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Data dependence of solutions for Fredholm-Volterra integral equations in $L^2[a, b]$

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Dedicated to the memory of Professor Antal Bege

Abstract. In this paper we study the continuous dependence and the differentiability with respect to the parameter $\lambda \in [\lambda_1, \lambda_2]$ of the solution operator $S : [\lambda_1, \lambda_2] \to L^2[a, b]$ for a mixed Fredholm-Volterra type integral equation. The main tool is the fiber Picard operators theorem (see [9], [8], [11], [3] and [2]).

1 Introduction

We study the solution operator of the equations

$$y(x) = f(x) + \int_a^x K_1(x, s, y(s); \lambda)ds + \int_a^b K_2(x, s, y(s); \lambda)ds, \qquad (1)$$

and

$$y(x) = f(x) + \int_a^x K_1(x, s, y(g_1(s)); \lambda)ds + \int_a^b K_2(x, s, y(g_2(s)); \lambda)ds, \qquad (2)$$

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where $\lambda \in [\lambda_1, \lambda_2]$ is a real parameter. The existence and uniqueness of the solutions of such equations in $C[a,b]$ was studied by many authors [5], [6], [1], we recall the results from [1]. If the functions K_i and f satisfy the conditions under which the existence and uniqueness (in $C[a,b]$) is guaranteed then the differentiability of the functions K_i with respect to the parameter guarantees the differentiability of the solution. This property was proved in [1] using the following fiber Picard operator theorem:

Theorem 1 (Fiber Picard operator's) [9] *Let* (V, d) *be a generalized metric space with* $d(v_1, v_2) \in \mathbb{R}_+^p$, *and* (W, ρ) *a complete generalized metric space with* $\rho(w_1, w_2) \in \mathbb{R}_+^m$. *Let* $A : V \times W \to V \times W$ *be a continuous operator. If we suppose that:*

a) $A(v, w) = (B(v), C(v, w))$ *for all* $v \in V$ *and* $w \in W$;

b) *the operator* $B : V \to V$ *is a weakly Picard operator;*

c) *there exists a matrix* $Q \in M_m(\mathbb{R}_+)$ *convergent to zero, such that the operator* $C(v, \cdot) : W \to W$ *is a* Q *contraction for all* $v \in V$,

then the operator A *is a weakly Picard operator. Moreover, if* B *is a Picard operator, then the operator* A *is a Picard operator.*

In this paper we use the same technique to give some modified Carathéodory type conditions which guarantee the continuity and differentiability with respect to the parameter of the solution operator. We study these equations both in bounded and unbounded intervals.

2 Fredholm-Volterra equations on a compact interval

We need the following lemma.

Lemma 1 *If* $I = [a, b]$, $k \in L^2(I^2)$ *and the function* $u \in L^2(I)$ *has nonnegative values then the inequality*

$$u(t) \leq \alpha + \int_a^b k(t,s)u(s)ds, \ \text{a.e.}\ t \in I, \tag{3}$$

where $\alpha > 0$ *and* $\|k\|_{L^2(I^2)} < 1$, *implies*

$$\|u\|_{L^2(I)} \leq \frac{\alpha\sqrt{2(b-a)}}{1 - \|k\|_{L^2(I^2)}}.$$

Proof. Consider the sets

$$A = \{t \in I \,|\, u(t) \leq \alpha\} \text{ and } B = \{t \in I \,|\, u(t) > \alpha\}.$$

These sets are measurable because u is measurable. If $t \in B$, from the Cauchy-Buniakovski inequality we have

$$(u(t) - \alpha)^2 \leq \left(\int_a^b k(t,s)u(s)ds\right)^2 \leq \int_a^b k^2(t,s)ds \cdot \int_a^b u^2(s)ds.$$

By integrating on B we deduce

$$\begin{aligned}\int_B u^2(s)ds &\leq 2\alpha \textstyle\int_B u(t)dt - \alpha^2 \cdot \mu(B) + \int_B\int_a^b k^2(t,s)dsdt \cdot \|u\|^2_{L^2(I)} \\ &\leq 2\alpha \textstyle\int_B u(t)dt - \alpha^2 \cdot \mu(B) + \int_a^b\int_a^b k^2(t,s)dsdt \cdot \|u\|^2_{L^2(I)} \\ &\leq 2\alpha\sqrt{\mu(B)\textstyle\int_a^b u^2(t)dt} - \alpha^2 \cdot \mu(B) + \|k\|^2_{L^2(I^2)} \cdot \|u\|^2_{L^2(I)}.\end{aligned}$$

By the other hand $u^2(t) \leq \alpha^2$, for $t \in A$, so

$$\int_A u^2(t)dt \leq \alpha^2 \cdot \mu(A).$$

From these inequalities we have

$$\left(\|u\|_{L^2(I)} - \alpha\sqrt{\mu(B)}\right)^2 \leq \alpha^2\mu(A) + \|k\|^2_{L^2(I^2)} \cdot \|u\|^2_{L^2(I)},$$

so

$$\|u\|_{L^2(I)} - \alpha\sqrt{\mu(B)} \leq \sqrt{\alpha^2\mu(A) + \|k\|^2_{L^2(I^2)} \cdot \|u\|^2_{L^2(I)}}.$$

From

$$\sqrt{\alpha^2\mu(A) + \|k\|^2_{L^2(I^2)} \cdot \|u\|^2_{L^2(I)}} \leq \alpha\sqrt{\mu(A)} + \|k\|_{L^2(I^2)} \cdot \|u\|_{L^2(I)}$$

and

$$\sqrt{\mu(A)} + \sqrt{\mu(B)} \leq \sqrt{2(b-a)}$$

we deduce the desired inequality. □

Remark 1 *By using both the Minkovski and the Cauchy-Buniakovski inequality we can prove a sharpened version:*

$$\|u\|_{L^2(I)} \leq \frac{\alpha\sqrt{(b-a)}}{1 - \|k\|_{L^2(I^2)}}.$$

Indeed (3) *implies*

$$\|u\|_{L^2(I)} \leq \left\| \alpha + \sqrt{\int_a^b k^2(t,s)ds \cdot \int_a^b u^2(s)ds} \right\|_{L^2(I)} \leq \alpha\sqrt{b-a} + \|k\|_{L^2(I^2)} \cdot \|u\|_{L^2(I)}.$$

By an analogous reasoning we have the following property: If $k \in L^2(I^2)$, $g \in L^2(I)$ *and the function* $u \in L^2(I)$ *has nonnegative values then the inequality*

$$u(t) \leq g(t) + \int_a^b k(t,s)u(s)ds, \ \text{a.e.}\, t \in I,$$

where $\|k\|_{L^2(I^2)} < 1$, *implies*

$$\|u\|_{L^2(I)} \leq \frac{\|g\|_{L^2(I)}}{1 - \|k\|_{L^2(I^2)}}.$$

These inequalities are in fact Gronwall type inequalities and they can be proved also by using the abstract Gronwall lemma from [10].

We use the usual definition of differentiability for functions with values in a Banach space and a generalized Weierstrass type theorem. To avoid any misunderstanding we recall this definition and we prove the above mentioned theorem.

Definition 1 *If* $S : [\lambda_1, \lambda_2] \to L^2(I)$ *is a continuous function then we call it differentiable at the point* λ, *if exists* $z_\lambda \in L^2(I)$ *such that*

$$\lim_{\overline{\lambda} \to \lambda} \frac{\|S(\overline{\lambda}) - S(\lambda) - (\overline{\lambda} - \lambda)z_\lambda\|_{L^2(I)}}{\overline{\lambda} - \lambda} = 0.$$

For the simplicity we identify the function $t \to tz_\lambda$ *(the differential) with the element* z_λ.

Theorem 2 *If the sequence* $y_n(\cdot, \lambda) \in L^2(I)$, $n \geq 0$ *converges in* $L^2(I)$ *to* $y^*(\cdot, \lambda)$ *for all* $\lambda \in [\lambda_1, \lambda_2]$, *the operators* $S_n : [\lambda_1, \lambda_2] \to L^2(I)$ *defined by* $S_n(\lambda)(t) = y_n(t, \lambda)$, $\forall\, t \in I$, $\forall\, \lambda \in [\lambda_1, \lambda_2]$ *are differentiable, their differentials converge in* $L^2(I)$ *to* $z^*(\cdot, \lambda)$, *and these convergencies are uniform with respect to* λ, *then the operator* $S : [\lambda_1, \lambda_2] \to L^2(I)$ *defined by* $S(\lambda)(t) = y^*(t, \lambda)$, $\forall\, t \in I$, $\forall\, \lambda \in [\lambda_1, \lambda_2]$ *is differentiable and* $z^*(\cdot, \lambda)$ *is its differential in* λ.

Proof. Due to the mean theorem for functions with values in a Banach space (see [4] 2-5) we have the following inequality:

$$\frac{\|[y_m(\cdot,\overline{\lambda}) - y_n(\cdot,\overline{\lambda})] - [y_m(\cdot,\lambda) - y_n(\cdot,\lambda)]\|_{L^2(I)}}{\overline{\lambda}-\lambda}$$
$$\leq \sup_{\lambda\in[\lambda_1,\lambda_2]} \|z_m(\cdot,\lambda) - z_n(\cdot,\lambda)\|_{L^2(I)},$$

where $z_m(\cdot,\lambda)$ is the differential of $S_n(\lambda)(\cdot)$.

The condition $\|z_n(\cdot,\lambda)-z^*(\cdot,\lambda)\|_{L^2(I)} \to 0$ uniform with respect to λ, implies that for every $\varepsilon > 0$ exists $n_1(\varepsilon) \in \mathbb{N}$ such that

$$\frac{\|[y^*(\cdot,\overline{\lambda}) - y^*(\cdot,\lambda)] - [y_n(\cdot,\overline{\lambda}) - y_n(\cdot,\lambda)]\|_{L^2(I)}}{\overline{\lambda}-\lambda} \leq \frac{\varepsilon}{3},\ \forall n \geq n_1(\varepsilon). \tag{4}$$

By the other hand for all $\varepsilon > 0$ exists $n_2(\varepsilon) \in \mathbb{N}$ such that

$$\|z_n(\cdot,\lambda) - z^*(\cdot,\lambda)\|_{L^2(I)} \leq \frac{\varepsilon}{3},\ \forall n \geq n_2(\varepsilon) \tag{5}$$

and there exists $\delta > 0$ such that

$$\frac{\|y_n(\cdot,\overline{\lambda}) - y_n(\cdot,\lambda) - (\overline{\lambda}-\lambda)z_n(\cdot,\lambda)\|_{L^2(I)}}{\overline{\lambda}-\lambda} \leq \frac{\varepsilon}{3}, \tag{6}$$

if $|\overline{\lambda}-\lambda| < \delta$. From these relations we deduce

$$\lim_{\overline{\lambda}\to\lambda} \frac{\|y^*(\cdot,\overline{\lambda}) - y^*(\cdot,\lambda) - (\overline{\lambda}-\lambda)z^*(\cdot,\lambda)\|_{L^2(I)}}{\overline{\lambda}-\lambda} = 0,$$

so S is differentiable in λ and its differential is $z^*(\cdot,\lambda)$. □

For equation (1) we have the following theorem (some parts of this theorem are classical):

Theorem 3 *If*

I. (*Carathéodory type conditions*) *the functions* $K_i : I^2 \times [\lambda_1,\lambda_2] \times \mathbb{R} \to \mathbb{R}$, $i \in \{1,2\}$ *with* $I = [a,b]$ *satisfy the conditions*

   a) $K_i(\cdot,\cdot,\lambda,u)$ *is measurable on* $I^2 = [a,b] \times [a,b]$ *for all* $u \in \mathbb{R}$ *and* $\lambda \in [\lambda_1,\lambda_2]$;

   b) $K_i(x,s,\lambda,\cdot)$ *is continuous on* $\mathbb{R}$ *for almost every pairs* $(x,s) \in I^2$ *and every* $\lambda \in [\lambda_1,\lambda_2]$.

II. (*space invariance*) $f \in L^2(I)$, $K_i(\cdot,\cdot,\lambda,0) \in L^2(I^2)$ *for all* $\lambda \in [\lambda_1,\lambda_2]$, $i \in \{1,2\}$ *and exists* $M_1 > 0$ *such that* $\|K_i(\cdot,\cdot,\lambda,0)\|_{L^2(I^2)} < M_1$ *for all* $\lambda \in [\lambda_1,\lambda_2]$;

III. (*Lipschitz type conditions*) *exists* $k_i \in L^2(I^2)$, $i \in \{1,2\}$, *such that*

$$|K_i(t,s,\lambda,u) - K_i(t,s,\lambda,v)| \le k_i(t,s)|u-v|,\ \forall\, t,s \in I, \lambda \in [\lambda_1,\lambda_2], u,v \in \mathbb{R};$$

IV. (*contraction condition*)

$$L^2 := \int_a^b\int_a^t (k_1(t,s)+k_2(t,s))^2 dsdt + \int_a^b\int_t^b k_2^2(t,s)dsdt < 1 \tag{7}$$

then

1. *for all* $\lambda \in [\lambda_1,\lambda_2]$ *exists a unique solution* $y^*(\cdot,\lambda) \in L^2(I)$ *of the equation (1);*

2. *the sequence of successive approximation*

$$y_{n+1}(x) = f(x) + \int_a^x K_1(x,s,\lambda,y_n(s))ds + \int_a^b K_2(x,s,\lambda,y_n(s))ds$$

   converges in $L^2(I)$ *to* $y^*(\cdot,\lambda)$, *for all* $y_0(\cdot) \in L^2(I)$ *and every* $\lambda \in [\lambda_1,\lambda_2]$*;*

3. *for every* $n \in \mathbb{N}$ *we have*

$$\|y_n(\cdot) - y^*(\cdot,\lambda)\|_{L^2(I)} \le \frac{L^n}{1-L}\|y_1(\cdot) - y_0(\cdot)\|_{L^2(I)}.$$

Moreover if

I.c) *the functions* $(K_i(x,s,\cdot,u))_{x,s\in I,u\in\mathbb{R}}$ *are equally continuous,*

then the operator $S : [\lambda_1,\lambda_2] \to L^2(I)$ *defined by* $S(\lambda)(x) = y^*(x,\lambda)$, $\forall\, x \in I$, $\forall\,\lambda \in [\lambda_1,\lambda_2]$ *is continuous.*

If instead of I.b), I.c) *and* III. *we have the conditions*

I.b') $K_i(x,s,\lambda,\cdot)$ *is in* $C^1(\mathbb{R})$ *for all* $\lambda \in [\lambda_1,\lambda_2]$, *a.e.* $(x,s) \in I^2$, *and there exist* $k_i \in L^2(I^2)$, $i \in \{1,2\}$, *such that*

$$\left|\frac{\partial K_i(t,s,\lambda,u)}{\partial u}\right| \le k_i(t,s),\ \forall\, t,s \in I, \forall\,\lambda \in [\lambda_1,\lambda_2], \forall\, u \in \mathbb{R};$$

I.c') $K_i(x,s,\cdot,u)$ *is in* $C^1[\lambda_1,\lambda_2]$ *for all* $u\in\mathbb{R}$, *a.e.* $(x,s)\in I^2$, *the partial derivatives satisfy condition* I., $\frac{\partial K_i}{\partial\lambda}(\cdot,\cdot,\lambda,u)\in L^2(I^2), i\in\{1,2\}$ *and there exists* $M_2>0$ *such that*

$$\left\|\frac{\partial K_i}{\partial\lambda}(\cdot,\cdot,\lambda,u)\right\|_{L^2(I^2)} < M_2,\ \forall\lambda\in[\lambda_1,\lambda_2],\ \forall u\in\mathbb{R},$$

then the operator S *is differentiable.*

Proof. First we prove that for a fixed λ the operator $T: L^2(I)\to L^2(I)$ defined by

$$T[y](x) = f(x) + \int_a^x K_1(x,s,\lambda,y(s))ds + \int_a^b K_2(x,s,\lambda,y(s))ds$$

is a contraction. From the Lipschitz condition we have

$$\int_a^b K_2(t,s,\lambda,y(s))ds \le \int_a^b K_2(t,s,\lambda,0) + k_2(t,s)|y(s)|ds.$$

Due to Minkovski and Cauchy-Buniakovski inequality we deduce

$$\int_a^b \left(\int_a^b K_2(t,s,\lambda,y(s))ds\right)^2 dt$$

$$\le \left(\sqrt{b-a}\|K_2(\cdot,\cdot,\lambda,0)\|_{L^2(I^2)} + \sqrt{b-a}\|k_2\|_{L^2(I^2)}\cdot\|y\|_{L^2(I)}\right)^2 < \infty.$$

Analogously

$$\int_a^b \left(\int_a^t K_1(t,s,\lambda,y(s))ds\right)^2 dt < \infty,$$

so because of $f\in L^2(I)$ we have $T[y]\in L^2(I)$. On the other hand

$$\begin{aligned}|T[y_1](t) - T[y_2](t)| &\le \int_a^t |K_1(t,s,\lambda,y_1(s)) - K_1(t,s,\lambda,y_2(s))|ds \\ &\quad + \int_a^b |K_2(t,s,\lambda,y_1(s)) - K_2(t,s,\lambda,y_2(s))|ds \\ &\le \int_a^t k_1(t,s)|y_1(s)-y_2(s)|ds + \int_a^b k_2(t,s)|y_1(s)-y_2(s)|ds \\ &= \int_a^b (\overline{k}_1(t,s)+k_2(t,s))|y_1(s)-y_2(s)|ds,\end{aligned}$$

where $\overline{k}_1(t,s) = \begin{cases} k_1(t,s), & t \geq s \\ 0, & t < s \end{cases}$. From the Cauchy-Buniakovski inequality we obtain

$$\|T[y_1](\cdot) - T[y_2](\cdot)\|^2_{L^2(I)} \leq L^2 \cdot \|y_1(\cdot) - y_2(\cdot)\|^2_{L^2(I)},$$

where L^2 is defined by (7). Hence T is a contraction and from the contractions principle we have the conclusions.

If we have condition I.c), then for every $\varepsilon > 0$ there exists $\varepsilon_1 = \frac{(1-L)\varepsilon}{2(b-a)\sqrt{2(b-a)}}$ and $\delta > 0$ such that for $|\lambda - \overline{\lambda}| < \delta$ we have

$$|K_i(t,s,\lambda,u) - K_i(t,s,\overline{\lambda},u)| \leq \varepsilon_1,$$

for all $u \in \mathbb{R}$ and a.e. $(t,s) \in I^2$. If y^*_λ and $y^*_{\overline{\lambda}}$ are the corresponding unique solutions to λ, and $\overline{\lambda}$, then

$$\begin{aligned} |y^*_\lambda(t) - y^*_{\overline{\lambda}}(t)| \leq & \int_a^t |K_1(t,s,\lambda,y^*_\lambda(s)) - K_1(t,s,\overline{\lambda},y^*_{\overline{\lambda}}(s))|ds \\ & + \int_a^b |K_2(t,s,\lambda,y^*_\lambda(s)) - K_2(t,s,\overline{\lambda},y^*_{\overline{\lambda}}(s))|ds \\ & \leq 2(b-a)\varepsilon_1 + \int_a^t |K_1(t,s,\lambda,y^*_\lambda(s)) - K_1(t,s,\lambda,y^*_{\overline{\lambda}}(s))|ds \\ & + \int_a^b |K_2(t,s,\lambda,y^*_\lambda(s)) - K_2(t,s,\lambda,y^*_{\overline{\lambda}}(s))|ds \\ & \leq 2(b-a)\varepsilon_1 + \int_a^b (\overline{k}_1(t,s) + k_2(t,s))|y^*_\lambda(s) - y^*_{\overline{\lambda}}(s)|ds. \end{aligned}$$

From this inequality and Lemma 1 we obtain

$$\|y^*_\lambda(\cdot) - y^*_{\overline{\lambda}}(\cdot)\|_{L^2(I)} \leq \frac{2(b-a)\varepsilon_1\sqrt{2(b-a)}}{1-L},$$

where L is defined in (7). So for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\lambda - \overline{\lambda}| < \delta \Rightarrow \|y^*_\lambda(\cdot) - y^*_{\overline{\lambda}}(\cdot)\|_{L^2(I)} < \varepsilon,$$

this is the continuity of the operator S.

If we have I.b') and I.c'), we use the fiber Picard theorem to study the differentiability of the operator S. Consider the spaces $V = W = L^2(I)$ and the operators $B : V \to V$, $C : V \times W \to W$ defined by the following relations

$$B[v](t) = f(t) + \int_a^t K_1(t, s, \lambda, y(s))ds + \int_a^b K_2(t, s, \lambda, y(s))ds$$

and

$$C[(v, w)](t) = \int_a^t \frac{\partial K_1(t, s, v(s); \lambda)}{\partial \lambda} ds + \int_a^b \frac{\partial K_2(t, s, v(s); \lambda)}{\partial \lambda} ds$$

$$+ \int_a^t \frac{\partial K_1(t, s, v(s); \lambda)}{\partial v} w(s)ds + \int_a^b \frac{\partial K_2(t, s, v(s); \lambda)}{\partial v} w(s)ds.$$

Due to the given conditions the operator B is a Picard operator (condition I.b') implies condition III.) and the operator C satisfies

$$\|C[(v, w_1)] - C[(v, w_2)]\|_{L^2(I)} \le L_1 \|w_1 - w_2\|_{L^2(I)},$$

where $L_1 = \sqrt{\int_a^b \int_a^t (k_1(t, s) + k_2(t, s))^2 dsdt + \int_a^b \int_t^b k_2^2(t, s)dsdt}$. Theorem 1 implies that the triangular operator $A[v, w] = (B[v], C[v, w])$ is a Picard operator and so the sequence of successive approximations constructed by the relations $(y_{n+1}, z_{n+1}) = A[y_n, z_n]$ converges in $(L^2(I))^2$ to the unique fixed point. If we choose for $y_0(\cdot, \lambda)$ a C^1 function in its last variable and $z_0 = \frac{\partial y_0}{\partial \lambda}$, then from the definition of the operator C we deduce $z_n = \frac{\partial y_n}{\partial \lambda}$. By the other hand the operators $S_n : [\lambda_1, \lambda_2] \to L^2(I)$ defined by $S_n(\lambda)(t) = y_n(t), \forall t \in I, \forall \lambda \in [\lambda_1, \lambda_2]$ are differentiables and the differential of S_n in λ is z_n, hence we can apply Theorem 2 and we obtain the differentiability of the operator S. □

Remark 2 *We can prove the same results working in the space*

$$Y = \left\{ y : I \times \Lambda \to \mathbb{R} \;\middle|\; y(\cdot, \lambda) \in L^2[I], \forall \lambda \in \Lambda, \; y(t, \cdot) \in C(\Lambda) \text{ a.e. } t \in I \right\},$$

where $\Lambda = [\lambda_1, \lambda_2]$ *and the norm is defined by* $\|y\|_Y = \max_{\lambda \in \Lambda} \|y(\cdot, \lambda)\|_{L^2(I)}$.

Using the same arguments we can prove the following theorem for equation (2).

Theorem 4 *If*

a) *the functions* $K_i : I \times I \times [\lambda_1, \lambda_2] \times \mathbb{R} \to \mathbb{R}, i = \overline{1,2}$ *satisfy conditions* I.-IV. *from Theorem 3;*

b) *the functions* $g_1, g_2 : [a,b] \to \mathbb{R}$ *are injective and measurable and they satisfy the conditions* $\operatorname{Im}(g_1) = [a_1, a_2]$, $\operatorname{Im}(g_2) = [b_2, b_1]$ *with* $a_1 \le a \le a_2 \le b$, *and* $a \le b_2 \le b \le b_1$;

c) $\varphi_1 \in L^2([a_1, a])$ *and* $\varphi_2 \in L^2([b, b_1])$;

then

1) *equation* (2) *has a unique solution* $y^*(\cdot, \lambda)$ *in* $L^2(I_1)$ *for all* $\lambda \in [\lambda_1, \lambda_2]$, *where* $I_1 = [a_1, b_1]$;

2) *the sequence of successive approximations converges* $L^2(I_1)$ *to* $y^*(\cdot, \lambda)$ *for every admissible initial function* $y_0(\cdot, \lambda)$, *where the set of admissible functions is*

$$Y_a = \{y(\cdot, \lambda) \in L^2(I_1) \,|\, y_0(t, \lambda) = \varphi_1(t), \forall\, t \in [a_1, a], \, y_0(t, \lambda) = \varphi_2(t), \forall\, t \in [b, b_1]\};$$

3) *we have the following estimation:*

$$\|y_n(\cdot) - y^*(\cdot, \lambda)\|_{L^2(I_1)} \le \frac{L^n}{1-L} \|y_1(\cdot) - y_0(\cdot)\|_{L^2(I_1)},$$

where L *is defined by relation* (7).

Moreover if condition I.c) *holds, then the operator* $S : [\lambda_1, \lambda_2] \to L^2(I_1)$ *defined by* $S(\lambda)(x) = y^*(x, \lambda)$, $\forall\, x \in [a_1, b_1]$, $\forall\, \lambda \in [\lambda_1, \lambda_2]$ *is continuous.*

If instead of conditions I.b), I.c) *and* III. *the conditions* I.b') *and* I.c') *are satisfied, then* S *is differentiable.*

Remark 3 *The differentiability of* S *implies the existence of the partial derivative* $\frac{\partial y^*(\cdot,\lambda)}{\partial \lambda}$ *and so from the construction of the operator* C *we deduce that this partial derivative satisfies the equation*

$$\frac{\partial y^*(t,\lambda)}{\partial \lambda} = \int_a^t \frac{\partial K_1(t,s,\lambda,y^*(s,\lambda))}{\partial \lambda} ds + \int_a^b \frac{\partial K_2(t,s,\lambda,y^*(s,\lambda))}{\partial \lambda} ds + \int_a^t \frac{\partial K_1(t,s,\lambda,y^*(s,\lambda))}{\partial y^*} \frac{\partial y^*(s,\lambda)}{\partial \lambda} ds + \int_a^b \frac{\partial K_2(t,s,\lambda,y^*(s,\lambda))}{\partial y^*} \frac{\partial y^*(s,\lambda)}{\partial \lambda} ds;$$

in the case of Theorem 3 and the equation

$$\frac{\partial y^*(t,\lambda)}{\partial\lambda} = \int_a^t \frac{\partial K_1(t,s,\lambda,y^*(g_1(s),\lambda))}{\partial\lambda}ds + \int_a^b \frac{\partial K_2(t,s,\lambda,y^*(g_2(s),\lambda))}{\partial\lambda}ds$$
$$+ \int_a^t \frac{\partial K_1(t,s,\lambda,y^*(g_1(s),\lambda))}{\partial y^*} \cdot \frac{\partial y^*(g_1(s),\lambda)}{\partial\lambda}ds$$
$$+ \int_a^b \frac{\partial K_2(t,s,\lambda,y^*(g_2(s),\lambda))}{\partial y^*} \cdot \frac{\partial y^*(g_2(s),\lambda)}{\partial\lambda}ds$$

in the case of Theorem 4.

3 Fredholm-Volterra equations on an unbounded interval

If $I = [a, \infty)$, we can't use the same inequalities because in Lemma 1 and in some estimations we used it was essential the finite length of the interval. Due to this problem we need other conditions to guarantee the same properties of the solution operator.

Theorem 5 *If conditions* I.-III. *from Theorem 3 are satisfied with* $I = [a, \infty)$ *and*

$$L^2 := \int_a^\infty \int_a^t (k_1(t,s) + k_2(t,s))^2 ds dt + \int_a^\infty \int_t^\infty k_2^2(t,s) ds dt < 1, \quad (8)$$

then

1. *for every* $\lambda \in [\lambda_1, \lambda_2]$ *there exists an unique solution* $y^*(\cdot,\lambda) \in L^2(I)$;

2. *the sequence of successive approximations*

$$y_{n+1}(x) = f(x) + \int_a^x K_1(x,s,\lambda,y_n(s))ds + \int_a^\infty K_2(x,s,\lambda,y_n(s))ds$$

converges in $L^2(I)$ *to* $y^*(\cdot,\lambda)$, *for all* $y_0(\cdot) \in L^2(I)$;

3. *for every* $n \in \mathbb{N}$ *we have*

$$\|y_n(\cdot) - y^*(\cdot,\lambda)\|_{L^2(I)} \le \frac{L^n}{1-L}\|y_1(\cdot) - y_0(\cdot)\|_{L^2(I)}.$$

Moreover if

I.c) *there exist* $\Lambda_i : [\lambda_1,\lambda_2] \times [\lambda_1,\lambda_2] \to \mathbb{R}$, *and* $g_i : I^2 \to \mathbb{R}$, $i \in \{1,2\}$ *such that*

i)
$$|K_i(x,s,\lambda,u) - K_i(x,s,\overline{\lambda},u)| \le \Lambda_i(\lambda,\overline{\lambda}) \cdot g_i(t,s), \tag{9}$$
$\forall\, u \in \mathbb{R}, \lambda, \overline{\lambda} \in [\lambda_1,\lambda_2]$, a.e.$(t,s) \in I^2, i \in \{1,2\}$;

ii) $\lim\limits_{\overline{\lambda}\to\lambda} \Lambda(\lambda,\overline{\lambda}) = 0$;

iii) $\int\limits_a^\infty \left[\left(\int\limits_a^t g_1(s,t)ds\right)^2 + \left(\int\limits_a^\infty g_2(s,t)\right)^2\right] dt < +\infty$

then the operator $S : [\lambda_1,\lambda_2] \to L^2(I)$ *defined by* $S(\lambda)(x) = y^*(x,\lambda)$, $\forall\, x \in I$, $\forall\, \lambda \in [\lambda_1,\lambda_2]$ *is continuous.*

If instead of the conditions I.b) *and* III. *condition* I.b') *from Theorem 3 is fulfilled and*

I.c') $K_i(x,s,\cdot,u)$ *is a* $C^1[\lambda_1,\lambda_2]$ *function for all* $u \in \mathbb{R}$, *a.e.* $(x,s) \in I^2$, *the partial derivatives satisfy condition* I., *and there exists* $M_3 > 0$ *such that*

$$\int_a^\infty \left(\int_a^t \frac{\partial K_1}{\partial\lambda}(t,s,\lambda,u)ds\right)^2 dt + \int_a^\infty \left(\int_a^t \frac{\partial K_2}{\partial\lambda}(t,s,\lambda,u)ds\right)^2 dt < M_3^2,$$

for all $\lambda \in [\lambda_1,\lambda_2]$ *and for all* $u \in \mathbb{R}$,

then S *is differentiable.*

Proof. As in Theorem 3 for a fixed λ the operator $T : L^2(I) \to L^2(I)$ defined by

$$T[y](x) = f(x) + \int\limits_a^x K_1(x,s,\lambda,y(s))ds + \int\limits_a^\infty K_2(x,s,\lambda,y(s))ds$$

is a contraction with Lipschitz constant L. If y^*_λ and $y^*_{\overline{\lambda}}$ are the unique solutions corresponding to λ and $\overline{\lambda}$, from I.c) we deduce:

$$\int\limits_a^\infty\left(\int\limits_a^t |K_1(t,s,\lambda,y^*_{\overline{\lambda}}(s)) - K_1(t,s,\overline{\lambda},y^*_{\overline{\lambda}}(s))|ds\right)^2 dt \le \Lambda_1^2(\lambda,\overline{\lambda})\cdot \int\limits_a^\infty\left(\int\limits_a^t g_1(t,s)ds\right)^2 dt$$

and

$$\int_a^\infty\left(\int_a^\infty |K_2(t,s,\lambda,y^*_\lambda(s)) - K_2(t,s,\overline{\lambda},y^*_{\overline{\lambda}}(s))|ds\right)^2 dt \le \Lambda_2^2(\lambda,\overline{\lambda})\cdot \int_a^\infty\left(\int_a^\infty g_2(t,s)ds\right)^2 dt.$$

From

$$\begin{aligned}
|y^*_\lambda(t) - y^*_{\overline{\lambda}}(t)| &\le \int_a^t |K_1(t,s,\lambda,y^*_\lambda(s)) - K_1(t,s,\overline{\lambda},y^*_{\overline{\lambda}}(s))|ds \\
&\quad + \int_a^b |K_2(t,s,\lambda,y^*_\lambda(s)) - K_2(t,s,\overline{\lambda},y^*_{\overline{\lambda}}(s))|ds \\
&\le \int_a^t |K_1(t,s,\lambda,y^*_{\overline{\lambda}}(s)) - K_1(t,s,\overline{\lambda},y^*_{\overline{\lambda}}(s))|ds \\
&\quad + \int_a^b |K_2(t,s,\lambda,y^*_{\overline{\lambda}}(s)) - K_2(t,s,\overline{\lambda},y^*_{\overline{\lambda}}(s))|ds \\
&\quad + \int_a^t |K_1(t,s,\lambda,y^*_\lambda(s)) - K_1(t,s,\lambda,y^*_{\overline{\lambda}}(s))|ds \\
&\quad + \int_a^b |K_2(t,s,\lambda,y^*_\lambda(s)) - K_2(t,s,\lambda,y^*_{\overline{\lambda}}(s))|ds \\
&\le \int_a^t |K_1(t,s,\lambda,y^*_{\overline{\lambda}}(s)) - K_1(t,s,\overline{\lambda},y^*_{\overline{\lambda}}(s))|ds \\
&\quad + \int_a^b |K_2(t,s,\lambda,y^*_{\overline{\lambda}}(s)) - K_2(t,s,\overline{\lambda},y^*_{\overline{\lambda}}(s))|ds \\
&\quad + \int_a^b (\overline{k}_1(t,s) + k_2(t,s))|y^*_\lambda(s) - y^*_{\overline{\lambda}}(s)|ds
\end{aligned}$$

we deduce (using Minkovski inequality)

$$\|y^*_\lambda(\cdot) - y^*_{\overline{\lambda}}(\cdot)\|_{L^2(I)} \le \frac{\Lambda}{1-L},$$

where L is defined in (8) and

$$\Lambda = \Lambda_1(\lambda,\overline{\lambda})\sqrt{\int_a^\infty\left(\int_a^t k_1(s,t)ds\right)^2 dt} + \Lambda_2(\lambda,\overline{\lambda})\sqrt{\int_a^\infty\left(\int_a^\infty k_2(s,t)\right)^2 dt}.$$

This inequality implies the continuity of the operator S.

If conditions I.b') and I.c') are satisfied we can use the fiber Picard theorem again. Consider the spaces $V = W = L^2(I)$ and the operators $B : V \to V$, $C : V \times W \to W$ defined by the following relations

$$B[v](t) = f(t) + \int_a^t K_1(t, s, \lambda, y(s))ds + \int_a^\infty K_2(t, s, \lambda, y(s))ds$$

and

$$C[(v, w)](t) = \int_a^t \frac{\partial K_1(t, s, v(s); \lambda)}{\partial \lambda} ds + \int_a^\infty \frac{\partial K_2(t, s, v(s); \lambda)}{\partial \lambda} ds + \int_a^t \frac{\partial K_1(t, s, v(s); \lambda)}{\partial v} w(s)ds + \int_a^\infty \frac{\partial K_2(t, s, v(s); \lambda)}{\partial v} w(s)ds.$$

Due to the given conditions B is a Picard operator (condition I.b') implies condition III.) and C satisfies the uniform contraction condition:

$$\|C[(v, w_1)] - C[(v, w_2)]\|_{L^2(I)} \leq L_1 \|w_1 - w_2\|_{L^2(I)},$$

where $L_1 = \sqrt{\int_a^\infty \int_a^t (k_1(t, s) + k_2(t, s))^2 ds dt + \int_a^\infty \int_t^\infty k_2^2(t, s) ds dt}$. Theorem 1 implies that the triangular operator $A[v, w] = (B[v], C[v, w])$ is a Picard operator. Hence the sequence of successive approximation $(y_{n+1}, z_{n+1}) = A[y_n, z_n]$ converges in $(L^2(I))^2$ to the unique fixed point. If we choose $y_0(\cdot, \lambda)$ continuously differentiable (with respect to λ) and $z_0 = \frac{\partial y_0}{\partial \lambda}$, then from the construction of the operator C we obtain $z_n = \frac{\partial y_n}{\partial \lambda}$. On the other hand the operators $S_n : [\lambda_1, \lambda_2] \to L^2(I)$ defined by $S_n(\lambda)(t) = y_n(t, \lambda)$, $\forall\, t \in I$, $\forall\, \lambda \in [\lambda_1, \lambda_2]$ are differentiables and the differential of S_n in λ is z_n, so from Theorem 2 we obtain the differentiability of S. □

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