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New family of bi-univalent functions with respect to symmetric conjugate points associated with Borel distribution

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Abstract. In this paper, we introduce and investigate a new family, denoted by $\mathcal{W}_{\Sigma}^{sc}(\lambda, \eta, \delta, r)$, of normalized holomorphic and bi-univalent functions with respect to symmetric conjugate points, defined in $\mathbb{U}$, by making use the Borel distribution series, which is associated with the Horadam polynomials. We derive estimates on the initial Taylor-Maclaurin coefficients and solve the Fekete-Szegő type inequalities for functions in this family.

1 Introduction and preliminaries

We denote by $\mathcal{A}$ the family of functions which are holomorphic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

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and have the following normalized form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

We denote by $\mathcal{S}$ stands for the sub-family of the set $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$. According to the Koebe one-quarter theorem [11], every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U})$$

and

$$f\big(f^{-1}(w)\big) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \tag{2}$$

We say that a function $f \in \mathcal{A}$ is *bi-univalent* in $\mathbb{U}$ if both f and it inverse f^{-1} are univalent in $\mathbb{U}$. Let Σ stand for the family of bi-univalent functions in $\mathbb{U}$ given by (1). Beginning with Srivastava *et al.* pioneering work [31] on the subject, a large number of works related to the subject have been presented (see, for example, [6, 7, 8, 9, 10, 15, 23, 25, 27, 28, 30, 32, 35, 36, 39, 40, 41, 42, 45, 46]). From the work of Srivastava *et al.* [31], we choose to recall the following examples of functions in the family Σ :

$$\frac{z}{1-z}, \quad -\log(1-z) \qquad \text{and} \qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

We notice that the family Σ is not empty. However, the Koebe function is not a member of Σ. The problem to find the general coefficient bounds on the Taylor-Maclaurin coefficients

$$|a_n| \qquad (n \in \mathbb{N};\ n \geqq 3)$$

for functions $f \in \Sigma$ is still not completely addressed for many of the subfamilies of the bi-univalent function family Σ (see, for example, [27, 32, 33, 35]).

On this subject in geometric function theory, the so-called Fekete-Szegő type inequalities (or problems) which estimate some upper bounds for $\left|a_3 - \mu a_2^2\right|$ $(f \in \mathcal{S})$. Its origin was in the disproof by Fekete and Szegő [14] of the Littlewood-Paley conjecture that the coefficients of odd univalent functions are bounded by unity. The functional has since received great attention, particularly in the study of many subfamilies of the family of univalent functions. This topic has become of considerable interest among researchers in Geometric Function Theory of Complex Analysis (see, for example, [29, 34]).

A function $f \in \mathcal{A}$ is called starlike with respect to symmetric conjugate points in $\mathbb{U}$ if (see [12])

$$\Re\left\{\frac{zf'(z)}{\left(f(z)-\overline{f(-\overline{z})}\right)'}\right\} > 0 \qquad (z \in \mathbb{U}).$$

On the other hand, a function $f \in \mathcal{A}$ is called a convex with respect to symmetric conjugate points in $\mathbb{U}$ if

$$\Re\left\{\frac{z\left(f'(z)\right)'}{\left(f(z)-\overline{f(-\overline{z})}\right)'}\right\} > 0 \qquad (z \in \mathbb{U}).$$

The families of all starlike with respect to symmetric conjugate points and all convex functions with respect to symmetric conjugate points are denoted by S^*_{sc} and C_{sc}, respectively.

The elementary distributions such as the Poisson, the Pascal, the Logarithmic, the Binomial, the beta negative binomial have been partially studied in Geometric Function Theory from a theoretical point of view (see for example [5, 13, 22, 24, 44]).

In the paper [47] are introduced the following power series whose coefficients are probabilities of the Borel distribution:

$$\mathcal{M}(\delta, z) = z + \sum_{n=2}^{\infty} \frac{(\delta(n-1))^{n-2}\, e^{-\delta(n-1)}}{(n-1)!} z^n \qquad (z \in \mathbb{U};\ 0 < \delta \leqq 1).$$

We note by the familiar Ratio Test that the radius of convergence of the above series is infinity.

The linear operator $\mathcal{B}_\delta : \mathcal{A} \longrightarrow \mathcal{A}$ is defined as follows (see [47])

$$\mathcal{B}_\delta f(z) = \mathcal{M}(\delta, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(\delta(n-1))^{n-2}\, e^{-\delta(n-1)}}{(n-1)!} a_n z^n \qquad z \in \mathbb{U},$$

where the symbol $*$ stands the Hadamard product (or convolution) of two series.

For two functions $f, g \in \mathcal{A}$, we say that the function f is subordinate to g, if there exists a Schwarz function ω, which is holomorphic in $\mathbb{U}$ with the property

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(\omega(z)\big).$$

This subordination is symbolically written as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

It is well known that, if the function g is univalent in $\mathbb{U}$, then (see [21])

$$f \prec g \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subseteq g(\mathbb{U}).$$

Hörçum and Koçer in the paper [18] considered the familiar Horadam polynomials $h_n(r)$, which are given by Definition 1 below, in Geometric Function Theory of Complex Analysis.

Definition 1 (see [17] and [18]) The Horadam polynomials $h_n(r)$ are given by the following recurrence relation:

$$h_n(r) = prh_{n-1}(r) + qh_{n-2}(r) \quad (r \in \mathbb{R};\ n \in \mathbb{N} = \{1, 2, 3, \cdots\}) \tag{3}$$

with

$$h_1(r) = a \quad \text{and} \quad h_2(r) = br,$$

for some real constants a, b, p and q. Moreover, the characteristic equation of the recurrence relation (3) is given by

$$t^2 - prt - q = 0,$$

which has the following two real roots:

$$t_1 = \frac{pr + \sqrt{p^2r^2 + 4q}}{2} \quad \textit{and} \quad t_2 = \frac{pr - \sqrt{p^2r^2 + 4q}}{2}.$$

Remark 1 By selecting the particular values of the parameters a, b, p and q, the Horadam polynomial $h_n(r)$ reduces to several known polynomials. Some of these special cases are recorded below.

1. Taking $a = b = p = q = 1$, we obtain the Fibonacci polynomials $F_n(r)$.

2. Taking $a = 2$ and $b = p = q = 1$, we get the Lucas polynomials $L_n(r)$.

3. Taking $a = q = 1$ and $b = p = 2$, we have the Pell polynomials $P_n(r)$.

4. Taking $a = b = p = 2$ and $q = 1$, we find the Pell-Lucas polynomials $Q_n(r)$.

5. Taking $a = b = 1$, $p = 2$ and $q = -1$, we obtain the Chebyshev polynomials $T_n(r)$ of the first kind.

6. Taking $a = 1$, $b = p = 2$ and $q = -1$, we have the Chebyshev polynomials $U_n(r)$ of the second kind.

The families of orthogonal polynomials and other special functions and specific polynomials, as well as their extensions and generalizations, are potentially useful in a variety of disciplines in many branches of science, especially in the mathematical, statistical and physical sciences. For more information associated with these polynomials, see [16, 17, 19, 20].

The generating function of the Horadam polynomials $h_n(r)$ is given as follows (see [18]):

$$\Pi(r,z) = \sum_{n=1}^{\infty} h_n(r) z^{n-1} = \frac{a + (b - ap)rz}{1 - prz - qz^2}. \tag{4}$$

In fact, Srivastava *et al.* [26] have already applied the Horadam polynomials in a similar context involving holomorphic and bi-univalent functions in recent years, it was followed by such works as those by Al-Amoush [2], Wanas and Lupas [48], Abirami *et al.* [1] and others (see, for example, [1, 3, 4, 37, 38, 43]).

2 Main results

We begin this section by defining the new family denoted by $\mathcal{W}_{\Sigma}^{sc}(\lambda, \eta, \delta, r)$.

Definition 2 For $0 \leqq \eta \leqq \lambda \leqq 1$, $0 < \delta \leqq 1$ and $r \in \mathbb{R}$, a function $f \in \Sigma$ is said to be in the family $\mathcal{W}_{\Sigma}^{sc}(\lambda, \eta, \delta, r)$ if it fulfills the following subordination conditions:

$$\frac{2\left[\lambda\eta z^3\left(\mathcal{B}_\delta f(z)\right)''' + \left(\lambda+\eta(2\lambda-1)\right) z^2 \left(\mathcal{B}_\delta f(z)\right)'' + z\left(\mathcal{B}_\delta f(z)\right)'\right]}{\lambda\eta z^2\left(\mathcal{B}_\delta f(z) - \overline{\mathcal{B}_\delta f(-\overline{z})}\right)'' + (\lambda-\eta)z\left(\mathcal{B}_\delta f(z) - \overline{\mathcal{B}_\delta f(-\overline{z})}\right)' + (1-\lambda+\eta)\left(\mathcal{B}_\delta f(z) - \overline{\mathcal{B}_\delta f(-\overline{z})}\right)}$$
$$\prec \Pi(r,z) + 1 - a$$

and

$$\frac{2\left[\lambda\eta w^3\left(\mathcal{B}_\delta g(w)\right)''' + \left(\lambda+\eta(2\lambda-1)\right) w^2 \left(\mathcal{B}_\delta g(w)\right)'' + w\left(\mathcal{B}_\delta g(w)\right)'\right]}{\lambda\eta w^2\left(\mathcal{B}_\delta g(w) - \overline{\mathcal{B}_\delta g(-\overline{w})}\right)'' + (\lambda-\eta)w\left(\mathcal{B}_\delta g(w) - \overline{\mathcal{B}_\delta g(-\overline{w})}\right)' + (1-\lambda+\eta)\left(\mathcal{B}_\delta g(w) - \overline{\mathcal{B}_\delta g(-\overline{w})}\right)}$$
$$\prec \Pi(r,w) + 1 - a,$$

where a is real constant and the function $g = f^{-1}$ is given by (2).

Our first main result is asserted by Theorem 1 below.

Theorem 1 *For* $0 \leqq \eta \leqq \lambda \leqq 1$, $0 < \delta \leqq 1$ *and* $r \in \mathbb{R}$, *let* $f \in \mathcal{A}$ *be in the family* $\mathcal{W}_\Sigma^{sc}(\lambda,\eta,\delta,r)$. *Then*

$$|a_2| \leqq \frac{e^\delta\,|br|\sqrt{|br|}}{\sqrt{2\left|\left[\delta\left(6\lambda\eta+2(\lambda-\eta)+1\right)b - 2p\left(2\lambda\eta+\lambda-\eta+1\right)^2\right]br^2 - 2qa\left(2\lambda\eta+\lambda-\eta+1\right)^2\right|}}$$

and

$$|a_3| \leqq \frac{|br|\,e^{2\delta}}{2\delta\left(6\lambda\eta+2(\lambda-\eta)+1\right)} + \frac{b^2r^2e^{2\delta}}{4\left(2\lambda\eta+\lambda-\eta+1\right)^2},$$

where

$$\Omega(\delta,\gamma,\lambda,\mathfrak{s},t,q) = \left[(1-\delta)(\gamma+2)+\delta(3\lambda-1)\right]\left[1+t\right]_q^{\mathfrak{s}}\left[2+t\right]_q^{2\mathfrak{s}}, \tag{5}$$

$$\Gamma(\delta,\gamma,\lambda,\mathfrak{s},t,q) = \left[\frac{1}{2}(1-\delta)(\gamma+2)(\gamma-1)+\delta\left(2\lambda(\lambda-2)+1\right)\right]\left[3+t\right]_q^{\mathfrak{s}}\left[1+t\right]_q^{2\mathfrak{s}} \tag{6}$$

and

$$\Upsilon(\delta,\gamma,\lambda,\mathfrak{s},t,q) = \left[(1-\delta)(\gamma+1)+\delta(2\lambda-1)\right]^2\left[3+t\right]_q^{\mathfrak{s}}\left[1+t\right]_q^{2\mathfrak{s}}. \tag{7}$$

Proof. Let $f \in \mathcal{W}_\Sigma^{sc}(\lambda,\eta,\delta,r)$. Then there are two holomorphic functions $u, v : \mathbb{U} \longrightarrow \mathbb{U}$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \qquad (z \in \mathbb{U}) \tag{8}$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \qquad (w \in \mathbb{U}), \tag{9}$$

with

$$u(0) = v(0) = 0 \qquad \text{and} \qquad \max\{|u(z)|, |v(w)|\} < 1 \quad (z, w \in \mathbb{U}),$$

such that

$$\frac{2\left[\lambda\eta z^3 (\mathcal{B}_\delta f(z))''' + (\lambda + \eta(2\lambda - 1)) z^2 (\mathcal{B}_\delta f(z))'' + z (\mathcal{B}_\delta f(z))'\right]}{\lambda\eta z^2 \left(\mathcal{B}_\delta f(z) - \overline{\mathcal{B}_\delta f(-\overline{z})}\right)'' + (\lambda - \eta) z \left(\mathcal{B}_\delta f(z) - \overline{\mathcal{B}_\delta f(-\overline{z})}\right)' + (1 - \lambda + \eta) \left(\mathcal{B}_\delta f(z) - \overline{\mathcal{B}_\delta f(-\overline{z})}\right)}$$
$$= \Pi(r, u(z)) - a$$

and

$$\frac{2\left[\lambda\eta w^3 (\mathcal{B}_\delta g(w))''' + (\lambda + \eta(2\lambda - 1)) w^2 (\mathcal{B}_\delta g(w))'' + w (\mathcal{B}_\delta g(w))'\right]}{\lambda\eta w^2 \left(\mathcal{B}_\delta g(w) - \overline{\mathcal{B}_\delta g(-\overline{w})}\right)'' + (\lambda - \eta) w \left(\mathcal{B}_\delta g(w) - \overline{\mathcal{B}_\delta g(-\overline{w})}\right)' + (1 - \lambda + \eta) \left(\mathcal{B}_\delta g(w) - \overline{\mathcal{B}_\delta g(-\overline{w})}\right)}$$
$$= \Pi(r, v(w)) - a$$

or, equivalently, that

$$\frac{2\left[\lambda\eta z^3 (\mathcal{B}_\delta f(z))''' + (\lambda + \eta(2\lambda - 1)) z^2 (\mathcal{B}_\delta f(z))'' + z (\mathcal{B}_\delta f(z))'\right]}{\lambda\eta z^2 \left(\mathcal{B}_\delta f(z) - \overline{\mathcal{B}_\delta f(-\overline{z})}\right)'' + (\lambda - \eta) z \left(\mathcal{B}_\delta f(z) - \overline{\mathcal{B}_\delta f(-\overline{z})}\right)' + (1 - \lambda + \eta) \left(\mathcal{B}_\delta f(z) - \overline{\mathcal{B}_\delta f(-\overline{z})}\right)}$$
$$= 1 + h_1(r) + h_2(r)u(z) + h_3(r)u^2(z) + \cdots \tag{10}$$

and

$$\frac{2\left[\lambda\eta w^3 (\mathcal{B}_\delta g(w))''' + (\lambda + \eta(2\lambda - 1)) w^2 (\mathcal{B}_\delta g(w))'' + w (\mathcal{B}_\delta g(w))'\right]}{\lambda\eta w^2 \left(\mathcal{B}_\delta g(w) - \overline{\mathcal{B}_\delta g(-\overline{w})}\right)'' + (\lambda - \eta) w \left(\mathcal{B}_\delta g(w) - \overline{\mathcal{B}_\delta g(-\overline{w})}\right)' + (1 - \lambda + \eta) \left(\mathcal{B}_\delta g(w) - \overline{\mathcal{B}_\delta g(-\overline{w})}\right)}$$
$$= 1 + h_1(r) + h_2(r)v(w) + h_3(r)v^2(w) + \cdots . \tag{11}$$

Combining (8), (9), (10) and (11) yields

$$\frac{2\left[\lambda\eta z^3 (\mathcal{B}_\delta f(z))''' + (\lambda + \eta(2\lambda - 1)) z^2 (\mathcal{B}_\delta f(z))'' + z (\mathcal{B}_\delta f(z))'\right]}{\lambda\eta z^2 \left(\mathcal{B}_\delta f(z) - \overline{\mathcal{B}_\delta f(-\overline{z})}\right)'' + (\lambda - \eta) z \left(\mathcal{B}_\delta f(z) - \overline{\mathcal{B}_\delta f(-\overline{z})}\right)' + (1 - \lambda + \eta) \left(\mathcal{B}_\delta f(z) - \overline{\mathcal{B}_\delta f(-\overline{z})}\right)}$$
$$= 1 + h_2(r)u_1 z + \left[h_2(r)u_2 + h_3(r)u_1^2\right] z^2 + \cdots \tag{12}$$

and

$$\frac{2\left[\lambda\eta w^3 (\mathcal{B}_\delta g(w))''' + (\lambda + \eta(2\lambda - 1)) w^2 (\mathcal{B}_\delta g(w))'' + w (\mathcal{B}_\delta g(w))'\right]}{\lambda\eta w^2 \left(\mathcal{B}_\delta g(w) - \overline{\mathcal{B}_\delta g(-\overline{w})}\right)'' + (\lambda - \eta) w \left(\mathcal{B}_\delta g(w) - \overline{\mathcal{B}_\delta g(-\overline{w})}\right)' + (1 - \lambda + \eta) \left(\mathcal{B}_\delta g(w) - \overline{\mathcal{B}_\delta g(-\overline{w})}\right)}$$
$$= 1 + h_2(r)v_1 w + \left[h_2(r)v_2 + h_3(r)v_1^2\right] w^2 + \cdots . \tag{13}$$

It is well-known that, if

$$\max\{|u(z)|, |v(w)|\} < 1 \qquad (z, w \in \mathbb{U}),$$

then

$$|u_j| \leqq 1 \qquad \text{and} \qquad |v_j| \leqq 1 \qquad (\forall\, j \in \mathbb{N}). \tag{14}$$

Now, by comparing the corresponding coefficients in (12) and (13), we find that

$$2e^{-\delta}\left(2\lambda\eta + \lambda - \eta + 1\right)\ a_2 = h_2(r)u_1, \tag{15}$$

$$2\delta e^{-2\delta}\left(6\lambda\eta + 2(\lambda - \eta) + 1\right)\ a_3 = h_2(r)u_2 + h_3(r)u_1^2, \tag{16}$$

$$-2e^{-\delta}\left(2\lambda\eta + \lambda - \eta + 1\right)\ a_2 = h_2(r)v_1 \tag{17}$$

and

$$2\delta e^{-2\delta}\left(6\lambda\eta + 2(\lambda - \eta) + 1\right)\ \left(2a_2^2 - a_3\right) = h_2(r)v_2 + h_3(r)v_1^2. \tag{18}$$

It follows from (15) and (17) that

$$u_1 = -v_1 \tag{19}$$

and

$$8e^{-2\delta}\left(2\lambda\eta + \lambda - \eta + 1\right)^2\ a_2^2 = h_2^2(r)(u_1^2 + v_1^2). \tag{20}$$

If we add (16) to (18), we find that

$$4\delta e^{-2\delta}\left(6\lambda\eta + 2(\lambda - \eta) + 1\right)\ a_2^2 = h_2(r)(u_2 + v_2) + h_3(r)(u_1^2 + v_1^2). \tag{21}$$

Upon substituting the value of $u_1^2 + v_1^2$ from (20) into the right-hand side of (21), we deduce that

$$a_2^2 = \frac{e^{2\delta} h_2^3(r)(u_2 + v_2)}{4\left[\delta h_2^2(r)\left(6\lambda\eta + 2(\lambda - \eta) + 1\right) - 2h_3(r)\left(2\lambda\eta + \lambda - \eta + 1\right)^2\right]}, \tag{22}$$

By further computations using (3), (14) and (22), we obtain

$$|a_2| \leqq \frac{e^{\delta}\,|br|\sqrt{|br|}}{\sqrt{2\left|\left[\delta\left(6\lambda\eta + 2(\lambda - \eta) + 1\right) b - 2p\left(2\lambda\eta + \lambda - \eta + 1\right)^2\right] br^2 - 2qa\left(2\lambda\eta + \lambda - \eta + 1\right)^2\right|}}.$$

Next, if we subtract (18) from (16), we can easily see that

$$4\delta e^{-2\delta}\left(6\lambda\eta + 2(\lambda - \eta) + 1\right)\ \left(a_3 - a_2^2\right) = h_2(r)(u_2 - v_2) + h_3(r)(u_1^2 - v_1^2). \tag{23}$$

In the light of (19) and (20), we conclude from (23) that

$$a_3 = \frac{e^{2\delta} h_2(r)(u_2 - v_2)}{4\delta\left(6\lambda\eta + 2(\lambda - \eta) + 1\right)} + \frac{e^{2\delta} h_2^2(r)(u_1^2 + v_1^2)}{8\left(2\lambda\eta + \lambda - \eta + 1\right)^2}.$$

Thus, by applying (3), we obtain

$$|a_3| \leqq \frac{|br|\, e^{2\delta}}{2\delta\,(6\lambda\eta + 2(\lambda-\eta)+1)} + \frac{b^2 r^2 e^{2\delta}}{4\,(2\lambda\eta+\lambda-\eta+1)^2}.$$

This completes the proof of Theorem 1. □

In the next theorem, we present the Fekete-Szegő inequality for $\mathcal{W}_{\Sigma}^{sc}(\lambda,\eta,\delta,r)$.

Theorem 2 *For* $0 \leqq \eta \leqq \lambda \leqq 1$, $0 < \delta \leqq 1$ *and* $r, \mu \in \mathbb{R}$, *let* $f \in \mathcal{A}$ *be in the family* $\mathcal{W}_{\Sigma}^{sc}(\lambda,\eta,\delta,r)$. *Then*

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} \frac{e^{2\delta}|br|}{2\delta(6\lambda\eta+2(\lambda-\eta)+1)} \\ \left(|\varphi-1| \leqq \frac{\left|\left[\delta(6\lambda\eta+2(\lambda-\eta)+1)b-2p(2\lambda\eta+\lambda-\eta+1)^2\right]br^2-2qa(2\lambda\eta+\lambda-\eta+1)^2\right|}{\delta b^2 r^2(6\lambda\eta+2(\lambda-\eta)+1)}\right) \\ \frac{e^{2\delta}|br|^3|\mu-1|}{2\left|\left[\delta(6\lambda\eta+2(\lambda-\eta)+1)b-2p(2\lambda\eta+\lambda-\eta+1)^2\right]br^2-2qa(2\lambda\eta+\lambda-\eta+1)^2\right|} \\ \left(|\varphi-1| \geqq \frac{\left|\left[\delta(6\lambda\eta+2(\lambda-\eta)+1)b-2p(2\lambda\eta+\lambda-\eta+1)^2\right]br^2-2qa(2\lambda\eta+\lambda-\eta+1)^2\right|}{\delta b^2 r^2(6\lambda\eta+2(\lambda-\eta)+1)}\right). \end{cases}$$

Proof. It follows from (22) and (23) that

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{e^{2\delta} h_2(r)(u_2 - v_2)}{4\delta\,(6\lambda\eta + 2(\lambda-\eta)+1)} + (1-\mu)\, a_2^2 \\ &= \frac{e^{2\delta} h_2(r)(u_2 - v_2)}{4\delta\,(6\lambda\eta + 2(\lambda-\eta)+1)} \\ &\quad + \frac{e^{2\delta} h_2^3(r)(u_2+v_2)\,(1-\mu)}{4\left[\delta h_2^2(r)\,(6\lambda\eta+2(\lambda-\eta)+1) - 2h_3(r)\,(2\lambda\eta+\lambda-\eta+1)^2\right]} \\ &= \frac{h_2(r)}{4}\left[\left(\varphi(\mu,r) + \frac{e^{2\delta}}{\delta\,(6\lambda\eta+2(\lambda-\eta)+1)}\right)u_2 \right. \\ &\quad \left. + \left(\varphi(\mu,r) - \frac{e^{2\delta}}{\delta\,(6\lambda\eta+2(\lambda-\eta)+1)}\right)v_2\right], \end{aligned}$$

where

$$\varphi(\mu,r) = \frac{e^{2\delta} h_2^2(r)\,(1-\mu)}{\delta h_2^2(r)\,(6\lambda\eta+2(\lambda-\eta)+1) - 2h_3(r)\,(2\lambda\eta+\lambda-\eta+1)^2}.$$

Thus, according to (3), we have

$$\left|a_3-\mu a_2^2\right| \leqq \begin{cases} \frac{e^{2\delta}|br|}{2\delta(6\lambda\eta+2(\lambda-\eta)+1)} \\ \left(0 \leqq |\varphi(\mu,r)| \leqq \frac{e^{2\delta}}{\delta(6\lambda\eta+2(\lambda-\eta)+1)}\right) \\ \frac{1}{2}|br| \,.\, |\varphi(\mu,r)| \\ \left(|\varphi(\mu,r)| \geqq \frac{e^{2\delta}}{\delta(6\lambda\eta+2(\lambda-\eta)+1)}\right), \end{cases}$$

which, after simple computation, yields

$$\left|a_3-\mu a_2^2\right| \leqq \begin{cases} \frac{e^{2\delta}|br|}{2\delta(6\lambda\eta+2(\lambda-\eta)+1)} \\ \left(|\varphi-1| \leqq \frac{\left|\left[\delta(6\lambda\eta+2(\lambda-\eta)+1)b-2p(2\lambda\eta+\lambda-\eta+1)^2\right]br^2-2qa(2\lambda\eta+\lambda-\eta+1)^2\right|}{\delta b^2r^2(6\lambda\eta+2(\lambda-\eta)+1)}\right) \\ \frac{e^{2\delta}|br|^3|\mu-1|}{2\left|\left[\delta(6\lambda\eta+2(\lambda-\eta)+1)b-2p(2\lambda\eta+\lambda-\eta+1)^2\right]br^2-2qa(2\lambda\eta+\lambda-\eta+1)^2\right|} \\ \left(|\varphi-1| \geqq \frac{\left|\left[\delta(6\lambda\eta+2(\lambda-\eta)+1)b-2p(2\lambda\eta+\lambda-\eta+1)^2\right]br^2-2qa(2\lambda\eta+\lambda-\eta+1)^2\right|}{\delta b^2r^2(6\lambda\eta+2(\lambda-\eta)+1)}\right). \end{cases}$$

We have thus completed the proof of Theorem 2. □

By putting $\mu=1$ in Theorem 2, we are led to the following corollary.

Corollary 1 *For* $0 \leqq \eta \leqq \lambda \leqq 1$, $0 < \delta \leqq 1$ *and* $r \in \mathbb{R}$, *let* $f \in \mathcal{A}$ *be in the family* $\mathcal{W}_{\Sigma}^{sc}(\lambda,\eta,\delta,r)$. *Then*

$$\left|a_3-a_2^2\right| \leqq \frac{e^{2\delta}\,|br|}{2\delta\,(6\lambda\eta+2(\lambda-\eta)+1)}.$$

3 Conclusion

The fact that we can find many unique and effective uses of a large variety of interesting special functions and specific polynomials in Geometric Function Theory of Complex Analysis provided the primary inspiration for our analysis in this article. The primary objective was to create a new family $\mathcal{W}_{\Sigma}^{sc}(\lambda,\eta,\delta,r)$

of normalized analytic and bi-univalent function which is defined by means of the Borel distribution and also using the Horadam polynomial $h_n(r)$, which are given by the recurrence relation (3) and by generating function $\Pi(r,z)$ in (4). We generate Taylor-Maclaurin coefficient inequalities for functions belonging to this newly introduced bi-univalent function family $\mathcal{W}_{\Sigma}^{sc}(\lambda,\eta,\delta,r)$ and viewed the Fekete-Szegő problem.

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