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Galois covering and smash product of skew categories

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Dedicated to the memory of Professor Antal Bege

Abstract. In this paper we give a new proof of the famous result of E. L. Green [3], that gradings of a finite, path connected quiver are in one-to-one correspondence with Galois coverings. Namely we prove that the inverse construction to the skew group construction has as many solutions as the number of different gradings on the starting quiver.

1 Introduction

In mathematics, a k-category or an Abelian category is a category in which morphisms and objects can be added. The motivation for k-categories originated from examining the category of Abelian groups, $\mathcal{A}b$. The theory rises from a tentative attempt to unify several cohomology theories by A. Grothendieck.

In this paper we examine skew categories and their connection to skew group algebras. The paper has two parts. In the first part we recall the basic notions and results of this topic, for this we use as a basis literature [1] and [2]. We present the categorical machinery developed by the above mentioned authors among some reformulations of several coherence results, to fit better in our context. In the second part we give a new proof of the famous result of E. L. Green [3], that gradings of a finite, path connected quiver are in one-to-one correspondence with Galois coverings. More precisely we prove that the inverse construction to the skew group construction has as many solutions as the number of different gradings on the starting quiver.

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1.1 Basic notions

From now on we deal with small categories over a commutative field k, this means that the objects C_0 form a set and not a class, and the morphisms consist of modules over k. We set G to be an arbitrary group. We define the notions of group action and grading on k-categories by [2].

Definition 1 *A* G*-**category** is a category* C *with the propriety that* G *acts on the set of objects, in other words the elements of* G *are* k*-module morphisms, such that the following hold: for all* $s \in G$*, and for all* $x, y \in C_0$ *for* $s\colon {}_yC_x \to_{sy} C_{sx}$*, we have that*

- $s(gf) = (sg)(sf)$*, if* f *and* g *can be composed in* C*;*
- *If* $t, s \in G$ *and* f *is a morphism in* C*, then* $(ts)f = t(sf)$*;*
- $1f = f$*,* $1 \in G$ *the identity element.*

In other words G is a group of autofunctors of C.

Since we defined a G-action on our categories it makes sense to talk about graded categories. These gradings will play a crucial role in the inverse construction we deal with in the second section.

Definition 2 *A* G*-**graded category** is a category* C *for which the following hold:*

- *For all* $x, y \in C_0$ *we have* ${}_yC_x = \bigoplus_{s\in G}({}_yC_x^s)$ *and*
$$ {}_zC_y^t \; {}_yC_x^s \subset_y C_x^{ts} $$
- ${}_x1_x \in_x Cx^1$.

For the definition of Galois covering of categories we must define first what a quotient category is. For this we cite [2], Definition 2.1.

Definition 3 *If* C *is a free* G*-category over* k*, then the objects of the **quotient category*** C/G *are the* G*-orbits of* C_0 *and if* α, β *are two* G*-orbits, then the morphisms between them are:*

$$ {}_\beta(C/G)_\alpha = \left(\bigoplus_{x\in\alpha,\ y\in\beta} ({}_yC_x) \right) /G. $$

Now let $p : C \to C/G$ *be the projection functor, then we call* p *the **Galois covering** of* C *with Galois group* G*.*

Similarly to the above construction we are interested in the categorical definition of skew group algebras, namely skew categories ([2], Definition 2.3).

Definition 4 *Let* C *be a* G*-category. Then the objects of the* ***skew category*** $C[G]$ *are the objects of the category* C*, so we have* $(C[G])_0 = C_0$ *and the morphisms between them are*

$$_y(C[G])_x = \bigoplus_{s \in G} ({}_yC_{sx}).$$

It is natural to ask if the above construction gives back the classical notion of skew group algebras. For this we cite a coherence result with the usual skew algebra construction ([2], Proposition 2.4).

Proposition 1 *Let* G *be a finite group and let* C *be* G*-category over* k*, with finite number of objects. Let* $a(C)$ *be the* k*-algebra associated to* C*, namely*

$$a(C) = \bigoplus_{x,y \in C_0} {}_yC_x,$$

provided with the matrix product induced by the composition of morphisms. Then we have that

$$a(C[G]) \cong a(C)[G].$$

Now let us see how Galois coverings and skew group algebras are related. The following theorem is very important for the theory ([2], Theorem 2.8) and from now on we will use it implicitly without referring to it.

Proposition 2 *Let* C *be a free* G*-category over* k*. The quotient category* C/G *and the skew category* $C[G]$ *are equivalent.*

The main goal of this part is to present the necessary tools for developing the inverse construction to taking the quotient of a category, this will be the smash product category ([2], Definition 3.1).

Definition 5 *Let* G *be a group and let* C *be a* G*-graded category over* k*. Then the* ***smash product category*** $C\#G$ *has object set* $C_0 \times G$*. Let* $(x,s),(y,t) \in C_0 \times G$ *be two objects. The* k*-module morphisms are defined as follows:*

$$_{(y,t)}(C\#G)_{(x,s)} = {}_yC_x^{t^{-1}s}.$$

It is natural to ask again if this construction gives back the classical notion of the smash product of algebras. For this we present a coherence result in a form to serve better our further goals.

Proposition 3 *Let* G *be a finite group and let* C *be a* G*-category, then the* k*-algebras* $a(C)\#G$ *and* $a(C\#G)$ *are Morita equivalent.*

Since this is not the classical statement regarding smash product categories we present a proof for it.

Proof. If C is a G-category, then every morphism space ${}_yC_x$ is a G-module, but G being finite one can also regard these spaces as $(kG)^*$ modules. In this setting C can be thought as a $(kG)^*$-module category.

Now by Theorem 2.9 of [1] we have that the k-categories $C\#G$ and $C\#(kG)^*$ are Morita equivalent. Moreover we can derive from this that $a(C\#G)$ and $a(C\#(kG)^*)$ are Morita equivalent as k-algebras.

Combining this with Proposition 2.3 from [1], which claims that the k-algebras $a(C)\#(kG)^*$ and $a(C\#(kG)^*)$ are isomorphic, we get that $a(C)\#G$ and $a(C\#G)$ are Morita equivalent. □

A last definition before we reach to the main duality theorems of this section, is of a matrix category ([1], Definition 4.1).

Definition 6 *Let* C *be a* k*-category and let* n *be a sequence of positive integers* $(n_x)_{x\in C_0}$*. The object set of the* ***matrix category*** $M_n(C)$ *remains the same objects of* C*. The set of morphisms from* x *to* y *is the vector space of* n_x *columns and* n_y *rows rectangular matrices with entries in* ${}_yC_x$*. Composition of morphisms is given by the matrix product combined with the composition in* C*.*

A classical way of relating the matrix categories to the corresponding matrix algebras is to consider single object categories provided by an algebra A and then proving that the matrix category has one object with endomorphism algebra precisely the usual algebra of matrices $M_n(A)$. Unfortunately this approach is not sufficient for our further goal, so we need to develop a different correspondence between these categorical and ring theoretical objects. For this we have the following lemma.

Lemma 1 *Let* C *be a* k*-category and let* n *be a positive integer, then we have the following* k*-algebra isomorphism*

$$a\left(M_n(C)\right) \cong M_n(kC),$$

where in the right hand side kC *is regarded as the path algebra of the underlying quiver of C.*

Proof. Let us examine carefully the construction of the morphism spaces of the matrix category, we have that

$$\mathfrak{a}\,(M_n(C)) = \bigoplus_{x,y\in M_n(C)_0} {}_y(M_n(C))_x = \bigoplus_{x,y\in C_0} M_n({}_yC_x) =$$

$$= M_n\left(\bigoplus_{x,y\in C_0} {}_yC_x\right) = M_n(kC).$$

Here we consider the vertices as the identity morphisms on the corresponding object, hence the set of vertices is a subset of the set of all morphisms. In this respect we can consider $\mathfrak{a}(C)$ isomorphic to the path algebra kC. □

Going back for a moment to the matrix categories we want to recall the following equivalence ([1], Corollary 4.5).

Proposition 4 *Let* C *be a* k*-category and* n *a positive integer, then* C *and* $M_n(C)$ *are Morita equivalent.*

Now the last statement of this section is the categorical version of the Cohen-Mongomery duality ([2], Proposition 3.2).

Theorem 1 *Let* C *be a* G*-graded category over* k*. Then the category* $(C\#G)[G]$ *is equivalent to* C*.*

2 The inverse construction

Now that we presented the categorical machinery developed for skew categories and smash products, we pass to the main theorem of this paper, namely the inverse construction to the skew group construction. From now on we consider finite, path connected quivers as categories over k: the objects are the vertices of the quiver and morphisms between two vertices are free k-modules having a basis given by the paths between these vertices.

Theorem 2 *Let* C *be a finite, path connected quiver, and let* G *be a group acting on it. Given a* G*-grading on* C*, we have that the skew group algebra* $(kC_G)[G]$ *and the path algebra* kC *are Morita equivalent, where* C_G *is the quiver corresponding to* $C\#G$*.*

Proof. We are considering C as a k-category, then by the Cohen-Mongomery duality (Theorem 1) we have the following equivalence of categories

$$(C\#G)[G] \cong C.$$

Translating this to the language of k-algebras, via the functor $\mathfrak{a}$, we get that

$$\mathfrak{a}((C\#G)[G]) \cong \mathfrak{a}(C),$$

as k-algebras. Now by the remark in the proof of Lemma 1 we can consider $\mathfrak{a}(C)$ to be the path algebra kC.

Applying the coherence property of the skew group construction (Proposition 1), we get the following isomorphism of algebras

$$\mathfrak{a}(C\#G)[G] \cong kC.$$

From this point, by applying the coherence result of the smash product (Proposition 3), we pass to Morita equivalences. So we get that $(\mathfrak{a}(C)\#G)[G]$ is Morita equivalent to kC, where $\mathfrak{a}(C)\#G$ is a smash product of algebras.

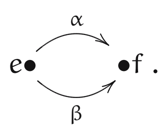
Finally applying again the remark from Lemma 1, we get that $(k(C\#G))[G]$ is Morita equivalent to kC, here $k(C\#G)$ is viewed as the path algebra of the quiver corresponding to $C\#G$.

Now putting everything in our notation we get the expected result, that the skew group algebra $(kC_G)[G]$ and the path algebra kC are Morita equivalent, where C_G is the quiver corresponding to $C\#G$. □

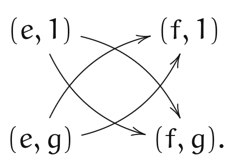
One can see from the above result that each different grading of C will lead to a different solution $C\#G$ to the inverse construction problem.

Finally we give an example to illustrate our result.

Example 1 *Let* $G = \langle g \rangle$ *be a cyclic group of order two and let* C *be the following quiver*



We can consider C *to be a* G*-graded quiver by setting degree* 1 *for the elements* $\{e, f\}$ *and degree* g *for the elements* $\{\alpha, \beta\}$*. In this case the quiver* C_G*, corresponding to the smash product of* C *and* G *is the following*



So we get that the skew group algebra $(kC_G)[G]$ *is Morita equivalent to the path algebra of* C*.*

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