

Related fixed point theorem for four complete metric spaces

Vishal Gupta

Maharishi Markandeshwar University

Department of Mathematics

Mullana, Ambala, Haryana (India)

email: vishal.gmn@gmail.com

Abstract. In the present paper, we obtain a new result on fixed point theorem for four metric spaces. Here we choose continuous mappings. In fact our result is the generalization of many results of fixed point theorem on two and three metric spaces. We also give some illustrative examples to justify our result.

1 Introduction

Related fixed point theorems on two metric spaces have been studied by B. Fisher [2]. Also some fixed point theorems on three metric spaces have been studied by B. Fisher *et al* [3], R. K. Jain *et al.* [4], R. K. Namdeo and B. Fisher [7], K. Kikina *et al.* [6], Z. Ansari *et al.* [1], and V. Gupta [8]. Also, the fixed point theorems on four metric spaces have been studied by L. Kikina *et al.* [5]. In the present paper a generalization is given for four complete metric space. Our theorem improves Theorem (2.1) of R. K. Jain *et al.* [4].

The following fixed point theorem was proved by R. K. Jain, H. K. Sahu, B. Fisher [4].

Theorem 1 *Let (X, d) , (Y, ρ) and (Z, σ) be complete metric spaces. If T is continuous mapping of $X \mapsto Y$, S is a continuous mapping of $Y \mapsto Z$ and R is mapping of $Z \mapsto X$ satisfying the inequalities*

2010 Mathematics Subject Classification: 54H25, 47H10

Key words and phrases: complete metric spaces, fixed points, Cauchy sequences, continuous mappings

$$\begin{aligned} d(RSTx, RSTx') &\leq c \max\{d(x, x'), d(x, RSTx), \\ &\quad d(x', RSTx'), \rho(Tx, Tx'), \sigma(STx, STx')\}, \end{aligned} \quad (1)$$

$$\begin{aligned} \rho(TRSy, TRSy') &\leq c \max\{\rho(y, y'), \rho(y, TRSy), \\ &\quad \rho(y', TRSy'), \sigma(Sy, Sy'), d(RSy, RSy')\}, \end{aligned} \quad (2)$$

$$\begin{aligned} \sigma(STRz, STRz') &\leq c \max\{\sigma(z, z'), \sigma(z, STRz) \\ &\quad \sigma(z', STRz'), d(Rz, Rz'), \rho(TRz, TRz')\}, \end{aligned} \quad (3)$$

$\forall x, x' \in X, y, y' \in Y$ and $z, z' \in Z$, where $0 \leq c < 1$, then RST has a unique fixed point $u \in X$, TRS has a unique fixed point $v \in Y$ and STR has a unique fixed point $w \in Z$. Further $Tu = v$, $Sv = w$ and $Rw = u$.

2 Main result

Theorem 2 Let $(Z_1, d_1), (Z_2, d_2), (Z_3, d_3)$, and (Z_4, d_4) be complete metric spaces. If A_1 is a continuous mapping of $Z_1 \mapsto Z_2$, A_2 is continuous mapping of $Z_2 \mapsto Z_3$, A_3 is continuous mapping of $Z_3 \mapsto Z_4$ and A_4 is a mappings of $Z_4 \mapsto Z_1$, satisfying the inequalities

$$\begin{aligned} d_1(A_4A_3A_2A_1z_1, A_4A_3A_2A_1z'_1) \\ \leq c \max\{d_1(z_1, z'_1), d_1(z_1, A_4A_3A_2A_1z_1), \\ d_1(z'_1, A_4A_3A_2A_1z'_1), d_2(A_1z_1, A_1z'_1) \\ d_3(A_2A_1z_1, A_2A_1z'_1), d_4(A_3A_2A_1z_1, A_3A_2A_1z'_1)\}, \end{aligned} \quad (4)$$

$$\begin{aligned} d_2(A_1A_4A_3A_2z_2, A_1A_4A_3A_2z'_2) \\ \leq c \max\{d_2(z_2, z'_2), d_2(z_2, A_1A_4A_3A_2z_2), \\ d_2(z'_2, A_1A_4A_3A_2z'_2), d_3(A_2z_2, A_2z'_2), \\ d_4(A_3A_2z_2, A_3A_2z'_2), d_1(A_4A_3A_2z_2, A_4A_3A_2z'_2)\}, \end{aligned} \quad (5)$$

$$\begin{aligned} d_3(A_2A_1A_4A_3z_3, A_2A_1A_4A_3z'_3) \\ \leq c \max\{d_3(z_3, z'_3), d_3(z_3, A_2A_1A_4A_3z_3), \\ d_3(z'_3, A_2A_1A_4A_3z'_3), d_4(A_3z_3, A_3z'_3), \\ d_1(A_4A_3z_3, A_4A_3z'_3), d_2(A_1A_4A_3z_3, A_1A_4A_3z'_3)\}, \end{aligned} \quad (6)$$

$$\begin{aligned} d_4(A_3A_2A_1A_4z_4, A_3A_2A_1A_4z'_4) \\ \leq c \max\{d_4(z_4, z'_4), d_4(z_4, A_3A_2A_1A_4z_4), \\ d_4(z'_4, A_3A_2A_1A_4z'_4), d_1(A_4z_4, A_4z'_4), \\ d_2(A_1A_4z_4, A_1A_4z'_4), d_3(A_2A_1A_4z_4, A_2A_1A_4z'_4)\}, \end{aligned} \quad (7)$$

$\forall z_1, z'_1 \in Z_1, z_2, z'_2 \in Z_2, z_3, z'_3 \in Z_3$ and $z_4, z'_4 \in Z_4$, where $0 \leq c < 1$, then $A_4A_3A_2A_1$ has a unique fixed point $\alpha_1 \in Z_1$, $A_1A_4A_3A_2$ has a unique fixed point $\alpha_2 \in Z_2$, $A_2A_1A_4A_3$ has a unique fixed point $\alpha_3 \in Z_3$ and $A_3A_2A_1A_4$ has a unique fixed point $\alpha_4 \in Z_4$.

Further $A_1\alpha_1 = \alpha_2, A_2\alpha_2 = \alpha_3, A_3\alpha_3 = \alpha_4, A_4\alpha_4 = \alpha_1$.

Proof. Let z_1^0 be an arbitrary point in Z_1 .

Define the sequence $\{z_n^1\}, \{z_n^2\}, \{z_n^3\}$ and $\{z_n^4\}$ in Z_1, Z_2, Z_3 and Z_4 respectively by

$$\begin{aligned} (A_4A_3A_2A_1)^n z_1^0 &= z_n^1 \\ A_1 z_{n-1}^1 &= z_n^2 \\ A_2 z_n^2 &= z_n^3 \\ A_3 z_n^3 &= z_n^4 \\ A_4 z_n^4 &= z_n^1 \quad \text{for } n = 1, 2, \dots \end{aligned}$$

Applying inequality (5), we get,

$$\begin{aligned} d_2(z_n^2, z_{n+1}^2) &= d_2(A_1A_4A_3A_2z_{n-1}^2, A_1A_4A_3A_2z_n^2) \\ &\leq c \max\{d_2(z_{n-1}^2, z_n^2), d_2(z_{n-1}^2, A_1A_4A_3A_2z_{n-1}^2), \\ &\quad d_2(z_n^2, A_1A_4A_3A_2z_n^2), d_3(A_2z_{n-1}^2, A_2z_n^2), \\ &\quad d_4(A_3A_2z_{n-1}^2, A_3A_2z_n^2), d_1(A_4A_3A_2z_{n-1}^2, A_4A_3A_2z_n^2)\} \\ d_2(z_n^2, z_{n+1}^2) &\leq c \max\{d_2(z_{n-1}^2, z_n^2), d_2(z_{n-1}^2, z_n^2), d_2(z_n^2, z_{n+1}^2), \\ &\quad d_3(z_{n-1}^3, z_n^3), d_4(z_{n-1}^4, z_n^4), d_1(z_{n-1}^1, z_n^1)\} \\ d_2(z_n^2, z_{n+1}^2) &\leq c \max\{d_1(z_{n-1}^1, z_n^1), d_2(z_{n-1}^2, z_n^2), d_3(z_{n-1}^3, z_n^3), d_4(z_{n-1}^4, z_n^4)\} \end{aligned} \quad (8)$$

Using inequality (6), we get,

$$\begin{aligned} d_3(z_n^3, z_{n+1}^3) &= d_3(A_2A_1A_4A_3z_{n-1}^3, A_2A_1A_4A_3z_n^3) \\ &\leq \max\{d_3(z_{n-1}^3, z_n^3), d_3(z_{n-1}^3, A_2A_1A_4A_3z_{n-1}^3), \\ &\quad d_3(z_n^3, A_2A_1A_4A_3z_n^3), d_4(A_3z_{n-1}^3, A_3z_n^3), \\ &\quad d_1(A_4A_3z_{n-1}^3, A_4A_3z_n^3), d_2(A_1A_4A_3z_{n-1}^3, A_1A_4A_3z_n^3)\} \\ d_3(z_n^3, z_{n+1}^3) &\leq c \max\{d_3(z_{n-1}^3, z_n^3), d_3(z_{n-1}^3, z_n^3), d_3(z_n^3, z_{n+1}^3), \\ &\quad d_4(z_{n-1}^4, z_n^4), d_1(z_{n-1}^1, z_n^1), d_2(z_{n-1}^2, z_{n+1}^2)\} \\ d_3(z_n^3, z_{n+1}^3) &\leq c \max\{d_1(z_{n-1}^1, z_n^1), d_2(z_{n-1}^2, z_n^2), \\ &\quad d_3(z_{n-1}^3, z_n^3), d_4(z_{n-1}^4, z_n^4)\} \end{aligned} \quad (9)$$

Using inequality (7), we have,

$$\begin{aligned}
 d_4(z_n^4, z_{n+1}^4) &= d_4(A_3A_2A_1A_4z_{n-1}^4, A_3A_2A_1A_4z_n^4) \\
 &\leq c \max\{d_4(z_{n-1}^4, z_n^4), d_4(z_{n-1}^4, A_3A_2A_1A_4z_{n-1}^4), \\
 &\quad d_4(z_n^4, A_3A_2A_1A_4z_n^4), d_1(A_4z_{n-1}^4, A_4z_n^4), \\
 &\quad d_2(A_1A_4z_{n-1}^4, A_1A_4z_n^4), d_3(A_2A_1A_4z_{n-1}^4, A_2A_1A_4z_n^4)\} \\
 d_4(z_n^4, z_{n+1}^4) &\leq c \max\{d_4(z_{n-1}^4, z_n^4), d_4(z_{n-1}^4, z_n^4), d_4(z_n^4, z_{n+1}^4), \\
 &\quad d_1(z_{n-1}^1, z_n^1), d_2(z_n^2, z_{n+1}^2), d_3(z_n^3, z_{n+1}^3)\} \\
 d_4(z_n^4, z_{n+1}^4) &\leq c \max\{d_1(z_{n-1}^1, z_n^1), d_2(z_{n-1}^2, z_n^2), \\
 &\quad d_3(z_{n-1}^3, z_n^3), d_4(z_{n-1}^4, z_n^4)\}
 \end{aligned} \tag{10}$$

Using inequality (4), we have,

$$\begin{aligned}
 d_1(z_n^1, z_{n+1}^1) &= d_1(A_4A_3A_2A_1z_{n-1}^1, A_4A_3A_2A_1z_n^1) \\
 &\leq c \max\{d_1(z_{n-1}^1, z_n^1), d_1(z_{n-1}^1, A_4A_3A_2A_1z_{n-1}^1), \\
 &\quad d_1(z_n^1, A_4A_3A_2A_1z_n^1), d_2(A_1z_{n-1}^1, A_1z_n^1), \\
 &\quad d_3(A_2A_1z_{n-1}^1, A_2A_1z_n^1), d_4(A_3A_2A_1z_{n-1}^1, A_3A_2A_1z_n^1)\} \\
 d_1(z_n^1, z_{n+1}^1) &\leq c \max\{d_1(z_{n-1}^1, z_n^1), d_1(z_{n-1}^1, z_n^1), d_1(z_n^1, z_{n+1}^1), \\
 &\quad d_2(z_n^2, z_{n+1}^2), d_3(z_n^3, z_{n+1}^3), d_4(z_n^4, z_{n+1}^4)\} \\
 d_1(z_n^1, z_{n+1}^1) &\leq c \max\{d_1(z_{n-1}^1, z_n^1), d_2(z_{n-1}^2, z_n^2), \\
 &\quad d_3(z_{n-1}^3, z_n^3), d_4(z_{n-1}^4, z_n^4)\}
 \end{aligned} \tag{11}$$

By induction on using inequalities (8), (9), (10) and (11), we have,

$$\begin{aligned}
 d_1(z_n^1, z_{n+1}^1) &\leq c^{n-1} \max\{d_1(z_1^1, z_2^1), d_2(z_1^2, z_2^2), \\
 &\quad d_3(z_1^3, z_2^3), d_4(z_1^4, z_2^4)\} \\
 d_2(z_n^2, z_{n+1}^2) &\leq c^{n-1} \max\{d_1(z_1^1, z_2^1), d_2(z_1^2, z_2^2), \\
 &\quad d_3(z_1^3, z_2^3), d_4(z_1^4, z_2^4)\} \\
 d_3(z_n^3, z_{n+1}^3) &\leq c^{n-1} \max\{d_1(z_1^1, z_2^1), d_2(z_1^2, z_2^2), \\
 &\quad d_3(z_1^3, z_2^3), d_4(z_1^4, z_2^4)\} \\
 d_4(z_n^4, z_{n+1}^4) &\leq c^{n-1} \max\{d_1(z_1^1, z_2^1), d_2(z_1^2, z_2^2), \\
 &\quad d_3(z_1^3, z_2^3), d_4(z_1^4, z_2^4)\}
 \end{aligned}$$

Since $c < 1$, it follows that $\{z_n^1\}, \{z_n^2\}, \{z_n^3\}$ and $\{z_n^4\}$ are Cauchy sequences with limit $\alpha_1, \alpha_2, \alpha_3$ and α_4 in Z_1, Z_2, Z_3 and Z_4 respectively.

Since A_1, A_2 and A_3 are continuous, we have,

$$\begin{aligned}\lim_{n \rightarrow \infty} z_n^2 &= \lim_{n \rightarrow \infty} A_1 z_n^1 = A_1 \alpha_1 = \alpha_2 \\ \lim_{n \rightarrow \infty} z_n^3 &= \lim_{n \rightarrow \infty} A_2 z_n^2 = A_2 \alpha_2 = \alpha_3 \\ \lim_{n \rightarrow \infty} z_n^4 &= \lim_{n \rightarrow \infty} A_3 z_n^3 = A_3 \alpha_3 = \alpha_4\end{aligned}$$

Using inequality (4), again, we get,

$$\begin{aligned}d_1(A_4 A_3 A_2 A_1 \alpha_1, z_n^1) &= d_1(A_4 A_3 A_2 A_1 \alpha_1, A_4 A_3 A_2 A_1 z_{n-1}^1) \\ &\leq c \max\{d_1(\alpha_1, z_{n-1}^1), d_1(\alpha_1, A_4 A_3 A_2 A_1 \alpha_1), \\ &\quad d_1(z_{n-1}^1, A_4 A_3 A_2 A_1 z_{n-1}^1), d_2(A_1 \alpha_1, A_1 z_{n-1}^1), \\ &\quad d_3(A_2 A_1 \alpha_1, A_2 A_1 z_{n-1}^1), d_4(A_3 A_2 A_1 \alpha_1, A_3 A_2 A_1 z_{n-1}^1)\}\end{aligned}$$

Since A_1, A_2 and A_3 are continuous, it follows on letting $n \rightarrow \infty$ that

$$d_1(A_4 A_3 A_2 A_1 \alpha_1, \alpha_1) \leq c \max\{d_1(\alpha_1, A_4 A_3 A_2 A_1 \alpha_1)\}$$

Thus, we have, $A_4 A_3 A_2 A_1 \alpha_1 = \alpha_1$. Since $c < 1$ and α_1 is the fixed point of $A_4 A_3 A_2 A_1$

$$A_1 A_4 A_3 A_2 \alpha_2 = A_1 A_4 A_3 A_2 A_1 \alpha_1 = A_1 \alpha_1 = \alpha_2$$

$$\text{and} \quad A_2 A_1 A_4 A_3 \alpha_3 = A_2 A_1 A_4 A_3 A_2 \alpha_2 = A_2 \alpha_2 = \alpha_3$$

$$\text{and} \quad A_3 A_2 A_1 A_4 \alpha_4 = A_3 A_2 A_1 A_4 A_3 \alpha_3 = A_3 \alpha_3 = \alpha_4$$

Hence α_2, α_3 and α_4 are fixed points of $A_1 A_4 A_3 A_2, A_2 A_1 A_4 A_3$ and $A_3 A_2 A_1 A_4$ respectively.

2.1 Uniqueness

Suppose that $A_4 A_3 A_2 A_1$ has a second fixed point α'_1 . Then, using inequality (4), we have,

$$\begin{aligned}d_1(\alpha_1, \alpha'_1) &= d_1(A_4 A_3 A_2 A_1 \alpha_1, A_4 A_3 A_2 A_1 \alpha'_1) \\ &\leq c \max\{d_1(\alpha_1, \alpha'_1), d_1(\alpha_1, A_4 A_3 A_2 A_1 \alpha_1), \\ &\quad d_1(\alpha'_1, A_4 A_3 A_2 A_1 \alpha'_1), d_2(A_1 \alpha_1, A_1 \alpha'_1), \\ &\quad d_3(A_2 A_1 \alpha_1, A_2 A_1 \alpha'_1), \\ &\quad d_4(A_3 A_2 A_1 \alpha_1, A_3 A_2 A_1 \alpha'_1)\}\end{aligned}$$

$$\begin{aligned}
d_1(\alpha_1, \alpha'_1) &\leq c \max\{d_1(\alpha_1, \alpha'_1), d_1(\alpha_1, \alpha_1), d_1(\alpha'_1, \alpha'_1), \\
&\quad d_2(A_1\alpha_1, A_1\alpha'_1), d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
&\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1)\} \\
d_1(\alpha_1, \alpha'_1) &\leq c \max\{d_2(A_1\alpha_1, A_1\alpha'_1), d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
&\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1)\}
\end{aligned} \tag{12}$$

Using inequality (5), we have

$$\begin{aligned}
d_2(A_1\alpha_1, A_1\alpha'_1) &= d_2(A_1A_4A_3A_2A_1\alpha_1, A_1A_4A_3A_2A_1\alpha'_1) \\
&\leq c \max\{d_2(A_1\alpha_1, A_1\alpha'_1), \\
&\quad d_2(A_1\alpha_1, A_1A_4A_3A_2A_1\alpha_1), \\
&\quad d_2(A_1\alpha'_1, A_1A_4A_3A_2A_1\alpha'_1), \\
&\quad d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
&\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1), \\
&\quad d_4(A_4A_3A_2A_1\alpha_1, A_4A_3A_2A_1\alpha'_1)\} \\
d_2(A_1\alpha_1, A_1\alpha'_1) &\leq c \max\{d_2(A_1\alpha_1, A_1\alpha'_1), d_2(A_1\alpha_1, A_1\alpha'_1), \\
&\quad d_1(A_1\alpha'_1, A_1\alpha'_1), \\
&\quad d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
&\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1), \\
&\quad d_1(A_4A_3A_2A_1\alpha_1, A_4A_3A_2A_1\alpha'_1)\} \\
d_2(A_1\alpha_1, A_1\alpha'_1) &\leq c \max\{d_2(A_1\alpha_1, A_1\alpha'_1), d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
&\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1), d_1(\alpha_1, \alpha'_1)\} \\
d_2(A_1\alpha_1, A_1\alpha'_1) &\leq c \max\{d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
&\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1), d_1(\alpha_1, \alpha'_1)\}
\end{aligned}$$

Now, we have

$$\begin{aligned}
d_2(A_1\alpha_1, A_1\alpha'_1) &\leq c \max\{d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
&\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1), \\
&\quad cd_2(A_1\alpha_1, A_1\alpha'_1), cd_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
&\quad cd_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1)\} \\
d_2(A_1\alpha_1, A_1\alpha'_1) &\leq c \max\{d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1) \\
&\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1)\}
\end{aligned} \tag{13}$$

Similarly on using inequality (6), we get

$$\begin{aligned}
 d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1) &\leq c \max\{d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
 &\quad d_3(A_2A_1\alpha_1, A_2A_1A_4A_3A_2A_1\alpha_1), \\
 &\quad d_3(A_2A_1\alpha'_1, A_2A_1A_4A_3A_2A_1\alpha'_1), \\
 &\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1), \\
 &\quad d_1(A_4A_3A_2A_1\alpha_1, A_4A_3A_2A_1\alpha'_1), \\
 &\quad d_2(A_1A_4A_3A_2A_1\alpha_1, A_1A_4A_3A_2A_1\alpha'_1)\} \\
 d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1) &\leq c \max\{d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
 &\quad d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1), \\
 &\quad d_3(A_2A_1\alpha'_1, A_2A_1\alpha'_1), \\
 &\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1), \\
 &\quad d_1(\alpha_1, \alpha'_1), d_2(A_1\alpha_1, A_1\alpha'_1)\} \\
 d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1) &\leq c \max\{d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1), \\
 &\quad d_1(\alpha_1, \alpha'_1), d_2(A_1\alpha_1, A_1\alpha'_1)\} \\
 d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1) &\leq c \max\{d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1), \\
 &\quad cd_2(A_1\alpha_1, A_1\alpha'_1), cd_3(A_2A_1\alpha_1, A_2A_1\alpha'_1) \\
 &\quad cd_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1)\}
 \end{aligned} \tag{14}$$

Using inequality (13) and (14), we have

$$d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1) \leq c \max\{d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1)\} \tag{15}$$

Similarly on using inequality (7), (13) and (15), we have,

$$d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1) \leq cd_1(\alpha_1, \alpha'_1) \tag{16}$$

Using inequality (12), (13), (15) and (16), we have

$$\begin{aligned}
 d_1(\alpha_1, \alpha'_1) &\leq cd_2(A_1\alpha_1, A_1\alpha'_1) \\
 &\leq c^2d_3(A_2A_1\alpha_1, A_2A_1\alpha'_1) \\
 &\leq c^3d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha'_1) \\
 &\leq c^4d_1(\alpha_1, \alpha'_1)
 \end{aligned}$$

Now we have

$$d_1(\alpha_1, \alpha'_1) \leq c^4d_1(\alpha_1, \alpha'_1)$$

Since $0 \leq c < 1$, we have

$$d_1(\alpha_1, \alpha'_1) = 0$$

$\Rightarrow \alpha_1 = \alpha'_1$, proving the uniqueness of α_1 .

We can similarly prove that $A_1A_4A_3A_2$ has a unique fixed point $d_2 \in Z_2$ and $A_2A_1A_4A_3$ has a unique fixed point $\alpha_3 \in Z_3$ and $A_3A_2A_1A_4$ has unique fixed point $\alpha_4 \in Z_4$. $\square$

Now, in support of our result, we give some examples.

Example 1 Let suppose $X = [0, 1], Y = [1, 2], Z = [2, 3]$ and $L = [3, 4]$ be complete metric spaces with usual metric. If $T : [0, 1] \rightarrow [1, 2], S : [1, 2] \rightarrow [2, 3]$ and $R : [2, 3] \rightarrow [3, 4]$ are continuous mappings and $U : [3, 4] \rightarrow [0, 1]$ is a mapping satisfying given conditions (in Theorem 2.1), where

$$T(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq \frac{3}{4} \\ \frac{4}{3}x, & \text{if } \frac{3}{4} < x \leq 1 \end{cases}, \quad S(y) = \begin{cases} 2, & \text{if } 1 \leq y \leq \frac{3}{2} \\ \frac{4}{3}y, & \text{if } \frac{3}{2} < y \leq 2 \end{cases}$$

$$R(z) = \begin{cases} 3, & \text{if } 2 \leq z \leq \frac{5}{2} \\ \frac{6}{5}z, & \text{if } \frac{5}{2} < z \leq 3 \end{cases}, \quad U(u) = \begin{cases} 1, & \text{if } 3 \leq u \leq \frac{7}{2} \\ \frac{3}{5}, & \text{if } \frac{7}{2} < u \leq 4 \end{cases}$$

then URST has fixed point 1 such that URST(1) = 1, TURS has fixed point $4/3$ such that TURS($4/3$) = $4/3$, STUR has fixed point 2 such that STUR(2) = 2 and RSTU has fixed point 3 such that RSTU(3) = 3. Also $T(1) = 4/3, S(4/3) = 2, R(2) = 3$ and $U(3) = 1$.

Remark 1 Below we give an example which satisfies all the condition of Theorem 2.1 but does not satisfies the condition of Theorem 1.1.

Example 2 Let $X = [0, 1], Y = [1, 2], Z = [2, 3]$ and $L = [3, 4]$ be complete metric space with usual metric. If $T : [0, 1] \rightarrow [1, 2], S : [1, 2] \rightarrow [2, 3]$ and $R : [2, 3] \rightarrow [3, 4]$ are continuous mappings and $U : [3, 4] \rightarrow [0, 1]$ is a mapping satisfying given conditions (in Theorem 2.1), where

$$T(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq \frac{1}{4} \\ 2x + \frac{1}{2}, & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4} \\ 2, & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases}, \quad S(y) = \begin{cases} 2, & \text{if } 1 \leq y \leq \frac{5}{4} \\ \frac{4}{5}y + 1, & \text{if } \frac{5}{4} \leq y \leq \frac{7}{4} \\ 2.4, & \text{if } \frac{7}{4} \leq y \leq 2 \end{cases}$$

$$R(z) = \begin{cases} 3, & \text{if } 2 \leq z \leq \frac{9}{4} \\ z + \frac{3}{4}, & \text{if } \frac{9}{4} \leq z \leq \frac{11}{4} \\ 3.5, & \text{if } \frac{11}{4} \leq z \leq 3 \end{cases}, \quad U(u) = \begin{cases} 0, & \text{if } 3 \leq u \leq \frac{7}{2} \\ \frac{2}{7}u - 1, & \text{if } \frac{7}{2} \leq u \leq 4 \end{cases}$$

then URST has fixed point 0 such that $\text{URST}(0) = 0$, TURS has fixed point 1 such that $\text{TURS}(1) = 1$, STUR has fixed point 2 such that $\text{STUR}(2) = 2$ and RSTU has fixed point 3 such that $\text{RSTU}(3) = 3$. Also $\text{T}(0)=1$, $\text{S}(1)=2$, $\text{R}(2)=3$ and $\text{U}(3)=0$.

Example 3 Let suppose $X = [0, 2]$, $Y = [1, 5]$, $Z = [0, 10]$ and $L = [1, 12]$ be complete metric spaces with usual metric. If $\text{T} : [0, 2] \rightarrow [1, 5]$, $\text{S} : [1, 5] \rightarrow [0, 10]$ and $\text{R} : [0, 10] \rightarrow [1, 12]$ are continuous mappings and $\text{U} : [1, 12] \rightarrow [0, 2]$ is a mapping satisfying given conditions (in Theorem 2.1), where

$$\text{T}(x) = [1 + x, 2], \quad \text{S}(y) = [2y + 1, 5]$$

$$\text{R}(z) = [1 + z, 10], \quad \text{U}(u) = \begin{cases} [\frac{u}{6}, 5] & \text{if } 1 \leq z \leq 6 \\ [1, \frac{u}{6}] & \text{if } 6 < z \leq 12 \end{cases}$$

then URST has fixed point 1 such that $\text{URST}(1) = 1$, TURS has fixed point 2 such that $\text{TURS}(2) = 2$, STUR has fixed point 5 such that $\text{STUR}(5) = 5$ and RSTU has fixed point 6 such that $\text{RSTU}(6) = 6$. Also $\text{T}(1) = 2$, $\text{S}(2) = 5$, $\text{R}(5) = 6$ and $\text{U}(6) = 1$.

Example 4 Let suppose $X = [0, 3]$, $Y = [1, 4]$, $Z = [4, 7]$ and $L = [3, 10]$ be complete metric spaces with usual metric. If $\text{T} : [0, 3] \rightarrow [1, 4]$, $\text{S} : [1, 4] \rightarrow [4, 7]$ and $\text{R} : [4, 7] \rightarrow [3, 10]$ be a continuous mappings and $\text{U} : [3, 10] \rightarrow [0, 3]$ be a mapping satisfying given conditions (in Theorem 2.1), where

$$\text{T}(x) = 1 + x, \quad \text{S}(y) = y + 2,$$

$$\text{R}(z) = z + 3, \quad \text{U}(u) = \begin{cases} \frac{u}{7} & \text{if } 3 \leq z \leq 5 \\ \frac{u+1}{8} & \text{if } 5 < z \leq 10 \end{cases}$$

then URST has fixed point 1 such that $\text{URST}(1) = 1$, TURS has fixed point 2 such that $\text{TURS}(2) = 2$, STUR has fixed point 4 such that $\text{STUR}(4) = 4$ and RSTU has fixed point 7 such that $\text{RSTU}(7) = 7$. Also $\text{T}(1) = 2$, $\text{S}(2) = 4$, $\text{R}(5) = 7$ and $\text{U}(7) = 1$.

References

- [1] Z. K. Ansari, M. Sharma, A. Garg, Related fixed points theorems on three metric spaces, *Int. J. Contemp. Math. Sci.*, **5** (42) (2010), 2059–2064.

- [2] B. Fisher, Related fixed points on two metric spaces, *Math. Sem. Notes, Kobe Univ.*, **10** 1982, 17–26.
- [3] B. Fisher, K. P. R. Rao, A related fixed point theorem on three metric spaces, *Hacet. J. Math. Stat.*, **32** (2) (2007), 143–146.
- [4] R. K. Jain, H. K. Sahu, B. Fisher, Related fixed point theorems for three metric spaces, *Novi Sad J. Math.*, **26** (1) (1996), 11–17.
- [5] L. Kikina, K. Kikina, A related fixed point theorem on four metric spaces, *Int. J. Math. Anal.*, **3** (32) (2009), 1559–1568.
- [6] K. Kikina, L. Kikina, *Generalized fixed point theorem in three metric spaces*, *Int. J. Math. Anal.*, **4** (40) (2010), 1995–2004.
- [7] R. K. Namdeo, B. Fisher, A related fixed point theorem for three pairs of mappings on three metric spaces, *Thai J. Math.*, **7** (1) (2009), 129–135.
- [8] V. Gupta, Fixed point theorems for pair of mappings on three metric spaces, *Int. J. Adv. Appl. Sci.*, **3** (5) (2012), 2733–2738.

Received: 26 July, 2013