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Partitioning to three matchings of given size is NP-complete for bipartite graphs

Dömötör PÁLVÖLGYI
Eötvös Loránd University, Institute of Mathematics
email: dom@cs.elte.hu

Abstract. We show that the problem of deciding whether the edge set of a bipartite graph can be partitioned into three matchings, of size k_1, k_2 and k_3 is NP-complete, even if one of the matchings is required to be perfect. We also show that the problem of deciding whether the edge set of a simple graph contains a perfect matching and a disjoint matching of size k or not is NP-complete, already for bipartite graphs with maximum degree 3. It also follows from our construction that it is NP-complete to decide whether in a bipartite graph there is a perfect matching and a disjoint matching that covers all vertices whose degree is at least 2.

Folkman and Fulkerson [2] described bipartite graphs whose edge set can be partitioned into l_1 matchings of size k_1 and l_2 matchings of size k_2. We complement this result by showing that it is NP-complete to decide whether the edge set of a bipartite graph can be partitioned into three matchings, of size k_1, k_2 and k_3. This will follow from the NP-completeness of the following "perfect matching + matching" problem.

Input: G bipartite graph with maximum degree 3, natural number k.

Goal: Decide whether G contains an edge-disjoint perfect matching and a matching of size k.

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Proof. First we show how the hardness of the partitioning problem follows from the hardness of this problem. Notice that if G contains an edge-disjoint perfect matching P and a matching M of size k, then it also contains another matching M' which is also edge-disjoint from P, has size at least k and is such that $E(G) - P - M'$ is also a matching, as we can start alternating paths from degree two vertices of $E(G) - P - M$. Therefore G contains an edge-disjoint perfect matching and a matching of size k if and only if its edges can be partitioned into three matchings, of size n, k' and $|E(G)| - n - k'$ for some $k' \geq k$. (This was only a Cook reduction, but from our construction below it can be easily made into a Karp reduction.)

Next we show that the "perfect matching + matching" problem is NP-complete. The reduction is from MAX-2-SAT, in which the input is a conjunctive normal form such that every clause contains at most 2 literals (2CNF) and a number s and the question is whether at least s clauses are satisfiable. This problem is well known to be NP-complete [3]. Let us denote the variables of our input Ψ by $x_1, \dots, x_n$ and the clauses by $C_1, \dots, C_m$. From this we make a bipartite graph G of maximum degree 3 on $N = 4mn + 4m$ vertices that will contain an edge-disjoint perfect matching P and a matching M of size $k = (N - 4m + 2s)/2$ if and only if at least s clauses of the input Ψ were satisfiable.

G consists of several smaller parts, which we now describe. To every variable x_i we associate a cycle of length $4m$, denoted by X_i. The vertices of X_i are denoted (in cyclic order) by $a_1^i, b_1^i, c_1^i, d_1^i, a_2^i, b_2^i, c_2^i, d_2^i, \dots, a_m^i, b_m^i, c_m^i, d_m^i$. To every clause C_j we associate four vertices, $u_j, v_j, u_j^{leaf}, v_j^{leaf}$ and two edges, $u_j u_j^{leaf}$ and $v_j v_j^{leaf}$.

These parts are connected as follows. If x_i is an unnegated variable of C_j, then a_j^i is connected to u_j and b_j^i is connected to v_j, while if x_i is a negated variable of C_j, then c_j^i is connected to u_j and b_j^i is connected to v_j. There are no other edges in the graph.

To see that G is bipartite, we can color all vertices $u_j, v_j^{leaf}, b_j^i, d_j^i$ with one color, and all vertices $v_j, u_j^{leaf}, a_j^i, c_j^i$ with the other.

Notice that G has exactly 2^n perfect matchings, as for each cycle X_i we can choose whether we select its edges $a_j^i b_j^i$ and $c_j^i d_j^i$ or $b_j^i c_j^i$ and $d_j^i a_{j+1}^i$ for all j (with circular indexing). The latter of these will correspond to x_i being true, the former to x_i being false.

Now, suppose that s clauses of Ψ are satisfiable. First we select an assignment of the variables satisfying s clauses and the corresponding perfect matching P, then we will show that a disjoint matching M of size s exists. For every satisfied clause C_j, we select a variable that satisfies it, some $x_{f(j)}$. Then the edges of M will be the symmetric difference of the following three sets. For all j where $x_{f(j)}$ is unnegated in C_j, take the path $u_j a_j^{f(j)} b_j^{f(j)} v_j$. For all j where $x_{f(j)}$ is negated in C_j, take the path $u_j c_j^{f(j)} b_j^{f(j)} v_j$. Finally, take $\bigcup_i X_i \setminus P$. So expressed with a formula

$$M = \left(\bigcup_{x_{f(j)} \in C_j} u_j c_j^{f(j)} b_j^{f(j)} v_j \right) \Delta \left(\bigcup_{\bar{x}_{f(j)} \in C_j} u_j a_j^{f(j)} b_j^{f(j)} v_j \right) \Delta \left(\bigcup_i X_i \setminus P \right).$$

The fact that P corresponds to the truth assignments of the variables guarantees that $a_j^i b_j^i \notin P$ if x_i is true and $b_j^i c_j^i \notin P$ if x_i is false, so we indeed obtained a matching covering all the vertices but the leafs and further the $2m - 2s$ vertices that belong to unsatisfied clauses.

On the other hand, suppose that we have an edge-disjoint perfect matching P and matching M covering $2k$ vertices. M must obviously miss the $2m$ leafs, as it is disjoint from P. We can also suppose that M covers each vertex of each X_i, as they can be covered by a matching even after the removal of P. Now we claim that the truth assignment that corresponds to P will satisfy at least s clauses of Ψ. Indeed, if u_j is connected to a vertex from X_i, then v_j must be also connected to X_i and x_i must satisfy C_j, or M would not be a matching covering the vertices of X_i.

This finishes the proof. □

Note that if instead of MAX-2-SAT we use 3-OCC-MAX-2SAT where every variable can appear in at most 3 clauses (total of unnegated and negated occurrences), then the number of vertices is only $12n + 4m$. This problem is also known to be NP-complete, in fact even inapproximable for some small constant, see [1].

A more interesting modification proves the NP-completeness of the following problem.

Input: G bipartite graph with maximum degree 4.

Goal: Decide whether G contains two edge-disjoint matchings, P and M, such that P is perfect and M covers every vertex whose degree is at least 2.

If in our construction instead of MAX-2-SAT we use 3-SAT, then such perfect matching P and matching M, covering all but the u_j^{leaf}, v_j^{leaf} vertices, exist if and only if the original formula is satisfiable, which proves the NP-completeness. As in this reduction the maximum degree grows to 4, we leave the maximum degree 3 case open.

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