Acta Univ. Sapientiae, Mathematica, **5**, 2 (2013) 113–131

DOI: 10.2478/ausm-2014-0008

Existence and uniqueness of a periodic solution to certain third order nonlinear delay differential equation with multiple deviating arguments

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Abstract. In this paper, we use Lyapunov's second method, by constructing a complete Lyapunov functional, sufficient conditions which guarantee existence and uniqueness of a periodic solution, uniform asymptotic stability of the trivial solution and uniform ultimate boundedness of solutions of Eq. (2). New results are obtained and proved, an example is given to illustrate the theoretical analysis in the work and to test the effectiveness of the method employed. The results obtained in this investigation extend many existing and exciting results on nonlinear third order delay differential equations.

1 Introduction

The importance of functional differential equations, in particular the delay differential equations, cannot be over emphasized as it creates a significant branch of nonlinear analysis and find numerous applications in physics, chemistry, biology, geography, economics, theory of nuclear reactors and in other fields of

2010 Mathematics Subject Classification: 34K12, 34K13, 34K20, 34K40, 34K60
Key words and phrases: third order, nonlinear neutral delay differential equation, uniform asymptotic stability, uniform ultimate boundedness, existence and uniqueness of periodic solution

engineering and natural sciences to mention few. The existence, uniqueness, boundedness and stability of solutions of the models derived from these applications are paramount to researchers in various fields of research.

Many work has been done by distinguished authors see for instance Burton [4, 5], Diver [7], Hale [9], Yoshizawa [21, 22] which contain general results on the subject matters. Other remarkable authors worked on stability, boundedness, asymptotic behaviour of solutions of third order delay differential include Ademola *et al* [2, 3], Omeike [10], Sadek [11], Tunç *et al* [13, 15, 16, 18] and the reference cited therein.

To the best of our knowledge few authors have discussed the existence and uniqueness of a periodic solution to delay differential equations (see the paper of Chukwu [6], Gui [8] and Zhu [23]). Also, in 2000, Tejumola and Tchegnani [12] discussed criteria for the existence of periodic solutions of third order differential equation with constant deviating arguments $\tau > 0$:

$$\dddot{x}+f(t,x,\dot{x},\ddot{x})\ddot{x}+g(t,x(t-\tau),\dot{x}(t-\tau))+h(x(t-\tau)) = p(t,x,x(t-\tau),\dot{x},\dot{x}(t-\tau),\ddot{x}).$$

In 2010, Tunç [17] established conditions on the existence of periodic solution for the nonlinear differential equation of third order with constant deviating argument $\tau > 0$:

$$\dddot{x} + \psi(\dot{x})\ddot{x} + g(\dot{x}(t-\tau)) + f(x) = p(t,x,x(t-\tau),\dot{x},\dot{x}(t-\tau),\ddot{x}).$$

Recently, in 2012, Abo-El-Ela *et al.* [1] discussed the existence and uniqueness of a periodic solutions for third order delay differential equation with two deviating arguments.

$$\begin{aligned} &\dddot{x} + \psi(\dot{x})\ddot{x} + f(x)\dot{x} + g_1(t,\dot{x}(t-\tau_1(t))) + g_2(t,\dot{x}(t-\tau_2(t))) \\ &= p(t) = p(t,x,x(t-\tau),\dot{x},\dot{x}(t-\tau),\ddot{x}). \end{aligned}$$

Also, in 2012, Tunç [14] considered the existence of periodic solutions to nonlinear differential equations of third order with multiple deviating arguments τ_i, $(i = 1, 2, \dots, n)$:

$$\begin{aligned} &\dddot{x} + \psi(\dot{x})\ddot{x} + \sum_{i=1}^{n} g_i(\dot{x}(t-\tau_i)) + f(x) \\ &= p(t,x,x(t-\tau_1),\dots,x(t-\tau_n),\dot{x},\dots x(t-\tau_1),\dots,\dot{x}(t-\tau_n),\dots x). \end{aligned}$$

However, the problem of uniform asymptotic stability, boundedness, existence and uniqueness of a periodic solution of third order neutral delay differential equation with multiple deviating arguments $\tau_i(t) \geq 0$ $(i = 1, 2, \ldots, n)$ and for all $t \geq 0$, has not been investigated. Therefore, the purpose of this paper is to establish criteria for uniform stability, boundedness, existence and uniqueness of a periodic solution for the third order nonlinear delay differential equation with multiple deviating arguments $\tau_i(t) \geq 0$ $(i = 1, 2, \ldots, n)$:

$$\begin{aligned} \dddot{x}(t) + f(t, x(t), \dot{x}(t), \ddot{x}(t))\ddot{x}(t) + \sum_{i=1}^{n} g_i(t, x(t - \tau_i(t)), \dot{x}(t - \tau_i(t))) \\ + \sum_{i=1}^{n} h_i(t, x(t - \tau_i(t))) = p(t, x, X, \dot{x}, \dot{X}, \ddot{x}), \end{aligned} \tag{1}$$

where $X = x(t-\tau_1(t)), \ldots, x(t-\tau_n(t))$ and $\dot{X} = \dot{x}(t-\tau_1(t)), \ldots, \dot{x}(t-\tau_n(t))$. Let $\dot{x}(t) = y(t)$ and $\ddot{x}(t) = z(t)$, (1) is equivalent to the system of first order differential equations

$$\begin{aligned} &\dot{x}(t) = y(t), \quad \dot{y}(t) = z(t) \\ &\dot{z}(t) = p(t, x(t), X, y(t), Y, z(t)) - f(t, x(t), y(t), z(t))z(t) \\ &- \sum_{i=1}^{n} g_i(t, x(t), y(t)) - \sum_{i=1}^{n} h_i(t, x(t)) + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} g_{it}(s, x(s), y(s))ds \\ &+ \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} g_{ix}(s, x(s), y(s))y(s)ds + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} g_{iy}(s, x(s), y(s))z(s)ds \\ &+ \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} h_{it}(s, x(s))ds + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} h_{ix}(s, x(s))y(s)ds, \end{aligned} \tag{2}$$

where $0 \leq \tau_i(t) \leq \gamma$, $\gamma > 0$ is a constant to be determined later, the functions f, g_i, h_i and p are continuous in their respective arguments on $\mathbb{R}^+ \times \mathbb{R}^3$, $\mathbb{R}^+ \times \mathbb{R}^2$, $\mathbb{R}^+ \times \mathbb{R}$ and $\mathbb{R}^+ \times \mathbb{R}^{2n+3}$ respectively with $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R} = (-\infty, \infty)$, periodic in t of period ω, and the derivatives $f_t(t, x, y, z)$, $f_x(t, x, y, z)$, $f_z(t, x, y, z)$, $g_{it}(t, x, y)$, $g_{ix}(t, x, y)$, $g_{iy}(t, x, y)$, $h_{it}(t, x)$, $h_{ix}(t, x)$, with respect to t, x, y, z, for all i, $(i = 1, 2, \ldots, n)$ exist and are continuous for all t, x, y, z with $h_i(t, 0) = 0$ for all t. The dots as usual, stands for differentiation with respect to t. Motivation for this study comes from the works [1, 8, 12, 14, 17] and the recent papers [2, 19]. These results are new and complement many existing and exciting latest results on third order delay differential equations.

2 Preliminary results

Consider the following general nonlinear non-autonomous delay differential equation

$$\dot{X} = \frac{dX}{dt} = F(t, X_t), \quad X_t = X(t+\theta), \quad -r \leq \theta < 0, \quad t \geq 0, \tag{3}$$

where $F : \mathbb{R}^+ \times C_H \to \mathbb{R}^n$ is a continuous mapping, $F(t+\omega, \phi) = F(t, \phi)$ for all $\phi \in C$ and for some positive constant ω. We assume that F takes closed bounded sets into bounded sets in $\mathbb{R}^n$. $(C, \|\cdot\|)$ is the Banach space of continuous function $\varphi : [-r, 0] \to \mathbb{R}^n$ with supremum norm, $r > 0$; for $H > 0$, we define $C_H \subset C$ by $C_H = \{\varphi \in C : \|\varphi\| < H\}$, C_H is the open H-ball in C, $C = C([-r, 0], \mathbb{R}^n)$.

Lemma 1 [22] *Suppose that* $F(t, \phi) \in \overline{C}_0(\phi)$ *and* $F(t, \phi)$ *is periodic in* t *of period* ω, $\omega \geq r$, *and consequently for any* $\alpha > 0$ *there exists an* $L(\alpha) > 0$ *such that* $\phi \in C_\alpha$ *implies* $|F(t, \phi)| \leq L(\alpha)$. *Suppose that a continuous Lyapunov functional* $V(t, \phi)$ *exists, defined on* $t \in \mathbb{R}^+$, $\phi \in S^*$, S^* *is the set of* $\phi \in C$ *such that* $|\phi(0)| \geq H$ (H *may be large*) *and that* $V(t, \phi)$ *satisfies the following conditions:*

(i) $a(|\phi(0)|) \leq V(t, \phi) \leq b(\|\phi\|)$, *where* $a(r)$ *and* $b(r)$ *are continuous, increasing and positive for* $r \geq H$ *and* $a(r) \to \infty$ *as* $r \to \infty$;

(ii) $\dot{V}_{(3)}(t, \phi) \leq -c(|\phi(0)|)$, *where* $c(r)$ *is continuous and positive for* $r \geq H$.

Suppose that there exists an $H_1 > 0$, $H_1 > H$, *such that*

$$hL(\gamma^*) < H_1 - H, \tag{4}$$

where $\gamma^* > 0$ *is a constant which is determined in the following way: By the condition on* $V(t, \phi)$ *there exist* $\alpha > 0$, $\beta > 0$ *and* $\gamma > 0$ *such that* $b(H_1) \leq a(\alpha)$, $b(\alpha) \leq a(\beta)$ *and* $b(\beta) \leq a(\gamma)$. γ^* *is defined by* $b(\gamma) \leq a(\gamma^*)$. *Under the above conditions, there exists a periodic solution of* (3) *with period* ω. *In particular, the relation* (4) *can always be satisfied if* h *is sufficiently small.*

Lemma 2 [22] *Suppose that* $F(t, \phi)$ *is defined and continuous on* $0 \leq t \leq c$, $\phi \in C_H$ *and that there exists a continuous Lyapunov functional* $V(t, \phi, \varphi)$ *defined on* $0 \leq t \leq c$, $\phi, \varphi \in C_H$ *which satisfy the following conditions:*

(i) $V(t, \phi, \varphi) = 0$ *if* $\phi = \varphi$;

(ii) $V(t,\phi,\varphi) > 0$ *if* $\phi \neq \varphi$;

(iii) *for the associated system*

$$\dot{x}(t) = F(t, x_t), \quad \dot{y}(t) = F(t, y_t) \tag{5}$$

we have $V'_{(5)}(t,\phi,\varphi) \leq 0$, *where for* $\|\phi\| = H$ *or* $\|\varphi\| = H$, *we understand that the condition* $V'_{(5)}(t,\phi,\varphi) \leq 0$ *is satisfied in the case* V' *can be defined.*

Then, for given initial value $\phi \in C_{H_1}$, $H_1 < H$, *there exists a unique solution of* (3).

Lemma 3 [22] *Suppose that a continuous Lyapunov functional* $V(t,\phi)$ *exists, defined on* $t \in \mathbb{R}^+$, $\|\phi\| < H$, $0 < H_1 < H$ *which satisfies the following conditions:*

(i) $a(\|\phi\|) \leq V(t,\phi) \leq b(\|\phi\|)$, *where* $a(r)$ *and* $b(r)$ *are continuous, increasing and positive,*

(ii) $\dot{V}_{(3)}(t,\phi) \leq -c(\|\phi\|)$, *where* $c(r)$ *is continuous and positive for* $r \geq 0$,

then the zero solution of (3) *is uniformly asymptotically stable.*

Lemma 4 [4] *Let* $V : \mathbb{R}^+ \times C \to \mathbb{R}$ *be continuous and locally Lipschitz in* ϕ. *If*

(i) $W_0(|X_t|) \leq V(t, X_t) \leq W_1(|X_t|) + W_2\left(\int\limits_{t-r(t)}^{t} W_3(X_t(s))ds\right)$ *and*

(ii) $\dot{V}_{(3)}(t, X_t) \leq -W_4(|X_t|) + N$, *for some* $N > 0$, *where* W_i $(i = 0,1,2,3,4)$ *are wedges.*

Then X_t *of* (3) *is uniformly bounded and uniformly ultimately bounded for bound* B.

3 Main results

Theorem 1 In addition to the basic assumptions on the functions f, g_i, h_i, p and τ_i, suppose that $a, a_1, b_i, B_i, c_i, \delta_i, E_i, K_i, M_i, \gamma$ $(i = 1, 2, \ldots, n)$ are positive constants and for all $t \geq 0$.

(i) $a \leq f(t,x,y,z) \leq a_1$ for all x, y, z;

(ii) $b_i \leq \dfrac{g_i(t,x,y)}{y} \leq \begin{cases} K_i t & \text{for all } t > 0, x \text{ and } y \neq 0, \\ B_i & \text{for all } t \geq 0, x \text{ and } y \neq 0, \end{cases}$ and $|g_{ix}(t,x,y)| \leq M_i$;

(iii) $h_i(t,0) = 0$, $\delta_i \leq \dfrac{h_i(t,x)}{x} \leq \begin{cases} E_i t & \text{for all } t > 0 \neq x, \\ c_i & \text{for all } t \geq 0 \neq x, \end{cases}$ and $ab_i > c_i$;

(iv) $\tau_i(t) \leq \gamma$, $\tau_i' \leq \rho$, $\rho \in (0,1)$, $0 \leq P(t) < \infty$;

if

$$\gamma < \min\left\{\frac{1}{2}\sum_{i=1}^{n}\beta\delta_i A_1^{-1}, \sum_{i=1}^{n}(\alpha b_i - c_i)A_2^{-1}, \frac{1}{2}(a-\alpha)A_3^{-2}\right\}, \tag{6}$$

where

$$A_1 := \frac{1}{2}\beta + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + (1-\rho)^{-1}(2+\alpha+\beta+a)\sum_{i=1}^{n}E_i$$

$$A_2 := \frac{1}{2}(\alpha+a) + \sum_{i=1}^{n}(B_i+c_i+E_i+K_i+M_i) + (1-\rho)^{-1}(2+\alpha+\beta+a)\sum_{i=1}^{n}(c_i+K_i+M_i)$$

and

$$A_3 := 1 + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + (1-\rho)^{-1}(2+\alpha+\beta+a)\sum_{i=1}^{n}B_i,$$

then (2) has a unique periodic solution of period ω.

Proof. Let (x_t, y_t, z_t) be any solution of (2) and the functional $V = V(t, x_t, y_t, z_t)$ be defined as

$$V = e^{-P(t)}U, \tag{7}$$

where

$$P(t) = \int_0^t |p(s,x,X,y,Y,z)|ds \tag{8a}$$

and $U = U(t, x_t, y_t, z_t)$ is defined as

$$2U = 2(\alpha + a)\sum_{i=1}^{n}\int_0^x h_i(t,\xi)d\xi + 4\sum_{i=1}^{n}\int_0^y g_i(t,x,\tau)d\tau + 4y\sum_{i=1}^{n} h_i(t,x)$$
$$+ 2(\alpha + a)yz + 2z^2 + 2(\alpha + a)\int_0^y \tau f(t,x,\tau,0)d\tau + \beta y^2 + \sum_{i=1}^{n} b_i x^2 + 2a\beta xy$$
$$+ 2\beta xz + \int_{-\tau(t)}^{0}\int_{t+s}^{t}(\lambda_0 x^2(\theta) + \lambda_1 y^2(\theta) + \lambda_2 z^2(\theta))d\theta ds, \tag{8b}$$

where α and β are fixed constants satisfying

$$\sum_{i=1}^{n} b_i^{-1}c_i < \alpha < a \tag{8c}$$

and

$$0 < \beta < \min\left\{\sum_{i=1}^{n} b_i, \sum_{i=1}^{n}(ab_i - c_i)A_4^{-1}, \frac{1}{2}(a-\alpha)A_5^{-1}\right\}, \tag{8d}$$

where

$$A_4 := 1 + a + \sum_{i=1}^{n}\delta_i^{-1}\left(\frac{g_i(t,x,y)}{y} - b_i\right)^2$$

and

$$A_5 := 1 + \sum_{i=1}^{n}\delta_i^{-1}\Big(f(t,x,y,z) - a\Big)^2.$$

Now, since $h_i(t,0) = 0$ for all $t \in \mathbb{R}^+$, (7) can be recast in the form

$$V = e^{-P(t)}\Bigg[\sum_{i=1}^{n} b_i^{-1}\int_0^x[(\alpha + a)b_i - 2h_{ix}(t,\xi)]h_i(t,\xi)d\xi + \frac{\beta}{2}y^2 + \frac{1}{2}(\alpha y + z)^2$$
$$+ 2\sum_{i=1}^{n}\int_0^y\left(\frac{g_i(t,x,\tau)}{\tau} - b_i\right)\tau d\tau + \sum_{i=1}^{n} b_i^{-1}\Big(h_i(t,x) + b_i y\Big)^2$$
$$+ \int_0^y\Big[(\alpha + a)f(t,x,\tau,0) - (\alpha^2 + a^2)\Big]\tau d\tau + \frac{1}{2}(\beta x + ay + z)^2$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\beta(b_i - \beta)x^2 + \frac{1}{2}\int_{-\tau(t)}^{0}\int_{t+s}^{t}\Big(\lambda_0 x^2(\theta) + \lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)\Big)d\theta ds\Bigg], \tag{9}$$

where $P(t)$ is the function defined by (8a). In view of the assumptions of Theorem 1 and the fact that the double integrals are non-negative, there exists a positive constant d_0 such that

$$V \geq d_0(x^2 + y^2 + z^2) \tag{10a}$$

for all $t \geq 0, x, y$ and z, where

$$\begin{aligned} d_0 = e^{-P_0} \min\Bigg\{ & \frac{1}{2}\sum_{i=1}^{n} b_i^{-1}(\alpha b_i - c_i + a b_i - c_i)\delta_i + \sum_{i=1}^{n} b_i^{-1}\min\{b_i, \delta_i\} \\ & + \frac{1}{2}\min\{1, a, \beta\} + \frac{1}{2}\sum_{i=1}^{n}\beta(b_i - \beta), \quad \frac{1}{2}\beta + \sum_{i=1}^{n} b_i^{-1}\min\{b_i, \delta_i\} + \frac{1}{2}\min\{1, \alpha\} \\ & + \frac{1}{2}\min\{1, a, \beta\} + \frac{1}{2}\alpha(a - \alpha), \quad \frac{1}{2}\min\{1, \alpha\} + \frac{1}{2}\min\{1, a, \beta\}\Bigg\}. \end{aligned}$$

Clearly, from(10a), we have $V(t, x, y, z) = 0$ if and only if $x^2 + y^2 + z^2 = 0$, $V(t, x, y, z) > 0$ if and only if $x^2 + y^2 + z^2 \neq 0$, it follows that

$$V(t, x, y, z) \to +\infty \quad \text{as} \quad x^2 + y^2 + z^2 \to \infty. \tag{10b}$$

Moreover, from the hypotheses of Theorem 1 and the obvious inequality $2|x_1 x_2| \leq x_2^2 + x_2^2$, Eq. (7) turns out to be

$$V(t, x, y, z) \leq d_1(x^2 + y^2 + z^2) + d_2\int_{-\tau(t)}^{0}\int_{t+s}^{t} d_3(x^2(\theta) + y^2(\theta) + z^2(\theta))d\theta ds \tag{11}$$

for all $t \geq 0, x, y, z$, and s, where

$$\begin{aligned} d_1 = \max\Bigg\{ & \frac{1}{2}\sum_{i=1}^{n}\Big[(\alpha + a + 2)c_i + \beta(1 + a + b_i)\Big], \\ & \frac{1}{2}\Big(2\sum_{i=1}^{n}(B_i + c_i) + (\alpha + a)(1 + a_1) + \beta(1 + a)\Big), \frac{1}{2}(2, \alpha + \beta + a)\Bigg\}, \end{aligned}$$

$d_2 = \dfrac{1}{2}$ and

$d_3 = \max\{\lambda_0, \lambda_1, \lambda_2\}$.

Next, the derivative of the function V with respect to t along a solution (x_t, y_t, z_t) of (2) is given by

$$\dot{V}_{(2)} = -e^{-P(t)}\Big[U\dot{P}(t) - \dot{U}_{(2)}\Big], \tag{12}$$

where $P(t)$ and U are defined by (8a) and (8b) respectively,

$$\dot{P}(t) = |p(t, x, X, y, Y, z)| \tag{13a}$$

and

$$\dot{U}_{(2)} = a\beta y^2 + 2\beta yz + [\beta x + (\alpha + a)y + 2z]p(t, x, X, y, Y, z) + \sum_{j=1}^{3} U_j - \sum_{j=4}^{5} U_j$$
$$- \beta[f(t, x, y, z) - a]xz - \beta \sum_{i=1}^{n} \left(\frac{g_i(t, x, y)}{y} - b_i\right)xy, \tag{13b}$$

where:

$$U_1 := (\alpha + a)\sum_{i=1}^{n}\int_0^x h_{it}(t, \xi)d\xi + 2\sum_{i=1}^{n}\int_0^y g_{it}(t, x, \tau)d\tau + 2y\sum_{i=1}^{n} h_{it}(t, x);$$

$$U_2 := (\alpha + a)\int_0^y \tau f_t(t, x, \tau, 0)d\tau + 2\sum_{i=1}^{n} y\int_0^y g_{ix}(t, x, \tau)d\tau$$
$$+ (\alpha + a)y\int_0^y \tau f_x(t, x, \tau, 0)d\tau;$$

$$U_3 := (\beta x + (\alpha + a)y + 2z)\Bigg[\sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} g_{it}(s, x(s), y(s))ds$$
$$+ \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} h_{it}(s, x(s))ds + \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} g_{iy}(s, x(s), y(s))z(s)ds$$
$$+ \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} g_{ix}(s, x(s), y(s))y(s)ds$$
$$+ \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} h_{ix}(s, x(s))y(s)ds\Bigg] + \Big(\lambda_0 x^2(t) + \lambda_1 y^2(t) + \lambda_2 z^2(t)\Big)\tau_i(t)$$
$$- \frac{1}{2}(1 - \tau_i'(t))\int_{t-\tau_i(t)}^{t}\Big(\lambda_0 x^2(\theta) + \lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)\Big)d\theta$$

$$U_4 := \beta x\sum_{i=1}^{n} h_i(t, x) + \sum_{i=1}^{n}\left[(\alpha + a)\frac{g_i(t, x, y)}{y} - 2h_{ix}(t, x)\right]y^2$$
$$+ \Big[2f(t, x, y, z) - (\alpha + a)\Big]z^2$$

and

$$U_5 := (\alpha + a)yz\Big[f(t,x,y,z) - f(t,x,y,0)\Big].$$

Now, $h_{it}(t,x) \leq E_i x$ for all $t \geq 0 \neq x$ and $g_{it}(t,x,y) \leq K_i y$ for all $t \geq 0, x, y \neq 0$, these inequalities imply the existence of a positive constant q_0 such that

$$U_1 \leq q_0(x^2 + y^2 + z^2),$$

where $q_0 = \max\{1,\ \frac{1}{2}\sum_{i=1}^{n}[(\alpha+a)E_i+2c_i],\ \ \sum_{i=1}^{n}(K_i+c_i)\}$. Also, $f(t,x,y,z) \leq a_1$ and $g_i(t,x,y) \leq B_i y$ these inequalities imply that

$$U_2 \leq 0$$

for all $t \geq 0, x, y$ and z.
Furthermore, in view of the assumptions of the theorem and the obvious inequality $2mn \leq m^2 + n^2$, we obtain

$$\begin{aligned}
U_3 \leq{}& \Big[\frac{1}{2}\beta + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + \lambda_0\Big]\tau_i(t)x^2 + \Big[\frac{1}{2}(\alpha + a)\\
&+ \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + \lambda_1\Big]\tau_i(t)y^2 + \Big[1 + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i)\\
&+ \lambda_2\Big]\tau_i(t)z^2 - \frac{1}{2}\Big[(1 - \tau_i'(t))\lambda_0 - (2 + \alpha + \beta + a)\sum_{i=1}^{n}E_i\Big]\int_{t-\tau_i(t)}^{t} x^2(s)ds\\
&- \frac{1}{2}\Big[(1 - \tau_i'(t))\lambda_1 - (2 + \alpha + \beta + a)\sum_{i=1}^{n}(c_i + K_i + M_i)\Big]\int_{t-\tau_i(t)}^{t} y^2(s)ds\\
&- \frac{1}{2}\Big[(1 - \tau_i'(t))\lambda_2 - (2 + \alpha + \beta + a)\sum_{i=1}^{n}B_i\Big]\int_{t-\tau_i(t)}^{t} z^2(s)ds,\\
U_4 \geq{}& \beta\sum_{i=1}^{n}\delta_i x^2 + \sum_{i=1}^{n}\Big(\alpha b_i - c_i + a b_i - c_i\Big)y^2 + (a - \alpha)z^2.
\end{aligned}$$

Finally, $f(t,x,y,z) \geq a$ implies that for $y > 0$, $yf_z(t,x,y,z) \geq 0$ for all $t \geq 0, x, y$ and z, so that

$$U_5 = (\alpha + a)yz^2 f_z(t,x,y,\theta_0 z) \geq 0$$

for all $t \geq 0, x, y, z$ and $(\alpha + a)yz^2 f_z(t, x, y, \theta_0 z) = 0$ when $z = 0$. Employing estimates U_i $(i = 1, \cdots, 5)$ in (13b), there exists a positive constant $q_1 = \max\{2, \alpha + a, \beta\}$ such that

$$\begin{aligned}
U_{(2)} &\leq (a+1)\beta y^2 + \beta z^2 + q_1(|x| + |y| + |z|)|p(t, x, X, y, Y, z)| \\
&+ q_0(x^2 + y^2 + z^2) + \left[\frac{1}{2}\beta + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + \lambda_0\right]\tau_i(t)x^2 \\
&- (a - \alpha)z^2 + \left[\frac{1}{2}(\alpha + a) + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + \lambda_1\right]\tau_i(t)y^2 \\
&+ \left[1 + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + \lambda_2\right]\tau_i(t)z^2 \\
&- \frac{1}{2}\left[(1 - \tau_i'(t))\lambda_0 - (2 + \alpha + \beta + a)\sum_{i=1}^{n} E_i\right]\int_{t-\tau_i(t)}^{t} x^2(s)ds \\
&- \frac{1}{2}\left[(1 - \tau_i'(t))\lambda_1 - (2 + \alpha + \beta + a)\sum_{i=1}^{n}(c_i + K_i + M_i)\right]\int_{t-\tau_i(t)}^{t} y^2(s)ds \\
&- \frac{1}{2}\left[(1 - \tau_i'(t))\lambda_2 - (2 + \alpha + \beta + a)\sum_{i=1}^{n} B_i\right]\int_{t-\tau_i(t)}^{t} z^2(s)ds - \beta\sum_{i=1}^{n}\delta_i x^2 \\
&- \sum_{i=1}^{n}\Big(\alpha b_i - c_i + a b_i - c_i\Big)y^2 - \beta[f(t, x, y, z) - a]xz \\
&- \beta\sum_{i=1}^{n}\left(\frac{g_i(t, x, y)}{y} - b_i\right)xy.
\end{aligned} \tag{14}$$

Since $\tau_i(t) \leq \gamma$ and $\tau_i'(t) \leq \rho$ for all $t \geq 0$, estimate (14) can be rearranged in the form

$$\begin{aligned}
U_{(2)} &\leq (a+1)\beta y^2 + \beta z^2 + q_1(|x| + |y| + |z|)|p(t, x, X, y, Y, z)| \\
&+ q_0(x^2 + y^2 + z^2) + \left[\frac{1}{2}\beta + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + \lambda_0\right]\gamma x^2 \\
&- (a - \alpha)z^2 + \left[\frac{1}{2}(\alpha + a) + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + \lambda_1\right]\gamma y^2
\end{aligned}$$

$$+\left[1+\sum_{i=1}^{n}(B_i+c_i+E_i+K_i+M_i)+\lambda_2\right]\gamma z^2$$
$$-\frac{1}{2}\left[(1-\rho)\lambda_0-(2+\alpha+\beta+a)\sum_{i=1}^{n}E_i\right]\int_{t-\tau_i(t)}^{t}x^2(s)ds$$
$$-\frac{1}{2}\left[(1-\rho)\lambda_1-(2+\alpha+\beta+a)\sum_{i=1}^{n}(c_i+K_i+M_i)\right]\int_{t-\tau_i(t)}^{t}y^2(s)ds$$
$$-\frac{1}{2}\left[(1-\rho)\lambda_2-(2+\alpha+\beta+a)\sum_{i=1}^{n}B_i\right]\int_{t-\tau_i(t)}^{t}z^2(s)ds-\frac{\beta}{2}\sum_{i=1}^{n}\delta_i x^2 \quad (15)$$
$$-\sum_{i=1}^{n}\Big(\alpha b_i-c_i+ab_i-c_i\Big)y^2-\frac{\beta}{4}\sum_{i=1}^{n}\delta_i\Big[x+2\delta_i^{-1}(f(t,x,y,z)-a)z\Big]^2$$
$$-\frac{\beta}{4}\sum_{i=1}^{n}\delta_i\left[x+2\delta_i^{-1}\left(\frac{g_i(t,x,y)}{y}-b_i\right)y\right]^2$$
$$+\beta\sum_{i=1}^{n}\delta_i^{-1}\left(\frac{g_i(t,x,y)}{y}-b_i\right)^2y^2+\beta\sum_{i=1}^{n}\delta_i^{-1}\Big(f(t,x,y,z)-a\Big)^2z^2,$$

for all $t\ge 0, x, y, z$. Choosing $\lambda_0=(1-\rho)^{-1}(2+\alpha+\beta+a)\sum_{i=1}^{n}E_i>0$, $\lambda_1=(1-\rho)^{-1}(2+\alpha+\beta+a)\sum_{i=1}^{n}(c_i+K_i+M_i)>0$ and $\lambda_2=(1-\rho)^{-1}(2+\alpha+\beta+a)\sum_{i=1}^{n}B_i>0$, and the fact that $[x+2\delta_i^{-1}(f(t,x,y,z)-a)z]^2\ge 0$ and $[x+2\delta_i^{-1}(\frac{g_i(t,x,y)}{y}-b_i)y]^2\ge 0$ for all $t\ge 0, x, y$ and z, the inequality in (15) yields

$$U_{(2)}\le q_1(|x|+|y|+|z|)|p(t,x,X,y,Y,z)|+q_0(x^2+y^2+z^2)$$
$$-\left\{\frac{1}{2}\sum_{i=1}^{n}\beta\delta_i-\left[\frac{1}{2}\beta+\sum_{i=1}^{n}(B_i+c_i+E_i+K_i+M_i)\right.\right.$$
$$\left.\left.+(1-\rho)^{-1}(2+\alpha+\beta+a)\ sum_{i=1}^{n}E_i\right]\gamma\right\}x^2$$
$$-\left\{\sum_{i=1}^{n}(\alpha b_i-c_i)-\left[\frac{1}{2}(\alpha+a)+\sum_{i=1}^{n}(B_i+c_i+E_i+K_i+M_i)\right.\right.$$
$$\left.\left.+(1-\rho)^{-1}(2+\alpha+\beta+a)\sum_{i=1}^{n}(c_i+K_i+M_i)\right]\gamma\right\}y^2$$
$$-\left\{\frac{1}{2}(a-\alpha)-\left[1+\sum_{i=1}^{n}(B_i+c_i+E_i+K_i+M_i)\right.\right.$$

$$+ (1-\rho)^{-1}(2+\alpha+\beta+a)\sum_{i=1}^{n} B_i\Big]\gamma\Big\}z^2$$
$$-\Big\{\sum_{i=1}^{n}(ab_i - c_i) - \beta\Big[1 + a + \sum_{i=1}^{n}\delta_i^{-1}\Big(\frac{g_i(t,x,y)}{y} - b_i\Big)^2\Big]\Big\}y^2$$
$$-\Big\{\frac{1}{2}(a-\alpha) - \beta\Big[1 + \sum_{i=1}^{n}\delta_i^{-1}\Big(f(t,x,y,z) - a\Big)^2\Big]\Big\}z^2.$$

In view of the estimates (6) and (8d) there exists a positive constant q_2 such that

$$U_{(2)} \le q_1(|x|+|y|+|z|)|p(t,x,X,y,Y,z)| + q_0(x^2+y^2+z^2) - q_2(x^2+y^2+z^2) \quad (16)$$

for all $t \ge 0, x, y$ and z. Applying the assumptions of Theorem 1, estimates (8c), (8d) in (8b), there exists a constant q_3 such that

$$U \ge q_3(x^2+y^2+z^2) \quad (17)$$

for all $t \ge 0, x, y$ and z, where $q_3 = d_0 e^{P_0} > 0$. Using (13a), (16) and (17) in (12) choosing $q_2 > q_0$ and $(x^2+y^2+z^2)^{1/2} \ge 3^{1/2} q_1 q_3^{-1}$ sufficiently large, there exists a constant $d_3 > 0$ such that

$$\dot{V}_{(2)} \le -d_3(x^2+y^2+z^2) \quad (18)$$

for all $t \ge 0, x, y$ and z, where $d_3 = e^{-P_0}(q_2 - q_0) > 0$. From inequalities (10a), (10b), (11) and (18), the assumptions of Lemma 1 hold, also by estimates (10) and (18) the hypotheses of Lemma 2 are satisfied. Hence by Lemma 1 and Lemma 2 Eq. (2) has a unique periodic solution of period ω. This completes the proof of Theorem 1. □

If $p(t,x,X,\dot{x},\dot{X},\ddot{x}) = 0$ in (1), Eq. (2) reduces to

$$\begin{aligned}
&\dot{x}(t) = y(t), \quad \dot{y}(t) = z(t) \quad \dot{z}(t) = -f(t,x(t),y(t),z(t))z(t)\\
&- \sum_{i=1}^{n} g_i(t,x(t),y(t)) - \sum_{i=1}^{n} h_i(t,x(t)) + \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} g_{it}(s,x(s),y(s))ds\\
&+ \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} g_{ix}(s,x(s),y(s))y(s)ds\\
&+ \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} g_{iy}(s,x(s),y(s))z(s)ds\\
&+ \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} h_{it}(s,x(s))ds + \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} h_{ix}(s,x(s))y(s)ds,
\end{aligned} \quad (19)$$

where f, g_i and h_i are the functions defined in section 1.

Theorem 2 If in addition to the hypotheses of Theorem 1, $g_i(t,0,0) = h_i(t,0) = p(t,x,X,y,Y,z) = 0$, then the trivial solution of (19) is uniformly asymptotically stable, provided that the inequality in (6) holds.

Proof. If $p(t,x,X,y,Y,z) = 0$, the function V defined in (7) reduces to $V = U$, where U is defined in (8b). With the assumptions of Theorem 2, it is not difficult to show that

$$V \geq d_4(x^2 + y^2 + z^2) \tag{20}$$

for all $t \geq 0, x, y, z$, where $d_4 = d_0 e^{P_0}$. Furthermore, in view of the assumptions of Theorem 2 estimate (11) holds.
Next, let (x_t, y_t, z_t) be any solution of (19), little calculation shows that

$$\dot{V}_{(19)} \leq -d_5(x^2 + y^2 + z^2) \tag{21}$$

for all $t \geq 0, x, y, z$, where $d_5 = d_3 e^{P_0}$. The inequalities in (11), (20) and (21) verify the assumptions of Lemma 3, thus by Lemma 3 the trivial solution of (19) is uniformly asymptotically stable. □

Theorem 3 If the hypothesis on the function p of Theorem 1 is replaced by

$$|p(t,x,X,y,Y,z)| \leq P_1, \qquad 0 < P_1 < \infty \tag{22}$$

for all $t \geq 0, x, X, y, Y$ and z, then the solutions of (2) is uniformly bounded and uniformly ultimately bounded.

Proof. If $t = 0$ in (8a), Eq. (7) becomes $V = U$, under the assumptions of Theorem 3, estimates (10a), (10b) and (11) hold. Let (x_t, y_t, z_t) be any solution of (2), the derivative of $V = U$ along a solution of (2) is estimated by (16). By (22), choosing q_2 sufficiently large such that $q_2 > q_0 + P_1 q_1$ there exist positive constants d_6 and d_7 such that

$$\dot{V}_{(2)} \leq -d_6(x^2 + y^2 + z^2) + d_7 \tag{23}$$

for all $t \geq 0, x, y, z$, where $d_6 = q_2 - q_0 - P_1 q_1 > 0$ and $d_7 = 3P_1 q_1 > 0$. In view of the inequality in (10), (11) and (23) all hypotheses of Lemma 4 hold true, thus by Lemma 4 the solutions of (2) are uniformly bounded and uniformly ultimately bounded. □

4 An example

Example 1 *Consider the following third order neutral delay differential equation*

$$\begin{aligned}
&\dddot{x} + \frac{3}{2}\ddot{x} + \frac{\ddot{x}}{1 + \sin t + |x\dot{x}| + \exp[(1 + \dot{x}\ddot{x})^{-1}]} \\
&+ 4\sum_{i=1}^{n} \dot{x}(t - \tau_i(t)) + \sum_{i=1}^{n} x(t - \tau_i(t)) \\
&+ \sum_{i=1}^{n} \frac{\dot{x}(t - \tau_i(t))}{3 + \sin(t/2) + |x(t - \tau_i(t))\dot{x}(t - \tau_i(t))|} + \sum_{i=1}^{n} \frac{x(t - \tau_i(t))}{4 + \sin t} \\
&= \frac{1}{4 + \sin t + |x| + |X| + |\dot{x}| + |\dot{X}| + |\ddot{x}|}.
\end{aligned} \tag{24}$$

(24) *is equivalent to system of first order differential equations*

$$\begin{aligned}
&\dot{x} = y, \quad \dot{y} = z, \\
&\dot{z} = \frac{1}{4 + \sin t + |x| + |X| + |y| + |Y| + |z|} - \left(1 + \frac{1}{4 + \sin t}\right) n x \\
&\quad + \left(\frac{3}{2} + \frac{1}{1 + \sin t + |xy| + \exp[(1 + |yz|)^{-1}]}\right) z \\
&\quad - \left(4 + \frac{1}{3 + \sin(t/2) + |xy| + y^2}\right) n y \\
&\quad + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \frac{y \cos(\mu/2) d\mu}{2[3 + \sin(\mu/2) + |xy| + y^2]^2} \\
&\quad - \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \frac{y^2 y(\mu) d\mu}{[3 + \sin(\mu/2) + |xy| + y^2]^2} \\
&\quad + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \left(4 + \frac{3 + \sin(\mu/2) - y^2}{[3 + \sin(\mu/2) + |xy| + y^2]^2}\right) z(\mu) d\mu \\
&\quad - \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \frac{x \cos \mu}{4 + \sin \mu} d\mu \\
&\quad + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \left(1 + \frac{1}{4 + \sin \mu}\right) y(\mu) d\mu.
\end{aligned} \tag{25}$$

In view of (2) *and* (25) *we have the following relations and estimates:*

(A) *The function* $f(t,x,y,z) = \dfrac{3}{2} + \dfrac{1}{1+\sin t + |xy| + \exp[(1+|yz|)^{-1}]}$, *it is not difficult to show that for all* $t \geq 0, x, y$ *and* z:

(i) $\dfrac{3}{2} \leq f(t,x,y,z) \leq \dfrac{5}{2}$, *where* $a = \dfrac{3}{2} > 0$ *and* $a_1 = \dfrac{5}{2} > 0$;

(ii) $f_t(t,x,y,z) = \dfrac{-\cos t}{[1+\sin t + |xy| + \exp[(1+|yz|)^{-1}]]^2} \leq 0$;

(iii) *for* $x > 0$, $yf_x(t,x,y,z) = \dfrac{-y^2}{[1+\sin t + |xy| + \exp[(1+|yz|)^{-1}]]^2} \leq 0$ *and*

(iv) *for* $z > 0$,

$$yf_z(t,x,y,z) = \frac{y^2 \exp[(1+|yz|)^{-1}]}{[1+|yz|]^2[1+\sin t + |xy| + \exp[(1+|yz|)^{-1}]]^2} \geq 0.$$

(B) *The function* $g_i(t,x,y) = \left(4 + \dfrac{1}{3+\sin(t/2)+|xy|+y^2}\right)y$, *which for all* $t \geq 0, x$ *and* y *we have:*

(i) $4 \leq \dfrac{g_i(t,x,y)}{y} \leq 5$, *where* $b_i = 4 > 0$ *and* $B_i = 5 > 0$;

(ii) *for* $x > 0$, $g_{ix}(t,x,y) = \dfrac{-y^2}{[3+\sin(t/2)+|xy|+y^2]^2} \leq 0$ *and*

(iii) $g_{it}(t,x,y) = \dfrac{-y\cos(t/2)}{2[3+\sin(t/2)+|xy|+y^2]^2} \leq \dfrac{|y||1-2\sin^2(t/4)|}{2[3+\sin(t/2)+|xy|+y^2]^2}$.
Now since $0 < \dfrac{|1-2\sin^2(t/4)|}{2[3+\sin(t/2)+|xy|+y^2]^2} < 1$ *for all* $t \geq 0, x, y$, *where* $K_i = 1 > 0$ *so that*

$$g_{it}(t,x,y) \leq |y|.$$

(C) *The function* $h_i(t,x) = x + \dfrac{x}{4+\sin t}$, *it is not difficult to show that*

(i) $h_i(t,0) = 0$;

(ii) $\dfrac{h_i(t,x)}{x} \geq 1$, *where* $\delta_i = 1 > 0$;

(iii) $h_{ix}(t,x) \leq 2 = c_i$;

(iv) $ab_i - c_i$ *implies that* $2 > 0$;

(v) $h_{it}(t,x) = \dfrac{-x\cos t}{4+\sin t} \leq |x|$ *since* $0 < \dfrac{|1-2\sin^2(t/2)|}{|4+\sin t|} < 1$, *where* $E_i = 1 > 0$.

(D) $p(t,x,x(t-\tau_i(t)),$

$$y(t-\tau_i(t)),z) = \frac{1}{4+\sin t+|x|+|x(t-\tau_i(t))|+|y|+|y(t-\tau_i(t))|+|z|}.$$

It is not difficult to show that $|p| \leq 1 < \infty$, *where* $P_1 = 1 > 0$.

(E) *Finally, it can be shown that* $0 < 1 = \alpha$, $\rho = \dfrac{1}{2} < 1$, $0 < \beta < \dfrac{1}{4}$ *and* $\gamma < \dfrac{1}{234}$.

All assumptions of the theorems are verified and hence the conclusions of these theorems follow.

Acknowledgment

The author would like to express his sincere gratitude to the referee for his or her invaluable corrections, comments and suggestions.

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Received: 17 July 2013