aligned}\mathbb{E}\left(K_1^2 var\left(H\left(\frac{y-Y_1}{h}\right)|X_1\right)\right) &= \mathbb{E}\left(K_1^2\mathbb{E}\left(\left(H\left(\frac{y-Y_1}{h}\right)\right)^2|X_1\right)\right)\\ &- \mathbb{E}\left(K_1^2\mathbb{E}^2\left(H\left(\frac{y-Y_1}{h}\right)|X_1\right)\right)\end{aligned}$$

By an integration par parts, followed by a change of variable, we get

$$\begin{aligned}\mathbb{E}(H^2\left(\frac{y-Y_1}{h_H}\right)|X_1) &= \frac{1}{h_H}\int H^2(t)dF(\theta,y-th_H,X_1)\\ &= 2\int H^{(1)}H(t)(F(\theta,y-th_H,X_1)-F(\theta,y,x))dt\\ &+ 2\int H^{(1)}H(t)F(\theta,y,x)dt\end{aligned}$$

Since

$$2\int H^{(1)}H(t)F(\theta,y,x)dt = F(\theta,y,x) \text{ as } n \to +\infty,$$

we deduce that, as $n \to +\infty$, we have

$$\mathbb{E}(K_{\theta,1}^2 H^2\left(\frac{y-Y_1}{h_H}\right)|X_1) \to \mathbb{E}(K_{\theta,1}^2]F(\theta,y,x)$$

and

$$\mathbb{E}(H\left(\frac{y-Y_1}{h_H}\right)|X_1) - F(\theta,y,x) \to 0,$$

so

$$\mathbb{E}(K_{\theta,1}^2\mathbb{E}^2(H^2\left(\frac{y-Y_1}{h_H}\right)|X_1)) \to \mathbb{E}(K_{\theta,1}^2]F^2(\theta,y,x)$$

finally, we obtain:

$$\mathbb{E}\left(K_1^2 var\left(H\left(\frac{y-Y_1}{h}\right)|X_1\right)\right) \to \mathbb{E}(K_1^2)F(\theta,y,x)(1-F(\theta,y,x))$$

Proof of Lemma 7.
We have

$$\begin{aligned}\sqrt{n\phi_{\theta,x}(h_K)}Q_n(\theta,x,y) &= \frac{\sqrt{n\phi_{\theta,x}(h_K)}}{n\mathbb{E}(\Omega_1 K_1)}\sum_{j=1}^{n}\Omega_j K_j(H_j - F(\theta,y,x)) \\ &- \frac{\sqrt{n\phi_{\theta,x}(h_K)}}{n\mathbb{E}(\Omega_1 K_1)}\mathbb{E}(\sum_{j=1}^{n}\Omega_j K_j(H_j - F(\theta,y,x))\end{aligned}$$

then, combined with (4) implies that

$$\begin{aligned}&\sqrt{n\phi_{\theta,x}(h_K)}\ Q_n(\theta,x,y) = \frac{1}{n\mathbb{E}(\beta_1^2 K_1)}\sum_{i=1}^{n}\beta_i^2 K_i\frac{\sqrt{n\phi_{\theta,x}(h_K)}\mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Omega_1 K_1)} \\ &\times\sum_{j=1}^{n}K_j(H_j - F(\theta,y,x)) \\ &- \frac{1}{n\mathbb{E}(\beta_1 K_1)}\sum_{i=1}^{n}\beta_i K_i\frac{\sqrt{n\phi_{\theta,x}(h_K)}\mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Omega_1 K_1)}\sum_{j=1}^{n}\beta_j K_j(H_j - F(\theta,y,x)) \\ &- \mathbb{E}\left(\frac{1}{n\mathbb{E}(\beta_1^2 K_1)}\sum_{i=1}^{n}\beta_i^2 K_i\frac{\sqrt{n\phi_{\theta,x}(h_K)}\mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Omega_1 K_1)}\sum_{j=1}^{n}K_j(H_j - F(\theta,y,x))\right) \\ &+ \mathbb{E}\left(\frac{1}{n\mathbb{E}(\beta_1 K_1)}\sum_{i=1}^{n}\beta_i K_i\frac{\sqrt{n\phi_{\theta,x}(h_K)}\mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Omega_1 K_1)}\sum_{j=1}^{n}\beta_j K_j(H_j - F(\theta,y,x))\right)\end{aligned}$$

Denote by

$$S_1 = \frac{1}{n\mathbb{E}(\beta_1^2 K_1)}\sum_{i=1}^{n}\beta_i^2 K_i \quad , \quad S_2 = \frac{\sqrt{n\phi_{\theta,x}(h_K)}\mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Omega_1 K_1)}\sum_{j=1}^{n}K_j(H_j - F(\theta,y,x))$$

$$S_3 = \frac{1}{n\mathbb{E}(\beta_1 K_1)}\sum_{i=1}^{n}\beta_i K_i \quad and \quad S_4 = \frac{\sqrt{n\phi_{\theta,x}(h_K)}\mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Omega_1 K_1)}\sum_{j=1}^{n}\beta_j K_j(H_j - F(\theta,y,x))$$

It remains to show that,

$$\begin{aligned}\sqrt{n\phi_{\theta,x}(h_K)}Q_n(\theta,x,y) &= S_1S_2 - S_3S_4 - \mathbb{E}(S_1S_2 - S_3S_4) \\ &= (S_1S_2 - \mathbb{E}(S_1S_2)) - (S_3S_4 - \mathbb{E}(S_3S_4))\end{aligned} \quad (13)$$

Hence by the Slutsky's theorem, to show (13), it suffices to prove the following two claims:

$$S_1S_2 - \mathbb{E}(S_1S_2) \xrightarrow{D} \mathcal{N}(0, V_{HK}(\theta,x,y)) \quad (14)$$

$$S_3S_4 - \mathbb{E}(S_3S_4) \xrightarrow{P} 0, \quad (15)$$

Proof of (14) We can write that

$$S_1S_2 - \mathbb{E}(S_1S_2) = S_2 - \mathbb{E}(S_2) + (S_1-1)S_2 - \mathbb{E}((S_1-1)S_2).$$

by the Slutsky's theorem, we get the following intermediate results,

$$(S_1-1)S_2 - \mathbb{E}((S_1-1)S_2) \xrightarrow{P} 0 \quad (16)$$

and

$$S_2 - \mathbb{E}(S_2) \xrightarrow{D} \mathcal{N}(0, V_{HK}(\theta,x,y)) \quad (17)$$

Concerning the proof of (16), by applying the Bienaymé-Tchebychv's inequality, we obtain for all $\epsilon > 0$

$$\mathbb{P}(|(S_1-1)S_2 - \mathbb{E}((S_1-1)S_2)| > \epsilon) \leq \frac{\mathbb{E}(|(S_1-1)S_2 - \mathbb{E}(S_1-1)S_2)|)}{\epsilon}.$$

Then, the Cauchy-Schwarz inequality implies that

$$\mathbb{E}(|(S_1-1)S_2 - \mathbb{E}((S_1-1)S_2)|) \leq 2\mathbb{E}(|(S_1-1)S_2)|) \leq 2\sqrt{\mathbb{E}((S_1-1)^2)}\sqrt{\mathbb{E}((S_2)^2)}$$

On one side, by using equations (7) and (8), we obtain

$$\begin{aligned}\mathbb{E}((S_1-1)^2) &= var(S_1) = \frac{1}{n^2\mathbb{E}^2(\beta_1^2K_1)} n\,var(\beta_1^2K_1) \\ &\leq \frac{1}{nO(h_K^4\phi_{\theta,x}^2(h_K))}\mathbb{E}(\beta_1^4K_1^2) = O\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right).\end{aligned}$$

and on the other side, we obtain

$$\begin{aligned}\mathbb{E}((S_2)^2) &= \frac{n\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}\mathbb{E}\left(\sum_{j=1}^{n} K_j(H_j - F(\theta,y,x))\right)^2 \\ &= \frac{n}{(n-1)^2O(\phi_{\theta,x}(h_K))}(nO(\phi_{\theta,x}(h_K)) + n(n-1)o(\phi_{\theta,x}^2(h_K))) \\ &= O(1) + o(n\phi_{\theta,x}(h_K)).\end{aligned}$$

Thus

$$\mathbb{E}(|(S_1-1)S_2-\mathbb{E}((S_1-1)S_2)|) \leq 2\sqrt{\mathbb{E}((S_1-1)^2)}\sqrt{\mathbb{E}((S_2)^2)}$$
$$2\sqrt{O\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right)(O(1)+o(n\phi_{\theta,x}(h_K)))}$$
$$= o(1),$$

which implies that $(S_1-1)S_2-\mathbb{E}(S_1-1)S_2) = o_p(1)$. Then, as $n\to\infty$, we get

$$\mathbb{P}(|(S_1-1)S_2-\mathbb{E}(S_1-1)S_2)|) > \epsilon) \leq \frac{\mathbb{E}(|(S_1-1)S_2-\mathbb{E}(S_1-1)S_2)|)}{\epsilon} \to 0.$$

Concerning the proof of (17), we denote

$$\begin{aligned} P_n &= S_2-\mathbb{E}(S_2) \\ &= \frac{\sqrt{n\phi_{\theta,x}(h_K)}\mathbb{E}(\beta_1^2K_1)}{\mathbb{E}(\Omega_1K_1)}\sum_{j=1}^{n} K_j(H_j-F(\theta,y,x))-\mathbb{E}(K_j(H_j-F(\theta,y,x))) \\ &= \frac{\sqrt{n\phi_{\theta,x}(h_K)}\mathbb{E}(\beta_1^2K_1)}{\mathbb{E}(\Omega_1K_1)}\sum_{j=1}^{n}\mu_{nj}(x,y), \end{aligned}$$

where

$$\mu_{nj}(x,y) = K_j(H_j-F(\theta,y,x))-\mathbb{E}(K_j(H_j-F(\theta,y,x)))$$

By the fact that $\mu_{nj}(x,y)$ are i.i.d., it follows that

$$\begin{aligned} var(P_n(x,y)) &= \frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}var(\mu_{n1}(x,y)) \\ &= \frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}\mathbb{E}(\mu_{n1}^2(x,y)) \end{aligned}$$

Thus

$$\begin{aligned} var(P_n(x,y)) = \frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}&(\mathbb{E}(K_1^2(H_1 \\ &-F(\theta,y,x))^2)-(\mathbb{E}(K_1(H_1-F(\theta,y,x)))^2). \end{aligned} \tag{18}$$

Concerning the second term on the right hand side of (18), we have

$$\begin{aligned} (\mathbb{E}(K_1(H_1-F(\theta,y,x)))^2 &= (\mathbb{E}(\mathbb{E}(K_1(H_1-F(\theta,y,x))|X_1))^2 \\ &= (\mathbb{E}(K_1\mathbb{E}((H_1|X_1)-F(\theta,y,x))))^2, \end{aligned}$$

where

$$\frac{1}{h_H}\mathbb{E}((H_1|X_1) - F(\theta, y, x)) \to 0 \quad as \quad n \to \infty \tag{19}$$

Now let us return to the first term of the right hand of (18). We have

$$\begin{aligned}
&\frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}(\mathbb{E}(K_1^2(H_1 - F(\theta, y, x))^2)\\
&= \frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}(\mathbb{E}(\mathbb{E}((H_1 - F(\theta, y, x))^2|X_1)K_1^2)\\
&= \frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}\mathbb{E}(var(H_1|X_1)K_1^2)\\
&+\frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}(\mathbb{E}(\mathbb{E}((H_1|X_1) - F(\theta, y, x))^2)K_1^2)
\end{aligned}$$

By using (19), that allows to have, as $n \to \infty$

$$\frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}(\mathbb{E}(\mathbb{E}((H_1|X_1) - F(\theta, y, x))^2)K_1^2) \to 0$$

Combining equations (7), (8) and (10), with lemma 6, we obtain as $n \to \infty$

$$\begin{aligned}
&\mathbb{E}(var(H_1|X_1)K_1^2) \to \mathbb{E}(K_1^2)F(\theta, y, x)(1 - F(\theta, y, x))\\
&= M_2F(\theta, y, x)(1 - F(\theta, y, x))\phi_{\theta,x}(h_K).
\end{aligned}$$

Therefore, by using equations (7), (8) and (10) , equation (18) becomes

$$\begin{aligned}
var(P_n(x, y)) &= \frac{n^2\phi_{\theta,x}(h_K)(N(1, 2)h_K^2\phi_{\theta,x}(h_K))^2}{((n - 1)N(1, 2)M_1h_K^2\phi_{\theta,x}(h_K))^2}\\
&\quad M_2F(\theta, y, x)(1 - F(\theta, y, x))\phi_{\theta,x}(h_K)\\
&= \frac{n^2M_2}{(n - 1)^2M_1^2}F(\theta, y, x)(1 - F(\theta, y, x))\\
&\to \frac{M_2}{M_1^2}F(\theta, y, x)(1 - F(\theta, y, x)) = V_{HK}(\theta, x, y) \quad as \quad n \to \infty
\end{aligned}$$

Now, in order to end the proof of (17), we focus on the central limit theorem. So, the proof of (14) is completed if the Lindberg's condition is verified. In fact, the Lindberg's condition holds since, for any $\eta > 0$

$$\sum_{j=1}^{n}\mathbb{E}(\mu_{nj}^2\mathbb{1}_{(|\mu_{nj}|>\eta)}) = n\mathbb{E}(\mu_{n1}^2\mathbb{1}_{(|\mu_{n1}|>\eta)}) = \mathbb{E}((\sqrt{n}\mu_{n1})^2\mathbb{1}_{(|\sqrt{n}\mu_{n1}|>\sqrt{n}\eta)})$$

as

$$\mathbb{E}((\sqrt{n}\mu_{n1})^2) = n\mathbb{E}(\mu_{n1}^2) \to \frac{M_2}{M_1^2} F(\theta, y, x)(1 - F(\theta, y, x)).$$

Proofs of (15). To use the same arguments as those invoked to prove (14), let us write

$$S_3S_4 - \mathbb{E}(S_3S_4) = S_4 - \mathbb{E}(S_4) + (S_3 - 1)S_4 - \mathbb{E}(S_3 - 1)S_4)).$$

By applying the Bienaymé-Tchebychv's inequality, we obtain for all $\epsilon > 0$

$$\mathbb{P}(|S_3S_4 - \mathbb{E}(S_3S_4)|) > \epsilon) \leq \frac{\mathbb{E}(|S_3S_4 - \mathbb{E}(S_3S_4)|)}{\epsilon}.$$

and the Cauchy-Schwarz inequality implies that

$$\mathbb{E}(|(S_3-1)S_4 - \mathbb{E}((S_3-1)S_4)|) \leq 2\mathbb{E}(|(S_3-1)S_4)|) \leq 2\sqrt{\mathbb{E}((S_3-1)^2)}\sqrt{\mathbb{E}((S_4)^2)}$$

Taking into account the equations (9) and (10), we get

$$\begin{aligned}
\mathbb{E}((S_3 - 1)^2 &= var(S_3) = \frac{n}{n^2\mathbb{E}^2(\beta_1 K_1)} var(\beta_1 K_1) \\
&\leq \frac{1}{nO(h_K^4 \phi_{\theta,x}^2(h_K))} \mathbb{E}(\beta_1^4 K_1^2) = O\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mathbb{E}((S_4)^2) &= \frac{n\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1 K_1)}{\mathbb{E}^2(\Omega_1 K_1)} \mathbb{E}\left(\sum_{j=1}^{n} \beta_j K_j (H_j - F(\theta, y, x))\right)^2 \\
&= \frac{n\phi_{\theta,x}(h_K)O(h_K^2\phi_{\theta,x}^2(h_K))}{(n-1)^2 O(h_K^4\phi_{\theta,x}^4(h_K))} (n\mathbb{E}(\beta_1 K_1(H_1 - F(\theta, y, x))))^2 \\
&\qquad + n(n-1)\mathbb{E}^2(\beta_1 K_1(H_1 - F(\theta, y, x))) \\
&= o(1) + o(n\phi_{\theta,x}(h_K))
\end{aligned}$$

It remains to show

$$\mathbb{E}(|(S_3 - 1)S_4 - \mathbb{E}((S_3 - 1)S_4)|) \leq 2\sqrt{\mathbb{E}((S_3 - 1)^2)}\sqrt{\mathbb{E}((S_4)^2)} = o(1)$$

which implies that

$$|(S_3 - 1)S_4 - \mathbb{E}((S_3 - 1)S_4)| = o_p(1)$$

Therefore,

$$\mathbb{P}(|S_3S_4 - \mathbb{E}(S_3S_4)|) > \epsilon) \leq \frac{\mathbb{E}(|S_3S_4 - \mathbb{E}(S_3S_4)|)}{\epsilon} \to 0 \quad as \quad n \to \infty.$$

So, to prove (15), it suffices to show $S_4 - \mathbb{E}(S_4) = o(1)$, while

$$\mathbb{E}(S_4 - \mathbb{E}(S_4))^2 = var(S_4) = \frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1 K_1)}{\mathbb{E}^2(\Omega_1 K_1)} var(\beta_1 K_1(H_1 - F(\theta, y, x)))$$

We arrive finally at

$$var(\beta_1 K_1(H_1 - F(\theta, y, x))) = F(\theta, y, x)(1 - F(\theta, y, x))\mathbb{E}(\beta_1^2 K_1^2)$$

This last result together equation (7), (8) and (10), lead directly to

$$\begin{aligned}\mathbb{E}(S_4 - \mathbb{E}(S_4))^2 &= \frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1 K_1)}{\mathbb{E}^2(\Omega_1 K_1)} F(\theta, y, x)(1 - F(\theta, y, x))\mathbb{E}(\beta_1^2 K_1^2) \\ &= (F(\theta, y, x)(1 - F(\theta, y, x)))o(1),\end{aligned}$$

which allows to finish the proof of Theorem.

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