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Some inequalities in bicentric quadrilateral

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Dedicated to the memory of Professor Antal Bege

Abstract. In this paper we prove some results concerning bicentric quadrilaterals. We offer a new proof of the Blundon-Eddy inequality, which we use to obtain other inequalities in bicentric quadrilaterals.

1 Introduction

Let $ABCD$ be a bicentric quadrilateral with $a = AB, b = BC, c = CD, d = AD$, $d_1 = AC$, $d_2 = BD$, $s = \frac{a+b+c+d}{2}$, R the radius of the circumscribed circle of the quadrilateral $ABCD$ and r the radius of the inscribed circle, F the area.

In [1] W. J. Blundon and R. H. Eddy proved that:

$$8r\left(\sqrt{4R^2+r^2}-r\right) \leq s^2 \leq \left(r+\sqrt{4R+r}\right)^2.$$

In the following we give a simple proof to this double inequality using the product

$$(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2,$$

then we deduce many other important new inequalities. We mention that the result concerning the above product is new.

We denote:

$$\sigma_1 = \sum a, \sigma_2 = \sum ab, \sigma_3 = \sum abc, x_1 = bc+ad, x_2 = ab+cd, x_3 = ac+bd.$$

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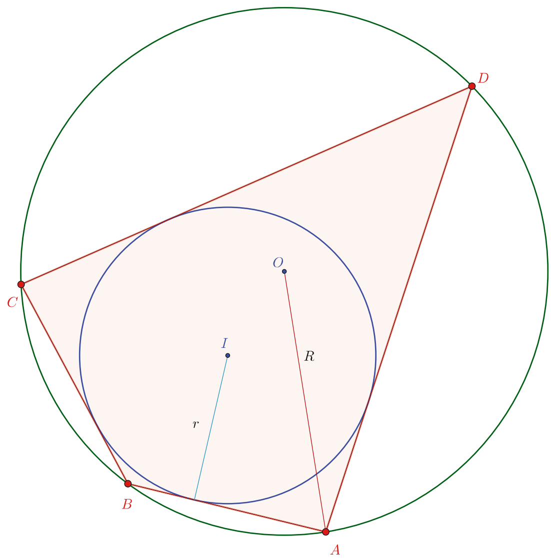

2 Main results

Lemma 1 *In every bicentric quadrilateral* $ABCD$ *the following equalities are true:*

1) $F^2 = (s-a)(s-b)(s-c)(s-d) = abcd$;
2) $x_1x_2x_3 = 16R^2r^2s^2$;
3) $x_1 + x_2 = s^2$;
4) $x_1 + x_2 + x_3 = s^2 + 2r^2 + 2r\sqrt{r^2+4R^2}$;
5) $x_3 = 2r\left(r + \sqrt{4R^2+r^2}\right)$;
6) $(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2 = (x_1-x_2)^2(x_2-x_3)^2(x_3-x_1)^2$.

Proof.

1) We have $a + c = b + d$. It results that $s - b = d$ and three similar equalities which imply

$$(s-a)(s-b)(s-c)(s-d) = abcd.$$

2) From Ptolemy's theorem it results that $x_3 = d_1d_2$. We have the equalities:

$$ad\sin A + bc\sin C = 2F,\ \ ab\sin B + dc\sin D = 2F.$$

We obtain $(ad+bc)\,d_1 = 4RF$, $(ab+dc)\,d_2 = 4RF$ which implies

$$(ad+bc)(ab+dc)\,d_1 d_2 = 16R^2F^2 \text{ or } x_1x_2x_3 = 16R^2r^2s^2. \qquad (1)$$

3) We have $x_1 + x_2 = ad + bc + ab + cd = (a+c)(d+b) = (a+c)^2 = \left(\frac{a+b+c+d}{2}\right) = s^2$.

4) From (1) it results that

$$\begin{aligned} &(ab+bc)(ad+dc)(ac+bd) = 16R^2F^2 \quad \text{or} \\ &abcd\sum a^2 + \sigma_3^2 - 2abcd\sigma_2 = 16R^2F^2 \quad \text{or} \\ &\sigma_3^2 - 4s^2r^2\sigma_2 + 4s^4r^2 = 16R^2r^2s^2v. \end{aligned} \qquad (2)$$

But $(s-a)(s-b)(s-c)(s-d) = s^2r^2$ or $-s^3 + \sigma_2 s - \sigma_3 = 0$ which implies

$$\sigma_3 = s\left(\sigma_2 - s^2\right). \qquad (3)$$

From (2) and (3) we have:

$$\begin{aligned} &s^2\left(\sigma_2 - s^2\right)^2 - 4s^2r^2\sigma_2 + 4s^4r^2 = 16R^2r^2s^2 \quad \text{or} \\ &\sigma_2^2 - \left(2s^2 + 4r^2\right)\sigma_2 + s^4 + 2s^2r^2 - 16r^2R^2 = 0. \end{aligned}$$

It results that: $\sigma_2 = s^2 + 2r^2 + 2r\sqrt{r^2+4R^2}$. But $\sigma_2 = x_1 + x_2 + x_3$, so it follows that

$$x_1 + x_2 + x_3 = s^2 + 2r^2 + 2r\sqrt{r^2+4R^2}. \qquad (4)$$

5) From 4) since $x_1 + x_2 = s^2$ it follows that $x_3 = 2r^2 + 2r\sqrt{4R^2+r^2}$.

6) We have $(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2 =$
$[(a-b)(c-d)]^2\,[(a-c)(b-d)]^2\,[(a-d)(b-c)]^2 =$
$(x_1-x_2)^2(x_2-x_3)^2(x_2-x_1)^2$.

□

Theorem 1 *In every bicentric quadrilateral* ABCD *the following equality is true:*

$$\begin{aligned} &(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2 \\ &= 16r^4s^2\left[s^2 - 8r\left(\sqrt{4R^2+r^2} - r\right)\right]\left[s^2 - \left(r + \sqrt{4R^2+r^2}\right)^2\right]^2. \end{aligned}$$

Proof. We denote $\triangle = (a-b)^2 (a-c)^2 (a-d)^2 (b-c)^2 (b-d)^2 (c-d)^2$.
From Lemma 1 6) we have:

$$\begin{aligned}\triangle &= (x_1 - x_2)^2 (x_3 - x_1)^2 (x_3 - x_2)^2 \\ &= \left[(x_1 + x_2)^2 - 4x_1x_2\right] \left[x_3^2 - x_3 (x_1 + x_2) + x_1x_2\right]^2.\end{aligned} \tag{5}$$

From Lemma 1 2) and 5) it results that:

$$x_1x_2 = \frac{8R^2r^2s^2}{r\left(r + \sqrt{4R^2 + r^2}\right)} = 2r\left(\sqrt{4R^2 + r^2} - r\right)s^2. \tag{6}$$

From Lemma 1 3), 5) and equalities (5), (6) we obtain:

$$\begin{aligned}\triangle = &\left[s^4 - 8r\left(\sqrt{4R^2 + r^2} - r\right)s^2\right]\left[4r^2\left(r + \sqrt{4R^2 + r^2}\right)^2\right. \\ &\left. - 2s^2r\left(r + \sqrt{4R^2 + r^2}\right) + 2r\left(\sqrt{4R^2 + r^2} - r\right)s^2\right]^2 \\ = &\, s^2\left[s^2 - 8r\left(\sqrt{4R^2 + r^2} - r\right)\right]\left[4r^2\left(r + \sqrt{4R^2 + r^2}\right)^2 - 4r^2s^2\right]^2 \\ = &\, 16r^4s^2\left[s^2 - 8r\left(\sqrt{4R^2 + r^2} - r\right)\right]\left[s^2 - \left(r + \sqrt{4R^2 + r^2}\right)^2\right].\end{aligned}$$

□

Theorem 2 *In every bicentric quadrilateral* ABCD *the following double inequality is true:* $8r\left(\sqrt{4R^2 + r^2} - r\right) \le s^2 \le \left(r + \sqrt{4R^2 + r^2}\right)^2$. *The equality holds in the case of two bicentric quadrilaterals* $A_1B_1C_1D_1$ *and* $A_2B_2C_2D_2$ *with the sides*

$$\begin{aligned}a_1 &= c_1 = \sqrt{2r\sqrt{4R^2 + r^2} - 2r^2} \\ b_1 &= \sqrt{2r\sqrt{4R^2 + r^2} - 2r^2} - \sqrt{2r\sqrt{4R^2 + r^2} - 6r^2} \\ d_1 &= \sqrt{2r\sqrt{4R^2 + r^2} - 2r^2} + \sqrt{2r\sqrt{4R^2 + r^2} - 6r^2} \\ a_2 &= d_2 = \frac{r + \sqrt{r^2 + 4R^2} - \sqrt{4R^2 - 2r^2 - 2r\sqrt{4R^2 + r^2}}}{2} \\ b_2 &= c_2 = \frac{r + \sqrt{r^2 + 4R^2} + \sqrt{4R^2 - 2r^2 - 2r\sqrt{4R^2 + r^2}}}{2}.\end{aligned}$$

Proof. We have $(x_3 - x_1)(x_3 - x_2) = (a-b)(b-c)(c-d)(d-a)$ and because $a+c = b+d$ it results that $(a-b)(b-c)(c-d)(d-a) = (a-b)^2$ $(b-c)^2 \geq 0$, which implies $(x_3 - x_1)(x_3 - x_2) \geq 0$ or

$$s^2 \leq \left(r + \sqrt{4R^2 + r^2}\right)^2.$$

But, from Theorem 1 since $\triangle \geq 0$, it results that

$$8r\left(\sqrt{4R^2 + r^2} - r\right) \leq s^2.$$

It remain to study the equality cases for $s_1 \leq s \leq s_2$ where

$$s_1 = \sqrt{8r\left(\sqrt{4R^2 + r^2} - r\right)},\ s_2 = r + \sqrt{4R^2 + r^2}.$$

From Theorem 1 it results that we may have the cases:

Case 1.

$$a = c.$$

We denote $a = x$. Then

$$a = x, b = y, c = x, d = 2x - y.$$

From Lemma 1 we have:

$$x_3 = 2r\left(r + \sqrt{4R^2 + r^2}\right) \text{ or } x^2 + y(2x - y) = 2r\left(r + \sqrt{4R^2 + r^2}\right).$$

But $F^2 = abcd$ or $(2x - y)y = 4r^2$. It results that $x^2 = 2r\sqrt{4R^2 + r^2} - 2r^2$. Since $s_1^2 = 4x^2 = 8r\left(\sqrt{4R^2 + r^2} - r\right)$ represents the left side of the inequality from the statement, so:

$$x = \sqrt{2r\sqrt{4R^2 + r^2} - 2r^2}$$

$$(y - x)^2 = 2r\sqrt{4R^2 + r^2} - 6r^2 \text{ or } |y - x| = \sqrt{2r\sqrt{4R^2 + r^2} - 6r^2}.$$

We denote $u_1 = 2r\sqrt{4R^2 + r^2} - 2r^2, u_2 = 2r\sqrt{4R^2 + r^2} - 6r^2$.
If $x \leq y$ we have

$$a = x = \sqrt{u_1},\ b = y = \sqrt{u_1} + \sqrt{u_2},\ c = \sqrt{u_1},\ d = 2x - y = \sqrt{u_1} - \sqrt{u_2}.$$

If $x > y$ we have

$$a = x = \sqrt{u_1},\ b = y = x - \sqrt{u_2} = \sqrt{u_1} - \sqrt{u_2},\ c = \sqrt{u_1},\ d = 2x - y = \sqrt{u_1} + \sqrt{u_2}.$$

It results that the equality from the left side of the inequality of the statement holds in the case of bicentric quadrilateral $A_1B_1C_1D_1$ with the sides

$$\sqrt{u_1}, \sqrt{u_1} - \sqrt{u_2}, \sqrt{u_1}, \sqrt{u_1} + \sqrt{u_2}.$$

Case 2.

$$a = d = x, b = c = y.$$

In this case $m(\measuredangle D) = m(\measuredangle B) = 90°$, $AC = 2R$. It results that $F = sr = 2\frac{xy}{2}$ or $xy = (x+y)r$.

We denote $\alpha = x + y, \beta = xy$.

We have $\beta = \alpha r$. But $x^2 + y^2 = 4R^2$ which implies $\alpha^2 - 2\beta = 4R^2$ so we have $\alpha^2 - 2\alpha r - 4R^2 = 0$.

It results that $\alpha = r + \sqrt{r^2 + 4R^2}$.

But $s_1 = x + y = \alpha = r + \sqrt{r^2 + 4R^2}$ which represents the right side of the inequality from the statement. We have $\begin{cases} x + y = \alpha \\ xy = r\alpha \end{cases}$, so x, y are the solutions of the equation $u^2 - \alpha u + r\alpha = 0$ which implies:

$$x = \frac{\alpha - \sqrt{\alpha^2 - 4r\alpha}}{2} = \frac{r + \sqrt{r^2 + 4R^2} - \sqrt{4R^2 - 2r^2 - 2r\sqrt{4R^2 + r^2}}}{2},$$

$$y = \frac{r + \sqrt{r^2 + 4R^2} + \sqrt{4R^2 - 2r^2 - 2r\sqrt{4R^2 + r^2}}}{2}.$$

So, the equality for the right side of the inequality from the statement is true in the case of bicentric quadrilateral $A_2B_2C_2D_2$ with the sides

$$a_2 = x, b_2 = x, c_2 = y, d_2 = y.$$

□

Theorem 3 *In every bicentric quadrilateral* $ABCD$ *the following inequalities are true:*

$$2r\left(r + \sqrt{4R^2 + r^2}\right) \le \min\{ab + cd, bc + ad\} \le 4r\left(\sqrt{4R^2 + r^2} - r\right)$$
$$\le \max\{ab + cd + bc + ad\} \le 4R^2.$$

Proof. We suppose that $x_1 \le x_2$, $x_1 + x_2 = s^2$, $x_1 x_2 = \alpha s^2$ where

$$\alpha = \frac{8R^2 r}{\sqrt{4R^2 + r^2} + r} = 2r\left(\sqrt{4R^2 + r^2} - r\right).$$

It results that: $x_1 = \frac{s^2 - \sqrt{s^4 - 4\alpha s^2}}{2}$, $x_2 = \frac{s^2 + \sqrt{s^4 - 4\alpha s^2}}{2}$. We consider the functions $f, g : (0, +\infty) \to R$.

$$f(s) = \frac{s^2 - \sqrt{s^4 - 4\alpha s^2}}{2}, g(s) = \frac{s^2 + \sqrt{s^4 - 4\alpha s^2}}{2}.$$

After differentiation we obtain:

$$f'(s) = \frac{s\left(\sqrt{s^4 - 4\alpha s^2} - s^2 + 2\alpha\right)}{\sqrt{s^4 - 4\alpha s^2}} \le 0, \ \ g'(s) = \frac{s\left(\sqrt{s^4 - 4\alpha s^2} + s^2 - 4\alpha\right)}{\sqrt{s^4 - 4\alpha s^2}} \ge 0.$$

From Theorem 2 it results that: $s^2 \ge 8r\left(\sqrt{4R^2 + r^2} - r\right) = 4\alpha$.

It results that f is a decreasing and g is an increasing function. Because $s \le r + \sqrt{4R^2 + r^2}$ we have $f\left(r + \sqrt{4R^2 + r^2}\right) \le f(s) = x_1$. If follows that

$$\begin{aligned}
x_1 &\ge \frac{1}{2}\Bigg[\left(r + \sqrt{4R^2 + r^2}\right)^2 \\
&\quad - \left(r + \sqrt{4R^2 + r^2}\right)\sqrt{\left(r + \sqrt{4R^2 + r^2}\right)^2 - 8r\left(\sqrt{4R^2 + r^2} - r\right)}\Bigg] \\
&= \frac{\left(r + \sqrt{4R^2 + r^2}\right)}{2}\Bigg[r + \sqrt{4R^2 + r^2} \\
&\quad - \sqrt{r^2 + 4R^2 + r^2 + 2r\sqrt{4R^2 + r^2} - 8r\sqrt{4R^2 + r^2} + 8r^2}\Bigg] \\
&= \frac{\left(r + \sqrt{4R^2 + r^2}\right)}{2}\left[r + \sqrt{4R^2 + r^2} - \sqrt{\left(\sqrt{4R^2 + r^2}\right)^2 + 9r^2 - 6r\sqrt{4R^2 + r^2}}\right] \\
&= 2r\left(r + \sqrt{4R^2 + r^2}\right).
\end{aligned}$$

It follows that

$$x_1 \ge 2r\left(r + \sqrt{4R^2 + r^2}\right). \tag{7}$$

From $s \leq r + \sqrt{4R^2 + r^2}$ it results also that

$$\begin{aligned} x_2 &= g(s) \leq g\left(r + \sqrt{4R^2 + r^2}\right) \\ &= \frac{1}{2}\Bigg[\left(r + \sqrt{4R^2 + r^2}\right)^2 \\ &\quad + \left(r + \sqrt{4R^2 + r^2}\right)\sqrt{\left(r + \sqrt{4R^2 + r^2}\right)^2 - 8r\left(\sqrt{4R^2 + r^2} - r\right)}\Bigg] \\ &= \left(\sqrt{4R^2 + r^2} + r\right)\left(\sqrt{4R^2 + r^2} - r\right) = 4R^2. \end{aligned}$$

Thus we get the following inequality

$$x_2 \leq 4R^2. \tag{8}$$

Since $8r\left(\sqrt{4R^2 + r^2} - r\right) \leq s^2$ we have $x_1 = f(s) \leq f\left(\sqrt{8r\left(\sqrt{4R^2 + r^2} - r\right)}\right)$ or in an equivalent form

$$\begin{aligned} x_1 &\leq \frac{1}{2}\Bigg[8r\left(\sqrt{4R^2 + r^2} - r\right) \\ &\quad - \sqrt{8r\left(\sqrt{4R^2 + r^2} - r\right)}\sqrt{8r\left(\sqrt{4R^2 + r^2} - r\right) - 8r\left(\sqrt{4R^2 + r^2} - r\right)}\Bigg] \\ &= 4r\left(\sqrt{4R^2 + r^2} - r\right). \end{aligned}$$

It follows that

$$x_1 \leq 4r\left(\sqrt{4R^2 + r^2} - r\right). \tag{9}$$

Because $8r\left(\sqrt{4R^2 + r^2} - r\right) \leq s^2$ and g is an increasing function it follows that:

$$g\left(\sqrt{8r\left(\sqrt{4R^2 + r^2} - r\right)}\right) \leq g(s) = x_2 \text{ or } x_2 \geq 4r\left(\sqrt{4R^2 + r^2} - r\right). \tag{10}$$

From (7) (8) (9) and (10) it results that:

$$x_3 = 2r\left(r + \sqrt{4R^2 + r^2}\right) \leq x_1 \leq 4r\left(\sqrt{4R^2 + r^2} - r\right) \leq x_2 \leq 4R^2.$$

□

Remark 1 *From Theorem 3 it results that* $2r\left(r+\sqrt{4R^2+r^2}\right) \leq 4r\left(\sqrt{4R^2+r^2}-r\right)$ *which, after performing some calculation, represent the well-known Fejes inequality* $R \geq \sqrt{2}r$.

Theorem 4 *In every bicentric quadrilateral* $ABCD$ *the following inequalities are true:*

$$\frac{r\left(\sqrt{4R^2+r^2}+r\right)}{R} \leq \min\{d_1, d_2\} \leq \frac{\sqrt{4R^2+r^2}+r}{R}\sqrt{\frac{\left(\sqrt{4R^2+r^2}-r\right)r}{2}}$$
$$\leq \max\{d_1, d_2\} \leq 2R.$$

Proof. We suppose that $x_1 \leq x_2$.

From Ptolemy's theorem it results that $\frac{x_1}{x_2} = \frac{d_1}{d_2}$ which implies $d_1 \leq d_2$.

Because $d_1 d_2 = x_3$ we have

$$d_1^2 = \frac{x_1}{x_2}x_3 = \frac{s^2-\sqrt{s^4-4\alpha s^2}}{s^2+\sqrt{s^4-4\alpha s^2}}x_3 = x_3\frac{\left(s^2-\sqrt{s^4-4\alpha s^2}\right)^2}{4\alpha s^2}$$
$$= x_3\frac{2s^4-4\alpha s^2-2s^2\sqrt{s^4-4\alpha s^2}}{4\alpha s^2} = \frac{x_3\left(s^2-2\alpha-\sqrt{s^4-4\alpha s^2}\right)}{2\alpha}$$
$$= \frac{2r\left(r+\sqrt{4R^2+r^2}\right)}{4r\left(\sqrt{4R^2+r^2}-r\right)}\left[s^2-\sqrt{s^4-4\alpha s^2}-2\alpha\right]$$
$$= \frac{\left(\sqrt{4R^2+r^2}+r\right)^2}{8R^2}\left[s^2-\sqrt{s^4-4\alpha s^2}-2\alpha\right] = B\left(2x_1-2\alpha\right),$$

where we denote $B = \frac{\left(\sqrt{4R^2+r^2}+r\right)^2}{8R^2}$.

But from Theorem 3 we have

$$4r\left(r+\sqrt{4R^2+r^2}\right) \leq 2x_1 \leq 8r\left(\sqrt{4R^2+r^2}-r\right).$$

We obtain

$$4r\left(r+\sqrt{4R^2+r^2}\right)-2\alpha \leq 2x_1-2\alpha \leq 8r\left(\sqrt{4R^2+r^2}-r\right)-2\alpha \text{ or}$$
$$8r^2 \leq 2x_1-2\alpha \leq 4r\left(\sqrt{4R^2+r^2}-r\right) \text{ or}$$
$$8r^2B \leq B\left(2x_1-2\alpha\right) \leq 4r\left(\sqrt{4R^2+r^2}-r\right)B \text{ or}$$
$$\frac{8r^2\left(\sqrt{4R^2+r^2}+r\right)^2}{8R^2} \leq d_1^2 \leq \frac{4r\left(\sqrt{4R^2+r^2}-r\right)\left(\sqrt{4R^2+r^2}+r\right)^2}{8R^2}.$$

It results that:

$$\frac{r\left(\sqrt{4R^2+r^2}+r\right)}{r} < d_1 \le \frac{\sqrt{4R^2+r^2}+r}{R}\sqrt{\frac{\left(\sqrt{4R^2+r^2}-r\right)r}{2}}. \tag{11}$$

Also:

$$\begin{aligned} d_2^2 &= \frac{x_2}{x_1}x_3 = \frac{s^2+\sqrt{s^4-4\alpha s^2}}{s^2-\sqrt{s^4-4\alpha s^2}}x_3 = \frac{\left(s^2+\sqrt{s^4-4\alpha s^2}\right)^2}{4\alpha s^2}x_3 \\ &= \frac{x_3}{4\alpha s^2}\left(2s^4-4\alpha s^2+2s^2\sqrt{s^4-4\alpha s^2}\right) = \frac{x_3}{2\alpha}\left(s^2+\sqrt{s^4-4\alpha s^2}-2\alpha\right) \\ &= \frac{x_3}{2\alpha}(2x_2-2\alpha) = \frac{\left(\sqrt{4R^2+r^2}+r\right)^2(2x_2-2\alpha)}{8R^2}. \end{aligned}$$

But we have proved that $4r\left(\sqrt{4R^2+r^2}-r\right) \le x_2 \le 4R$. It results that:

$$\begin{aligned} &4r\left(\sqrt{4R^2+r^2}-r\right) \le 2x_2-2\alpha \le 2\left(4R^2+2r^2-2r\sqrt{4R^2+r^2}\right) \text{ or} \\ &4r\left(\sqrt{4R^2+r^2}-r\right)\left(\frac{\sqrt{4R^2+r^2}+r}{2\sqrt{2}R}\right)^2 \le d_2^2 \\ &\qquad \le 2\left(\sqrt{4R^2+r^2}-r\right)^2\frac{\left(\sqrt{4R^2+r^2}+r\right)^2}{8R^2} \text{ or} \\ &\frac{\sqrt{4R^2+r^2}+r}{R}\sqrt{\frac{\left(\sqrt{4R^2+r^2}-r\right)r}{2}} \le d_2 \le 2R. \end{aligned} \tag{12}$$

From (11) and (12) it results the inequalities from the statement. □

Theorem 5 *Let be* $\alpha, \beta \in R$ *so that* $s \le \alpha R + \beta r$ *is true in every bicentric quadrilateral* ABCD. *Then* $2R + \left(4-2\sqrt{2}\right)r \le \alpha R + \beta r$ *is true in every bicentric quadrilateral* ABCD.

Proof. We consider the case of the square with the sides $a=b=c=d=1$. We have $2 \le \alpha\frac{1}{\sqrt{2}} + \beta\frac{1}{2}$. It results that

$$4 \le \sqrt{2}\alpha + \beta. \tag{13}$$

If $a = b = 1, c = d = 0$ it results that $R = \frac{1}{2}, r = 0$.

It follows that

$$1 \le \frac{\alpha}{2} \text{ or } \alpha \ge 2. \tag{14}$$

We know that

$$R \ge \sqrt{2}r. \tag{15}$$

From (13), (14) and (15) it results that

$$\begin{aligned}(\alpha - 2) R + \left(\beta - 4 + 2\sqrt{2}\right) r &\ge (\alpha - 2) \sqrt{2}r + \left(\beta - 4 + 2\sqrt{2}\right) r \\ &= \left(\alpha\sqrt{2} + \beta - 4\right) r \ge 0,\end{aligned}$$

therefore

$$\alpha R + \beta r \ge 2R + \left(4 - 2\sqrt{2}\right) r.$$

□

Theorem 6 *In every bicentric quadrilateral the following inequality is true:*

$$s \le 2R + \left(4 - 2\sqrt{2}\right) r.$$

Proof. From the Theorem 1 we have $s \le r + \sqrt{4R^2 + r^2}$. We denote $x = \frac{R}{r}$.

We prove that

$$r + \sqrt{4R^2 + r^2} \le 2R + \left(4 - 2\sqrt{2}\right) r,$$

or in an equivalent form

$$1 + \sqrt{4x^2 + 1} \le 2x + 4 - 2\sqrt{2} \text{ or } \sqrt{4x^2 + 1} \le 2x + 3 - 2\sqrt{2} \text{ or}$$
$$1 \le 4\left(3 - 2\sqrt{2}\right) x + \left(3 - 2\sqrt{2}\right)^2 \text{ or } x \ge \frac{\left(-2 + 2\sqrt{2}\right)\left(4 - 2\sqrt{2}\right)}{4\left(3 - 2\sqrt{2}\right)}.$$

After performing some calculation it results that $x \ge \sqrt{2}$ which represents just the Fejes's inequality [2]. □

Theorem 7 *In every bicentric quadrilateral* $ABCD$ *the following inequalities are true:*

1) $4r\left(3\sqrt{4R^2 + r^2} - 5r\right) \le a^2 + b^2 + c^2 + d^2 \le 8R^2$;

2) $2r\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}\left(7\sqrt{4R^2+r^2}-9r\right)\leq\sum a^2b\leq 8R^2+2r^2$;

3) $2r\left(5\sqrt{4R^2+r^2}-3r\right)\leq\sum ab\leq 4\left(R^2+r^2+r\sqrt{4R^2+r^2}\right)$;

4) $32r^2\sqrt{4R^2+r^2}\left(\sqrt{4R^2+r^2}-r\right)\leq\sum a^2bc$

$\leq 4r\sqrt{4R^2+r^2}\left(r+\sqrt{4R^2+r^2}\right)^2$;

5) $\left(2r^2+2r\sqrt{4R^2+r^2}\right)\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}\leq\sum abc$

$\leq 2r\left(r+\sqrt{4R^2+r^2}\right)^2$.

Proof. We have $\sigma_2=s^2+\alpha$, $\sigma_3=s\alpha$ where $\alpha=2r^2+2r\sqrt{r^2+4R^2}$.

1) $\sum a^2=(2s)^2-2\sigma_2=4s^2-2\sigma_2=4s^2-2s^2-4r^2-4r\sqrt{4R^2+r^2}$.
It results that: $\sum a^2=2s^2-4r^2-4r\sqrt{4R^2+r^2}$.
From Theorem 2 we obtain

$$4r\left(3\sqrt{4R^2+r^2}-5r\right)\leq a^2+b^2+c^2+d^2\leq 8R^2.$$

2) $a^2b=ab\left(2s-b-c-d\right)=2sab-ab^2-abc-abd$ or $a^2b+ab^2=2sab-abc-abd$.
It results that $\sum a^2b=2s\sigma_2-3\sigma_3=2s^3-s\alpha=s\left(2s^2-\alpha\right)$ which implies $\sum a^2b=s\left(2s^2-\alpha\right)$. We consider the increasing function

$$f:(0,+\infty)\to R, f\left(s\right)=2s^3-s\alpha,\text{ with } f'\left(s\right)=6s^2-\alpha\geq 0\text{ as}$$
$$s^2\geq 8r\left(\sqrt{4R^2+r^2}-r\right)\geq\frac{\alpha}{6}=\frac{2r^2+2r\sqrt{r^2+4R^2}}{6}.$$

The last inequality may be written as:

$$24\sqrt{4R^2+r^2}-24r\geq r+\sqrt{4R^2+r^2}\text{ or }23\sqrt{4R^2+r^2}\geq 25r.$$

But from inequality of Fejes it results that

$$23\sqrt{4R^2+r^2}\geq 25\sqrt{9r^2}=75r>25r.$$

From Theorem 2 it results that:

$$\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)\left(16\left(\sqrt{4R^2+r^2}-r\right)-2r^2-2r\sqrt{4R^2+r^2}\right)}$$
$$\leq \sum a^2b \leq \left(r+\sqrt{4R^2+r^2}\right)$$
$$\left(2r^2+8R^2+2r^2+2r\sqrt{4R^2+r^2}-2r^2-2r\sqrt{4R^2+r^2}\right)$$

which is equivalent with the inequality from the statement.

3) $\sigma_2 = \sum ab = s^2+\alpha$ or $8r\left(\sqrt{4R^2+r^2}-r\right)+2r^2+2r\sqrt{4R^2+r^2} \leq \sum ab \leq r^2+4R^2+r^2+2r\sqrt{4R^2+r^2}+2r^2+2r\sqrt{4R^2+r^2}$

which is equivalent with the inequality from the statement.

4) $a^2bc = a\ abc = (2s-b-c-d)\,abc = 2sabc - ab^2c - abc^2 - abcd$ or $a^2bc + ab^2c + abc^2 = 2sabc - abcd$ or $\sum a^2bc = 2s\sigma_3 - 4abcd = 2s\ s\alpha - 4s^2r^2$ or $\sum a^2bc = s^2\left(2\alpha-4r^2\right) = s^2\left(4r^2+4r\sqrt{4R^2+r^2}-4r^2\right) = 4r\sqrt{4R^2+r^2}s^2$.

From Theorem 2 it results the inequality from the statement.

5) $\sum abc = s\alpha$.

According to Theorem 2 it results the inequality from the statement.

□

Theorem 8 *Let be* $\alpha, \beta, \gamma \in R$, $\beta \geq 4$ *so that* $s^2 \leq \alpha R^2 + \beta Rr + \gamma r^2$ *is true in all bicentric quadrilateral. Then*

$$4R^2 + 4Rr + \left(8-4\sqrt{2}\right)r^2 \leq \alpha R^2 + \beta Rr + \gamma r^2$$

is true in all bicentric quadrilateral.

Proof. We consider the case of the bicentric quadrilateral with $a = b = c = d = 1$. It results that $4 \leq \frac{\alpha}{2} + \frac{\beta}{2\sqrt{2}} + \frac{\gamma}{4}$ or $16 \leq 2\alpha + \sqrt{2}\beta + \gamma$.

In the case of $a = b = 1$, $c = d = 0$ it results that $R = \frac{1}{2}$, $r = 0$ and $\alpha \geq 4$. But from inequality $R \geq \sqrt{2}r$ we have:

$$\begin{aligned}
&(\alpha-4)\,R^2+(\beta-4)\,Rr+\left(\gamma-8+4\sqrt{2}\right)r^2\\
&\geq(\alpha-4)\,2r^2+\sqrt{2}\,(\beta-4)\,r^2+\left(\gamma-8+4\sqrt{2}\right)r^2\\
&\geq(\alpha-4)\,2r^2+\sqrt{2}\,(\beta-4)\,r^2+\left(\gamma-8+4\sqrt{2}\right)r^2\\
&=\left(2\alpha+\sqrt{2}\beta+\gamma-16\right)r^2\geq 0.
\end{aligned}$$

□

Theorem 9 *In every bicentric quadrilateral* ABCD *the following inequality is true:*

$$s^2\leq 4R^2+4Rr+\left(8-4\sqrt{2}\right)r^2.$$

Proof. Since $s^2\leq\left(r+\sqrt{4R^2+r^2}\right)^2$ it is sufficient to prove that:

$\left(\sqrt{4x^2+1}+1\right)^2\leq 4x^2+4x+8-4\sqrt{2}$ or

$4x^2+1+1+2\sqrt{4x^2+1}\leq 4x^2+4x+8-4\sqrt{2}$ or

$2\sqrt{4x^2+1}\leq 4x+6-4\sqrt{2}$ or

$\sqrt{4x^2+1}\leq 2x+3-2\sqrt{2}$ or $4x^2+1\leq 4x^2+\left(12-8\sqrt{2}\right)x+\left(3-2\sqrt{2}\right)^2$ or

$$x\geq\frac{\left(1-3+2\sqrt{2}\right)\left(1+3-2\sqrt{2}\right)}{4\left(3-2\sqrt{2}\right)}=\frac{\left(\sqrt{2}-1\right)\left(2-\sqrt{2}\right)}{3-2\sqrt{2}}=\sqrt{2}.$$

□

Theorem 10 *In every bicentric quadrilateral* ABCD *the following inequalities are true:*

1) $\sum abc\leq 8R^2r+8Rr^2+\left(16-8\sqrt{2}\right)r^3;$

2) $\sum ab\leq 4\left[R^2+2Rr+\left(4-2\sqrt{2}\right)r^2\right];$

3) $\sum a^2bc\leq 32R^3r+16Rr^3+\left(80-32\sqrt{2}\right)R^2r^2+\left(32-16\sqrt{2}\right)r^4.$

Proof.

1) We proved that $\sum abc \le 2r\left(r+\sqrt{4R^2+r^2}\right)^2$, and

$$\left(r+\sqrt{4R^2+r^2}\right)^2 \le 4R^2+4Rr+\left(8-4\sqrt{2}\right)r^2.$$

It results that

$$\sum abc \le 2r\left(4R^2+4Rr+\left(8-4\sqrt{2}\right)r^2\right).$$

2) Since $\sqrt{4R^2+r^2} \le 2R+\left(3-2\sqrt{2}\right)r$, from Theorem 7 3) it results that:

$$\begin{aligned}\sum ab &\le 4\left(R^2+r^2+r\sqrt{4R^2+r^2}\right)\\ &\le 4\left[R^2+r^2+r\left(2R+\left(3-2\sqrt{2}\right)r\right)\right]\\ &= 4\left[R^2+r^2+2Rr+\left(3-2\sqrt{2}\right)r^2\right] \quad \text{or}\\ \sum ab &\le 4\left[R^2+2Rr+\left(4-2\sqrt{2}\right)r^2\right].\end{aligned}$$

3) From Theorem 7 4) it results that:

$$\begin{aligned}\sum a^2bc &\le 4r\sqrt{4R^2+r^2}\left(r+\sqrt{4R^2+r^2}\right)^2\\ &= 4r\sqrt{4R^2+r^2}\left(r^2+4R^2+r^2+2r\sqrt{4R^2+r^2}\right)\\ &= 8r\sqrt{4R^2+r^2}\left(2R^2+r^2+r\sqrt{4R^2+r^2}\right)\\ &= \left(16R^2r+8r^3\right)\sqrt{4R^2+r^2}+8r^2\left(4R^2+r^2\right)\\ &\le \left(16R^2r+8r^3\right)\left[2R+\left(3-2\sqrt{2}\right)r\right]+32R^2r^2+8r^4\\ &= 32R^3r+\left(48-32\sqrt{2}\right)R^2r^2+16Rr^3+\left(24-16\sqrt{2}\right)r^4\\ &\quad +32R^2r^2+8r^4,\end{aligned}$$

which is equivalent with the inequality from the statement.

□

Theorem 11 *In every bicentric quadrilateral* ABCD *the following inequalities are true:*

1) $2r\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}\left(5\sqrt{4R^2+r^2}-11r\right)\le\sum a^3$

$\le 2\left(r+\sqrt{4R^2+r^2}\right)\left(4R^2-r^2-r\sqrt{4R^2+r^2}\right);$

2) $352R^2r^2+208r^4-240r^3\sqrt{4R^2+r^2}$

$\le\sum a^3b\le\left(r+\sqrt{4R^2+r^2}\right)^2\left(8R^2-4r^2\right).$

Proof.

1) $a^3=a^2(2s-b-c-d)=2a^2s-a^2b-a^2c-a^2d$ or $\sum a^3=2s\sum a^2-\sum a^2b=2s\left(2s^2-2\alpha\right)-2s^3+s\alpha.$

It results that $\sum a^3=2s^3-3\alpha s.$

We consider the function $f:(0,+\infty)\to R$, $f(s)=2s^3-3\alpha s$, with the derivate $f'(s)=6s^2-3\alpha$. We prove that $f'(s)\ge 0$ or $s^2\ge\frac{\alpha}{2}$.

But $s^2\ge 8r\left(\sqrt{4R^2+r^2}-r\right)$. It will be sufficient to prove that:

$$8r\left(\sqrt{4R^2+r^2}-r\right)\ge r^2+r\sqrt{4R^2+r^2}\text{ or}$$

$$8\sqrt{4x^2+1}-8\ge 1+\sqrt{4x^2+1}\text{ or }\sqrt{4x^2+1}\ge\frac{9}{7},$$

which is true because $\sqrt{4x^2+1}\ge 2$ according to Fejes inequality.

Since f is an increasing function it results from Theorem 2 that:

$$\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}\left[16\left(\sqrt{4R^2+r^2}-r\right)-6r^2-6r\sqrt{4R^2+r^2}\right]$$

$$\le\sum a^3\le\left(r+\sqrt{4R^2+r^2}\right)\left[2r^2+8R^2+2r^2\right.$$

$$\left.+4r\sqrt{4R^2+r^2}-6r^2-6r\sqrt{4R^2+r^2}\right],$$

which is equivalent with the inequality from the statement.

2) $a^3b=ab\left(\sum a^2-b^2-c^2-d^2\right)=ab\sum a^2-ab^3-abc^2-abd^2$ or $a^3b+ab^3=ab\sum a^2-abc^2-abd^2$ or $\sum a^3b=\sum ab\sum a^2-\sum a^2bc=\left(s^2+\alpha\right)\left(2s^2-2\alpha\right)-\left(2\alpha-4r^2\right)s^2$ or $\sum a^3b=2s^4-\left(2\alpha-4r^2\right)s^2-2\alpha^2.$

We denote $s^2=t$ and consider the function: $f:(0,+\infty)\to R$,

$$f(t)=2t^2-\left(2a-4r^2\right)t-2a^2$$

and

$$t_v = \frac{2a - 4r^2}{4} = \frac{a - 2r^2}{2} = r\sqrt{4R^2 + r^2}.$$

We prove that $t \geq t_v$.

$s^2 \geq r\sqrt{4R^2 + r^2}$. But $s^2 \geq 8r\left(\sqrt{4R^2 + r^2} - r\right)$. It will be sufficient to prove that

$$8r\left(\sqrt{4R^2 + r^2} - r^2\right) \geq r\sqrt{4R^2 + r^2} \text{ or } \sqrt{4R^2 + r^2} \geq \frac{8}{7}$$

which is true because $\sqrt{4R^2 + r^2} \geq 3$.

It results that f is an increasing function which implies:

$$128r^2\left(4R^2 + 2r^2 - 2r\sqrt{4R^2 + r^2}\right) - 4r\sqrt{4R^2 + r^2}8r\left(\sqrt{4R^2 + r^2} - r\right)$$
$$- 2\left(2r^2 + 2r\sqrt{4R^2 + r^2}\right)^2 \leq \sum a^3b \leq 2\left(r + \sqrt{4R^2 + r^2}\right)^4$$
$$- 4r\sqrt{4R^2 + r^2}\left(r + \sqrt{4R^2 + r^2}\right)^2 - 2\left(2r^2 + 2r\sqrt{4R^2 + r^2}\right)^2$$

or

$$512R^2r^2 + 256r^4 - 256r^3\sqrt{4R^2 + r^2} - 32r^2\left(4R^2 + r^2\right) + 32r^3\sqrt{4R^2 + r^2}$$
$$- 8r^4 - 8r^2\left(4R^2 + r^2\right) - 16r^3\sqrt{4R^2 + r^2} \leq \sum a^3b \leq 2\left(r + \sqrt{4R^2 + r^2}\right)^2$$
$$\left(r^2 + 4R^2 + r^2 + 2r\sqrt{4R^2 + r^2} - 2r\sqrt{4R^2 + r^2}\right) - 8r^2\left(r + \sqrt{4R^2 + r^2}\right)^2$$

or

$$352R^2r^2 + 208r^4 - 240r^3\sqrt{4R^2 + r^2} \leq \sum a^3b \leq \left(r + \sqrt{4R^2 + r^2}\right)^2$$
$$\left(4r^2 + 8R^2 - 8r^2\right).$$

□

Theorem 12 *In every bicentric quadrilateral* ABCD *the following inequalities are true:*

1) $\sum a^3 \leq 16R^3 + \left(24 - 16\sqrt{2}\right)R^2r - 8Rr^2 - \left(16 - 8\sqrt{2}\right)r^3$;

2) $\sum a^3b \leq 32R^4 - 16R^2r^2 + 32R^3r + 16Rr^3 + \left(64 - 32\sqrt{2}\right)R^2r^2 - \left(32 - 16\sqrt{2}\right)r^4$;

3) $\sum a^3b \geq 352R^2r^2 + \left(480\sqrt{2} - 512\right) r^4 - 480Rr^3.$

Proof.

1) From Theorem 11 it results that:

$$\begin{aligned}\sum a^3 &\leq \left(r + \sqrt{4R^2 + r^2}\right)\left(8R^2 - 2r^2 - 2r\sqrt{4R^2 + r^2}\right)\\ &= 8R^2r - 2r^3 - 2r^2\sqrt{4R^2 + r^2} + 8R^2\sqrt{4R^2 + r^2}\\ &\quad - 2r^2\sqrt{4R^2 + r^2} - 8R^2r - 2r^3\end{aligned}$$

$$\begin{aligned}&= \left(8R^2 - 4r^2\right)\sqrt{4R^2 + r^2} - 4r^3\\ &\leq \left(8R^2 - 4r^2\right)\left[2R + \left(3 - 2\sqrt{2}\right) r\right] - 4r^3\\ &= 16r^3 + \left(24 - 16\sqrt{2}\right) R^2r - 8Rr^2 - \left(12 - 8\sqrt{2}\right) r^3 - 4r^3,\end{aligned}$$

which is equivalent with inequality from the statement.

2) From Theorem 11 it results that

$$\sum a^3b \leq \left(r + \sqrt{4R^2 + r^2}\right)^2\left(8R^2 - 4r^2\right)$$

and

$$\left(r + \sqrt{4R^2 + r^2}\right)^2 \leq 4R^2 + 4Rr + \left(8 - 4\sqrt{2}\right) r^2.$$

It results that:

$$\begin{aligned}\sum a^3b &\leq \left[4R^2 + 4Rr + \left(8 - 4\sqrt{2}\right) r^2\right]\left(8R^2 - 4r^2\right)\\ &= 32R^4 - 16R^2r^2 + 32R^3r - 16Rr^3 + \left(64 - 32\sqrt{2}\right) R^2r^2\\ &\quad - (32 - 16\sqrt{2})r^4,\end{aligned}$$

which is equivalent with the inequality from the statement.

3) We prove that:

$$\begin{aligned}\sum a^3b &\geq 352R^2r^2 + 208r^4 - 240r^3\sqrt{4R^2 + r^2}\\ &\geq 352R^2r^2 + 208r^4 - 240r^3\left[2R + \left(3 - 2\sqrt{2}\right) r\right]\\ &= 352R^2r^2 + 208r^4 - 480Rr^3 - \left(720 - 480\sqrt{2}\right) r^4,\end{aligned}$$

which is equivalent with the inequality from the statement.

□

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