

Contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold

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Abstract. In this paper we prove some properties of the indefinite Lorentzian para-Sasakian manifolds. Section 1 is introductory. In Section 2 we define D -totally geodesic and $D^\perp$ -totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold and deduce some results concerning such a manifold. In Section 3 we state and prove some results on mixed totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold. Finally, in Section 4 we obtain a result on the anti-invariant distribution of totally umbilic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold.

1 Introduction

Many valuable and essential results were given on differential geometry with contact and almost contact structure. In 1970 the geometry of cosymplectic

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manifold was studied by G. D. Ludden [14]. After them, in 1973 and 1974, B. Y. Chen and K. Ogiue introduced the geometry of submanifolds and totally real submanifolds in [8], [17], [7]. Then K. Ogiue expressed the differential geometry of Kaehler submanifolds in [17]. In 1976 contact manifolds in Riemannian geometry were discussed by D. E. Blair [5]. Later on, A. Bejancu discussed CR-submanifolds of a Kaehler manifold [1], [2], [4], and then, K. Yano and M. Kon gave the notion of invariant and anti invariant submanifold in [13] and [21]. M. Kobayashi studied CR-submanifolds of a Sasakian manifold in 1981 [12]. New classes of almost contact metric structures and normal contact manifold in [18], [6] were studied by J. A. Oubina, C. Calin and I. Mihai. A. Bejancu and K. L. Duggal introduced (ϵ)-Sasakian manifolds. Lightlike submanifold of semi Riemannian manifolds was introduced by K. L. Duggal and A. Bejancu [10], [9]. In 2003 and 2007, lightlike submanifolds and hypersurfaces of indefinite Sasakian manifolds were introduced [11]. Lastly, LP-Sasakian manifolds were studied by many authors in [15], [16], [19], [20].

In this paper we define D-totally and $D^\perp$ -totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold and prove some interesting results.

An n -dimensional differentiable manifold is called indefinite Lorentzian para-Sasakian manifold if the following conditions hold

$$\phi^2X = X + \eta(X)\xi, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad (1)$$

$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \epsilon\eta(X)\eta(Y), \quad (2)$$

$$\tilde{g}(X, \xi) = \epsilon\eta(X), \quad (3)$$

for all vector fields X, Y on $\tilde{M}$ [5] and where ϵ is 1 or -1 according to ξ is space-like or time-like vector field.

An indefinite almost metric structure $(\phi, \xi, \eta, \tilde{g})$ is called an indefinite Lorentzian para-Sasakian manifold if

$$(\tilde{\nabla}_X \phi)Y = g(X, Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi, \quad (4)$$

where $\tilde{\nabla}$ is the Levi-Civita (L - C) connection for a semi-Riemannian metric $\tilde{g}$. Also we have

$$\tilde{\nabla}_X \xi = \epsilon\phi X, \quad (5)$$

where $X \in T\tilde{M}$.

From the definition of contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold we have

Definition 1 An n -dimensional Riemannian submanifold M of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$ is called a contact CR-submanifold if

- i) ξ is tangent to M ,
- ii) there exists on M a differentiable distribution $D : x \longrightarrow D_x \subset T_x(M)$, such that D_x is invariant under ϕ ; i.e., $\phi D_x \subset D_x$, for each $x \in M$ and the orthogonal complementary distribution $D^\perp : x \longrightarrow D_x^\perp \subset T_x^\perp(M)$ of the distribution D on M is totally real; i.e., $\phi D_x^\perp \subset T_x^\perp(M)$, where $T_x(M)$ and $T_x^\perp(M)$ are the tangent space and the normal space of M at x .

D (resp. $D^\perp$) is the horizontal (resp. vertical) distribution. The contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold is called ξ -horizontal (resp. ξ -vertical) if $\xi_x \in D_x$ (resp. $\xi_x \in D_x^\perp$) for each $x \in M$ by [12].

The Gauss and Weingarten formulae are as follows

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (6)$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad (7)$$

for any $X, Y \in TM$ and $N \in T^\perp M$, where $\nabla^\perp$ is the connection on the normal bundle $T^\perp M$, h is the second fundamental form and A_N is the Weingarten map associated with N via

$$g(A_N X, Y) = g(h(X, Y), N). \quad (8)$$

The equation of Gauss is given by

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)), \quad (9)$$

where $\tilde{R}$ (resp. R) is the curvature tensor of $\tilde{M}$ (resp. M).

For any $x \in M$, $X \in T_x M$ and $N \in T_x^\perp M$, we write

$$X = PX + QX \quad (10)$$

$$\phi N = BN + CN, \quad (11)$$

where PX (resp. BN) denotes the tangential part of X (resp. ϕN) and QX (resp. CN) denotes the normal part of X (resp. ϕN) respectively.

Using (6), (7), (10), (11) in (4) after a brief calculation we obtain on comparing the horizontal, vertical and normal parts

$$P\nabla_X\phi PY - PA_{\phi QY}X = \phi P\nabla_XY + g(PX, Y)\xi + \epsilon\eta(Y)PX + 2\epsilon\eta(Y)\eta(X), \quad (12)$$

$$Q\nabla_X\phi PY + QA_{\phi QY}X = Bh(X, Y) + g(QX, Y)\xi + \epsilon\eta(Y)QX, \quad (13)$$

$$h(X, \phi PY) + \nabla_X^\perp\phi QY = \phi Q\nabla_XY + Ch(X, Y). \quad (14)$$

From (5) we have

$$\nabla_X\xi = \epsilon\phi PX, \quad (15)$$

$$h(X, \xi) = \epsilon\phi QX. \quad (16)$$

Also we have

$$h(X, \xi) = 0 \quad \text{if } X \in D, \quad (17)$$

$$\nabla_X\xi = 0, \quad (18)$$

$$h(\xi, \xi) = 0, \quad (19)$$

$$A_N\xi \in D^\perp. \quad (20)$$

2 D-totally geodesic and $D^\perp$ -totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold

First we define the D-totally (resp. $D^\perp$ -totally) geodesic contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold.

Definition 2 A contact CR-submanifold M of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$ is called D-totally geodesic (resp. $D^\perp$ -totally geodesic) if $h(X, Y) = 0$, $\forall X, Y \in D$ (resp. $X, Y \in D^\perp$).

From the above definition, the following propositions follow immediately.

Proposition 1 Let M be a contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold. Then M is a D-totally geodesic if and only if $A_NX \in D^\perp$ for each $X \in D$ and N a normal vector field to M .

Proof. Let M be D-totally geodesic. Then from (8) we get

$$g(h(X, Y), N) = g(A_NX, Y) = 0.$$

So if

$$h(X, Y) = 0, \quad \forall \quad X, Y \in D$$

i.e.,

$$A_N X \in D^\perp.$$

Conversely, let $A_N X \in D^\perp$. Then for $X, Y \in D$ we can obtain

$$g(A_N X, Y) = 0 = g(h(X, Y), N)$$

i.e.,

$$h(X, Y) = 0$$

$\forall \quad X, Y \in D$, which implies that M is D -totally geodesic. Thus our proof is complete. $\square$

Proposition 2 *Let M be a contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$. Then M is $D^\perp$ -totally geodesic if and only if $A_N X \in D$ for each $X \in D^\perp$ and N a normal vector field to M .*

Proof. The proof follows immediately from the above proposition. $\square$

Concerning the integrability of the horizontal distribution D and vertical distribution $D^\perp$ on M , we can state the following theorem:

Theorem 1 *Let M be a contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold. If M is ξ -horizontal, then the distribution D is integrable iff*

$$h(X, \phi Y) = h(\phi X, Y) \quad (21)$$

$\forall \quad X, Y \in D$. If M is ξ -vertical then the distribution $D^\perp$ is integrable iff

$$A_{\phi X} Y - A_{\phi Y} X = \epsilon[\eta(Y)X - \eta(X)Y] \quad (22)$$

$\forall \quad X, Y \in D^\perp$.

Proof. If M is ξ -horizontal, then using (14) we get

$$h(X, \phi PY) = \phi Q \nabla_X Y + Ch(X, Y)$$

$\forall \quad X, Y \in D$. Therefore $[X, Y] \in D$ iff $h(X, \phi Y) = h(Y, \phi X)$

Hence, if M is ξ -horizontal, $[X, Y] \in D$ iff $h(X, \phi Y) = h(\phi X, Y)$.

Again using (14) we get

$$\nabla_X^\perp \phi Y = \text{Ch}(X, Y) + \phi Q \nabla_X Y \quad (23)$$

for $X, Y \in D^\perp$.

After some calculations we see that

$$\begin{aligned} \tilde{\nabla}_X \phi Y &= g(X, Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(Y)\eta(X)\xi + \phi P \nabla_X Y \\ &\quad + \phi Q \nabla_X Y + \text{Bh}(X, Y) + \text{Ch}(X, Y). \end{aligned} \quad (24)$$

Again from (7) and (24) we get

$$\begin{aligned} \nabla_X^\perp \phi Y &= A_{\phi Y}X + g(X, Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(Y)\eta(X)\xi \\ &\quad + \phi P \nabla_X Y + \phi Q \nabla_X Y + \text{Bh}(X, Y) + \text{Ch}(X, Y) \end{aligned} \quad (25)$$

for $X, Y \in D^\perp$. From (24) and (25) we can write

$$\phi P \nabla_X Y = -A_{\phi Y}X - g(X, Y)\xi - \epsilon\eta(Y)X - 2\epsilon\eta(Y)\eta(X)\xi - \text{Bh}(X, Y). \quad (26)$$

Interchanging X and Y in (26) we get

$$\phi P \nabla_Y X = -A_{\phi X}Y - g(X, Y)\xi - \epsilon\eta(X)Y - 2\epsilon\eta(Y)\eta(X)\xi - \text{Bh}(X, Y). \quad (27)$$

Subtracting (27) from (26) we have

$$\phi P[X, Y] = -A_{\phi Y}X + A_{\phi X}Y - \epsilon\eta(Y)X + \epsilon\eta(X)Y. \quad (28)$$

Now since M is ξ -vertical, $[X, Y] \in D^\perp$ iff

$$A_{\phi X}Y - A_{\phi Y}X = \epsilon[\eta(Y)X - \eta(X)Y].$$

So the proof is complete. $\square$

D -umbilic (resp. $D^\perp$ -umbilic) contact CR-submanifold of indefinite Lorentzian para-Sasakian manifold is defined as follows:

Definition 3 A contact CR-submanifold M of an indefinite Lorentzian para-Sasakian manifold is said to be D -umbilic (resp. $D^\perp$ -umbilic) if $\text{h}(X, Y) = g(X, Y)L$ holds for all $X, Y \in D$ (resp. $X, Y \in D^\perp$), L being some normal vector field.

In view of the above definition we state and prove the following proposition:

Proposition 3 *Suppose M is a D -umbilic contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$. If M is ξ -horizontal (resp. ξ -vertical) then M is D -totally geodesic (resp. $D^\perp$ -totally geodesic).*

Proof. Consider M as D -umbilic ξ -horizontal contact CR-submanifold. Then we have from Definition 3

$$h(X, Y) = g(X, Y)L \quad \forall X, Y \in D,$$

L being some normal vector field on M . By putting $X = Y = \xi$ and using (19) we have

$$h(\xi, \xi) = g(\xi, \xi)L$$

i.e.

$$L = 0,$$

and consequently we get $h(X, Y) = 0$, which proves that M is D -totally geodesic.

Similarly, it can be easily shown that if M is $D^\perp$ -umbilic ξ -vertical contact CR-submanifold then it is $D^\perp$ -totally geodesic. $\square$

3 Mixed totally geodesic contact CR-submanifolds of indefinite Lorentzian para-Sasakian manifold

In this section we define mixed totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold (followed [12]).

Definition 4 *A contact CR-submanifold M of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$ is said to be mixed totally geodesic if $h(X, Y) = 0 \forall X \in D$ and $Y \in D^\perp$.*

Then we extract the following lemma and theorem

Lemma 1 *Let M be a contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold. Then M is mixed totally geodesic iff*

$$A_N X \in D, \quad \forall X \in D, \quad \text{and } \forall \text{ normal vector field } N, \quad (29)$$

$$A_N X \in D^\perp, \quad \forall X \in D^\perp \quad \text{and } \forall \text{ normal vector field } N. \quad (30)$$

Proof. If M is mixed totally geodesic, then from (8), we get

$$h(X, Y) = 0,$$

i.e., iff $A_N X \in D$, $\forall X \in D$ and $\forall$ normal vector field N . Conversely, if M is mixed totally geodesic, then using (8) we easily observe that $A_N X \in D^\perp$, $\forall X \in D^\perp$ and $\forall$ normal vector field N .

Hence the lemma is proved. $\square$

Using condition (29) we obtain the following theorem

Theorem 2 *If M is a mixed totally geodesic contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold, then*

$$A_{\phi N} X = -\phi A_N X, \quad (31)$$

$$\nabla_X^\perp \phi N = \phi \nabla_X^\perp N \quad (32)$$

$\forall X \in D$ and $\forall$ normal vector field N .

Proof. We get from (29), (6), (7) and after having some calculations we derive

$$\nabla_X \phi N = \phi \nabla_X^\perp N - \phi A_N X, \quad (33)$$

$$\nabla_X \phi N = -A_{\phi N} X + \nabla_X^\perp \phi N. \quad (34)$$

Comparing the above two equations we have the required theorem. Hence the proof follows. $\square$

Again we have the following definition

Definition 5 *A contact CR-submanifold M of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$ is called foliate contact CR-submanifold $\tilde{M}$ if D is involute. If M is a foliate ξ -horizontal contact CR-submanifold, we know from [3]*

$$h(\phi X, \phi Y) = h(\phi^2 X, Y) = -h(X, Y). \quad (35)$$

Considering the above definition we give the following proposition.

Proposition 4 *If M is a foliate ξ -horizontal mixed totally geodesic contact CR-submanifold M of an indefinite Lorentzian para-Sasakian manifold, then*

$$\phi A_N X = A_N \phi X \quad (36)$$

for all $X \in D$ and normal vector field N .

Proof. From (21) and (8) we compute the following:

$$g(h(X, \phi Y), N) = g(\phi A_N X, Y),$$

i.e.

$$g(h(\phi X, Y), N) = g(A_N \phi X, Y).$$

Therefore

$$\phi A_N X = A_N \phi X.$$

Hence the proof follows. $\square$

4 Anti-invariant distribution $D^\perp$ on totally umbilical contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold

Here we consider a contact CR-submanifold M of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$. Then we establish the following theorem.

Theorem 3 *Let M be a totally umbilical contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$. Then the anti invariant distribution $D^\perp$ is one dimensional, i.e. $\dim D^\perp = 1$.*

Proof. For an indefinite Lorentzian para-Sasakian structure we have

$$(\tilde{\nabla}_Z \phi)W = g(Z, W)\xi + \epsilon\eta(W)Z + 2\epsilon\eta(W)\eta(Z)\xi. \quad (37)$$

Also by the covariant derivative of tensor fields (for any $Z, W \in \Gamma(D^\perp)$ we know

$$\tilde{\nabla}_Z \phi W = (\tilde{\nabla}_Z \phi)W + \phi \tilde{\nabla}_Z W. \quad (38)$$

Using (37), (38), (6), (7) and (4) we obtain

$$\begin{aligned} \nabla_Z^\perp \phi W - g(H, \phi W)Z &= \phi[\nabla_Z W + g(Z, W)H] + g(Z, W)\xi \\ &\quad + \epsilon\eta(W)Z + 2\epsilon\eta(W)\eta(Z)\xi \end{aligned} \quad (39)$$

for any $Z, W \in \Gamma(D^\perp)$.

Taking the inner product with $Z \in \Gamma(D^\perp)$ in (39) we obtain

$$\begin{aligned} -g(H, \phi W)\|Z\|^2 &= g(Z, W)g(\phi H, Z) + \epsilon\eta(W)\|Z\|^2 + g(Z, W)g(\xi, Z) \\ &\quad + 2\eta(W)\eta(Z)g(Z, \xi). \end{aligned} \quad (40)$$

Using (2) after a brief calculation we have

$$\begin{aligned} g(H, \phi W) = & -\frac{g(Z, W)g(\phi H, Z)}{\|Z\|^2} - \frac{g(Z, W)g(\xi, Z)}{\|Z\|^2} \\ & - \epsilon g(W, \xi) - 2\frac{g(Z, \xi)^2 g(W, \xi)}{\|Z\|^2}. \end{aligned} \quad (41)$$

Interchanging Z and W we have

$$\begin{aligned} g(H, \phi Z) = & -\frac{g(Z, W)g(\phi H, W)}{\|W\|^2} - \frac{g(Z, W)g(\xi, W)}{\|W\|^2} \\ & - \epsilon g(Z, \xi) - 2\frac{g(W, \xi)^2 g(Z, \xi)}{\|W\|^2}. \end{aligned} \quad (42)$$

Substituting (41) in (40) and simplifying we get

$$\begin{aligned} g(H, \phi W) \left[1 - \frac{g(Z, W)^2}{\|Z\|^2 \|W\|^2} \right] - \frac{g(Z, W)}{\|Z\|^2} \left[\frac{g(Z, W)g(\xi, W)}{\|W\|^2} - g(Z, \xi) \right] \\ - \epsilon \left[\frac{g(Z, W)g(\xi, Z)}{\|Z\|^2} - g(W, \xi) \right] \\ - 2g(z, \xi)g(W, \xi) \left[\frac{g(Z, W)g(W, \xi)}{\|W\|^2 \|Z\|^2} - \frac{g(Z, W)}{\|Z\|^2} \right] = 0. \end{aligned} \quad (43)$$

The equation (43) has a solution if $Z \parallel W$, i.e. $\dim D^\perp = 1$.

Hence the theorem is proved. $\square$

Example 1 Let $\mathbf{R}^3$ be a 3-dimensional Euclidean space with rectangular coordinates (x, y, z) . In $\mathbf{R}^3$ we define

$$\eta = -dz - ydx \quad \xi = \frac{\partial}{\partial z}$$

$$\phi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}, \quad \phi\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \quad \phi\left(\frac{\partial}{\partial z}\right) = 0.$$

The Lorentzian metric g is defined by the matrix:

$$\begin{pmatrix} -\epsilon y^2 & 0 & \epsilon y \\ 0 & 0 & 0 \\ \epsilon y & 0 & -\epsilon \end{pmatrix}.$$

Then it can be easily seen that (ϕ, ξ, η, g) forms an indefinite Lorentzian para-Sasakian structure in $\mathbf{R}^3$ and the above results can be verified for this example.

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