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A note on convolution of Janowski type functions with q-derivative

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Abstract. The purpose of the present paper is to introduce and study new subclasses of analytic functions which generalize the classes of Janowski functions with q-derivative. We also study certain a convolution conditions, and apply the convolution conditions to get sufficient condition and the neighborhood results related to the functions in the class $\mathcal{S}^q(A, B, \alpha)$.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$.

For f and g be analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$, if there exists an analytic function w in $\mathcal{U}$ such that $|w(z)| < 1$ with $w(0) = 0$, and $f(z) = g(w(z))$, and we denote this by $f(z) \prec g(z)$. If g is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

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Using the principle of the subordination we define the class $\mathcal{P}$ of functions with positive real part.

Definition 1 *[1] Let $\mathcal{P}$ denote the class of analytic functions of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ defined on $\mathcal{U}$ and satisfying $p(0) = 1$, $\operatorname{Re} p(z) > 0$, $z \in \mathcal{U}$.*

Any function p in $\mathcal{P}$ has the representation $p(z) = \dfrac{1 + w(z)}{1 - w(z)}$, where $w \in \Omega$ and

$$\Omega = \{w \in \mathcal{A} : w(0) = 0, |w(z)| < 1\}. \tag{2}$$

Definition 2 *[2] Let $\mathcal{P}[A, B]$, with $-1 \leq B < A \leq 1$, denote the class of analytic function p defined on $\mathcal{U}$ with the representation $p(z) = \dfrac{1 + Aw(z)}{1 + Bw(z)}$, $z \in \mathcal{U}$, where $w \in \Omega$.*

Remark: $p \in \mathcal{P}[A, B]$ if and only if $p(z) \prec \dfrac{1 + Az}{1 + Bz}$.

In [3] the class $\mathcal{P}[A, B, \alpha]$ of generalized Janowski functions was introduced. For arbitrary numbers A, B, α, with $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$, a function p analytic in $\mathcal{U}$ with $p(0) = 1$ is in the class $\mathcal{P}[A, B, \alpha]$ if and only if

$$p(z) \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1 + Bz} \Leftrightarrow p(z) = \frac{1 + [(1-\alpha)A + \alpha B]w(z)}{1 + Bw(z)}, \quad w \in \Omega.$$

In order to define a new class of Janowski symmetrical functions associated with q- derivative defined in the open unit disk $\mathcal{U}$, we first recall the notion of q-derivative.

Jackson[4] initiated q-calculus and developed the concept of the q-integral and q-derivative.

For a function $f \in \mathcal{S}$ given by (1) and $0 < q < 1$, the q-derivative of f is defined by

Definition 3

$$\partial_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0, \qquad\qquad 0 < q < 1. \end{cases} \tag{3}$$

Equivalently (3), may be written as $\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0$ where

$[n]_q = \frac{1-q^n}{1-q}$. Note that as $q \to 1$, $[n]_q \to n$.
Under the hypothesis of the definition of q-difference operator, we have the following rules.
(i) $D_q(af(z) \pm bg(z)) = aD_q f(z) \pm bD_q g(z)$, where a and b any real (or complex) constants
(ii) $D_q(f(z)g(z)) = f(qz)D_q g(z) + g(z)D_q f(z) = f(z)D_q g(z) + g(qz)D_q f(z)$
(iii) $D_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)D_q f(z) - f(z)D_q g(z)}{g(qz)g(z)}$.
The convolution or Hadamard product of two analytic functions $f, g \in \mathcal{A}$ where f is defined by (1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, is

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

It can be easily seen that

$$zD_q f * g = f * zD_q g. \tag{4}$$

Using the generalized Janowski functions and the concept of q-derivative we will de
ne the following classes:

Definition 4 *A function* f *in* $\mathcal{A}$ *is said to belong to the class* $\mathcal{S}^q(A, B, \alpha)$, $(-1 \le B < A \le 1), 0 \le \alpha < 1$ *if*

$$\frac{zD_q f(z)}{f(z)} \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1 + Bz}, \qquad z \in \mathcal{U}.$$

We note that for special values of q, α, A and B yield the following classes. $\mathcal{S}^1(A, B, \alpha)$=$\mathcal{S}(A, B, \alpha)$ is the class introduced by Polatoglu, Bolcal, Sen and Yavuz, [3], $\mathcal{S}^1(A, B, 0)$=$\mathcal{S}(A, B)$ is the class studied by Janowski [2] and etc.

Definition 5 *A function* f *in* $\mathcal{A}$ *is said to belong to the class* $\mathcal{K}^q(A, B, \alpha)$, $(-1 \le B < A \le 1), 0 \le \alpha < 1$ *if*

$$\frac{D_q(zD_q f(z))}{D_q f(z)} \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1 + Bz}, \qquad z \in \mathcal{U}.$$

We need to recall the following neighborhood concept introduced by Goodman [5] and generalized by Ruscheweyh [6]

Definition 6 *For any* $f \in \mathcal{A}$*, which is of the form* (1)*,* ρ*-neighborhood of function* f *can be defined as:*

$$\mathcal{N}_{\rho}(f) = \left\{ g \in \mathcal{A} : g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad \sum_{n=2}^{\infty} n|a_n - b_n| \leq \rho \right\}. \tag{5}$$

For $e(z) = z$*, we can see that*

$$\mathcal{N}_{\rho}(e) = \left\{ g \in \mathcal{A} : g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad \sum_{n=2}^{\infty} n|b_n| \leq \rho \right\}. \tag{6}$$

Ruscheweyh [6] proved, among other results that for all $\eta \in \mathbb{C}$, with $|\eta| < \rho$,

$$\frac{f(z) + \eta z}{1 + \eta} \in \mathcal{S}^* \Rightarrow \mathcal{N}_{\rho}(f) \subset \mathcal{S}^*.$$

In this paper, we investigate a sufficient condition and convolution property. Finally motivated by Definition 6, we give analogous definition of neighborhood for the class $\mathcal{S}^q(A, B, \alpha, b)$, proof the convolution Lemma and then investigate related neighborhood result for this new class.

2 Main results

Theorem 1 *The function* $f \in \mathcal{K}^q(A, B, \alpha)$ *if and only if*

$$\frac{1}{z}\left[f * \frac{xz + \left(x + \frac{[2]_q(1+Ax)}{(B-A)(1-\alpha)} \right) qz^2 + \frac{(1+q-[2]_q)(1+Ax)}{(B-A)(1-\alpha)} qz^3}{(1-z)(1-qz)(1-q^2z)} \right] \neq 0$$

where $0 < q < 1, \ -1 \leq B < A \leq 1, 0 \leq \alpha < 1$ *and* $|z| < R \leq 1, |x| = 1$.

Proof. The function $f \in \mathcal{K}^q(A, B, \alpha)$ if and only if

$$\frac{D_q(zD_qf(z))}{D_qf(z)} \in P(A, B, \alpha), \qquad \text{for all } z \in \mathcal{U}. \tag{7}$$

Since $\frac{D_q(zD_qf)}{D_qf} = 1$ at $z = 0$, so (7) is equivalent to

$$\frac{D_q\left(zD_qf\right)}{D_qf} \neq \frac{1 + [(1-\alpha)A + \alpha B]x}{1 + Bx}, \quad (|z| < R, \ |x| = 1, \ x \neq -1)$$

which implies

$$(1+Bx)D_q\left(zD_qf\right)-(1+[(1-\alpha)A+\alpha B]x)D_qf\neq 0. \tag{8}$$

Setting $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, we have

$$D_qf=1+\sum_{n=2}^{\infty}[n]_qa_nz^{n-1}$$

$$D_q\left(zD_qf\right)=1+\sum_{n=2}^{\infty}[n]_q^2a_nz^{n-1}=D_qf*\frac{1}{(1-z)(1-qz)}.$$

The left hand side of (8) is equivalent to

$$(1{+}Bx)\left[D_qf*\sum_{n=1}^{\infty}[n]_qz^{n-1}\right]{-}D_qf{*}\sum_{n=1}^{\infty}(1{+}[(1{-}\alpha)A{+}\alpha B]x)z^{n-1}$$

$$=D_qf*\sum_{n=1}^{\infty}\left[(1+Bx)[n]_q-(1+[(1-\alpha)A+\alpha B]x)\right]z^{n-1}$$

$$=D_qf*\left(\frac{-(1+[(1-\alpha)A+\alpha B]x)}{1-z}+\frac{1+Bx}{(1-z)(1-qz)}\right)$$

$$=D_qf*\left(\frac{x((B-A)(1-\alpha)+(1+[(1-\alpha)A+\alpha B]x)qz}{(1-z)(1-qz)}\right).$$

Thus

$$\frac{1}{z}\left[zD_qf*\frac{xz+\frac{(1+[(1-\alpha)A+\alpha B]x)}{(B-A)(1-\alpha)}qz^2}{(1-z)(1-qz)}\right]\neq 0. \tag{9}$$

By using (4), we can write (9) as

$$\frac{1}{z}\left[f*\frac{xz+\left(x+\frac{[2]_q(1+[(1-\alpha)A+\alpha B]x)}{(B-A)(1-\alpha)}\right)qz^2+\frac{(1+q-[2]_q)(1+[(1-\alpha)A+\alpha B]x)}{(B-A)(1-\alpha)}qz^3}{(1-z)(1-qz)(1-q^2z)}\right]\neq 0$$

which completes the proof. □

As $q\to 1^-$, and $\alpha=0$ we have following result proved by Ganesan and et al. in [7].

Corollary 1 *The function* $f \in C(A, B)$ *in* $|z| < R \leq 1$ *if and only if*

$$\frac{1}{z}\left[f * \frac{xz + \frac{(Ax+Bx+2)}{B-A}z^2}{(1-z)^3}\right] \neq 0.$$

Remark 1 *As* $q \to 1^-, \alpha = 0$ *and* $A = 1,\ B = -1$*, we get convolution condition characterizing convex functions as in Silverman and et al. in [8] with a suitable modification.*

Theorem 2 *The function* $f \in \mathcal{S}^q(A, B, \alpha)$ *in* $|z| < R \leq 1$ *if and only if*

$$\frac{1}{z}\left[f * \frac{xz + \frac{1+[(1-\alpha)A+\alpha B]x}{(B-A)(1-\alpha)}qz^2}{(1-z)(1-qz)}\right] \neq 0, \quad (|z| < R,\ |x| = 1).$$

Proof. Since $f \in \mathcal{S}^q(A, B, \alpha)$ if and only if $g(z) = \int_0^z \frac{f(\zeta)}{\zeta} d_q\zeta \in \mathcal{K}^q(A, B, \alpha)$, we have

$$\frac{1}{z}\left[g * \frac{xz + \left(x + \frac{[2]_q(1+[(1-\alpha)A+\alpha B]x)}{(B-A)(1-\alpha)}\right)qz^2 + \frac{(1+q-[2]_q)(1+[(1-\alpha)A+\alpha B]x)}{(B-A)(1-\alpha)}qz^3}{(1-z)(1-qz)(1-q^2z)}\right]$$

$$= \frac{1}{z}\left[f * \frac{xz + \frac{1+[(1-\alpha)A+\alpha B]x}{(B-A)(1-\alpha)}qz^2}{(1-z)(1-qz)}\right].$$

Thus the result follows from Theorem 1. □

Remark 2 *Note that from The Theorem 2 we can easily obtain that the equivalent condition for a function* f *belonging to the class* $\mathcal{S}^q(A, B, \alpha)$ *if and only if*

$$\frac{(f * g)(z)}{z} \neq 0, \quad g \in \mathcal{A}, z \in \mathcal{U}, \tag{10}$$

where $g(z)$ *has the form*

$$g(z) = z + \sum_{n=2}^{\infty} t_n z^n,$$

$$t_n = \frac{[n]_q - 1 + ([n]_q B - [(1-\alpha)A + \alpha B])x}{(B-A)(1-\alpha)x}. \tag{11}$$

As $q \to 1^-$ and $\alpha = 0$ in Theorem 2 we have following result proved by Ganesan and et al. in [7].

Corollary 2 *The function* $f \in S^*(A, B)$ *in* $|z| < R \leq 1$ *if and only if*

$$\frac{1}{z}\left[f * \frac{xz + \frac{1+Ax}{B-A}z^2}{(1-z)^2}\right] \neq 0, \quad (|z| < R, \ |x| = 1).$$

Theorem 3 *Let* f *be a function defined* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$*, which is analytic in* $\mathcal{U}$*, for* $-1 \leq B < A \leq 1$*, and* $0 \leq \alpha < 1$*, if*

$$\sum_{n=2}^{\infty} \{([n]_q - 1) + |[(1-\alpha)A + \alpha B] - B[n]_q|\} |a_n| \leq (A - B)(1 - \alpha),$$

then $f(z) \in \mathcal{S}^q(A, B, \alpha)$*.*

Proof.

For the proof of Theorem 3, it suffices to show that $\frac{(f*g)(z)}{z} \neq 0$ where g is given by (11). Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} t_n z^n$. The convolution

$$\frac{(f*g)(z)}{z} = 1 + \sum_{n=2}^{\infty} t_n a_n z^{n-1}, z \in \mathcal{U}. \tag{12}$$

It is known from Theorem 2 that $f(z) \in \mathcal{S}^q(A, B, \alpha)$ if and only if $\frac{(f*g)(z)}{z} \neq 0$, for g given by (11). Using (11) and (12), we get

$$\left|\frac{(f*g)(z)}{z}\right| \geq 1 - \sum_{n=2}^{\infty} \frac{[n]_q - 1 + |[n]_q B - [(1-\alpha)A + \alpha B]|}{|(B-A)(1-\alpha)|} |a_n||z|^{n-1} > 0, \ z \in \mathcal{U}.$$

Thus, $f \in \mathcal{S}^q(A, B, \alpha)$.

□

To find some neighborhood results for the class $\mathcal{S}^q(A, B, \alpha)$ analogous to those obtained by Ruscheweyh [6], we need the following concept of neighborhood.

Definition 7 *For* $-1 \leq B < A \leq 1, 0 \leq \alpha < 1$ *and* $\rho \geq 0$ *we define* $\mathcal{N}^q(A, B, \alpha; f, \rho)$ *the neighborhood of a function* $f \in \mathcal{A}$ *as*

$$\mathcal{N}^q(A, B, \alpha; f, \rho) = \Bigg\{ g \in \mathcal{A} : g(z) = z + \sum_{n=2}^{\infty} b_n z^n, d(f, g) = \sum_{n=2}^{\infty} \frac{([n]_q - 1) + |[(1-\alpha)A + \alpha B] - B[n]_q|}{(1-\alpha)(A-B)} |b_n - a_n| \leq \rho \Bigg\}, \tag{13}$$

where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.

Remark 3 *For parametric values* $q \to 1, A = -B = 1,$ *and* $\alpha = 0$ (13) *reduces to* (5).

Theorem 4 *Let* f *be a function defined* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, *which is analytic in* $\mathcal{U}$, *and for all complex number* η, *with* $|\eta| < \rho$, *if*

$$\frac{f(z) + \eta z}{1 + \eta} \in \mathcal{S}^q(A, B, \alpha), \tag{14}$$

then

$$\mathcal{N}^q(A, B, \alpha; f, \rho) \subset \mathcal{S}^q(A, B, \alpha).$$

Proof. We assume that a function h defined by $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is in the class $\mathcal{N}^q(A, B, \alpha; f, \rho)$. In order to prove the theorem, we only need to prove that $h \in \mathcal{S}^q(A, B, \alpha)$. We would prove this claim in next three steps.
We first note that Theorem 2 is equivalent to

$$f \in \mathcal{S}^q(A, B, \alpha) \Leftrightarrow \frac{1}{z}[(f * g)(z)] \neq 0, \quad z \in \mathcal{U}, \tag{15}$$

where is given by (11). For $|x| = 1, -1 \leq B < A \leq 1$, and $0 \leq \alpha < 1$.
We can write $g(z) = z + \sum_{n=2}^{\infty} t_n z^n$,
where

$$t_n = \frac{([n]_q - 1) + |[(1-\alpha)A + \alpha B] - B[n]_q| x}{(1-\alpha)(B-A)x}, \tag{16}$$

Secondly we obtain that (14) is equivalent to

$$\left| \frac{f(z) * g(z)}{z} \right| \geq \rho, \tag{17}$$

because, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ and satisfy (14), then (15) is equivalent to

$$g \in \mathcal{S}^q(A, B, \alpha) \Leftrightarrow \frac{1}{z}\left[\frac{f(z) * g(z)}{1+\eta}\right] \neq 0, \qquad |\eta| < \rho.$$

Thirdly letting $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$ we notice that

$$\begin{aligned}
\left|\frac{h(z) * g(z)}{z}\right| &= \left|\frac{f(z) * g(z)}{z} + \frac{(h(z) - f(z)) * g(z)}{z}\right| \\
&\geq \rho - \left|\frac{(h(z) - f(z)) * g(z)}{z}\right|, \quad \text{(by using (17))} \\
&= \rho - \left|\sum_{n=2}^{\infty} (b_n - a_n) t_n z^n\right|, \\
&\geq \rho - |z| \sum_{n=2}^{\infty} \left[\frac{([n]_q - 1) + |[(1-\alpha)A + \alpha B] - B[n]_q|}{|(1-\alpha)(B-A)|}\right] |b_n - a_n| \\
&\geq \rho - \rho|z| > 0, \qquad \text{by applying (16).}
\end{aligned}$$

This prove that

$$\frac{h(z) * g(z)}{z} \neq 0, \qquad z \in \mathcal{U}.$$

In view of our observations (15), it follows that $h \in \mathcal{S}^q(A, B, \alpha)$. This completes the proof of the theorem. □

When $q \to 1, A = -B = 1$ and $\alpha = 0$ in the above theorem we get (6) proved by Ruscheweyh in [6].

Corollary 3 *Let $\mathcal{S}^*$ be the class of starlike functions. Let $f \in \mathcal{A}$ and for all complex numbers η, with $|\mu| < \rho$, if*

$$\frac{f(z) + \eta z}{1+\eta} \in \mathcal{S}^*, \tag{18}$$

then $\mathcal{N}_\sigma(f) \subset \mathcal{S}^*$.

Theorem 5 *Let $f \in \mathcal{S}^q(A, B, \alpha)$, for $\rho < c$. Then*

$$\mathcal{N}^q(A, B, \alpha; f, \rho) \subset \mathcal{S}^q(A, B, \alpha).$$

Where
c is a non-zero real number with $c \leq \left|\frac{(f*g)(z)}{z}\right|, z \in \mathcal{U}$ and g is defined in Remark 2.

Proof. Let $h = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{N}^q(A, B, \alpha; f, \rho)$. For the proof of Theorem 5, it suffices to show that $\frac{(h*g)(z)}{z} \neq 0$ where g is given by (11). Consider

$$\left|\frac{h(z) * g(z)}{z}\right| \geq \left|\frac{f(z) * g(z)}{z}\right| - \left|\frac{(h(z) - f(z)) * g(z)}{z}\right|. \tag{19}$$

Since $f \in \mathcal{S}^q(A, B, \alpha)$, therefore applying Theorem 3, we obtain

$$\left|\frac{(f * g)(z)}{z}\right| \geq c, \tag{20}$$

where c is a non-zero real number and $z \in \mathcal{U}$. Now

$$\begin{aligned}\left|\frac{(h(z) - f(z)) * g(z)}{z}\right| &= \left|\sum_{n=2}^{\infty}(b_n - a_n)t_n z^n\right| \\ &\leq \sum_{n=2}^{\infty}\left[\frac{([n]_q - 1) + |[(1-\alpha)A + \alpha B] - B[n]_q|}{|(1-\alpha)(B - A)|}\right]|b_n - a_n| = \rho, \end{aligned} \tag{21}$$

using (20) and (21) in (19), we obtain

$$\left|\frac{h(z) * g(z)}{z}\right| \geq c - \rho > 0,$$

where $\rho < c$. This completes the proof. □

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