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Meromorphic functions sharing fixed points and poles with finite weights

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Abstract. In the paper, with the aid of weighted sharing method we study the problems of meromorphic functions that share fixed points (or a nonzero finite value) and poles with finite weights. The results of the paper improve some recent results due to Y. H. Cao and X. B. Zhang [Journal of Inequalities and Applications, 2012:100].

1 Introduction, definitions and results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [8], [15] and [16]. For a nonconstant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic of f and by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ possibly outside a set of finite linear measure. A meromorphic function $\alpha(z)(\not\equiv\infty)$ is called a small function with respect to f, provided that $T(r,\alpha)=S(r,f)$.

We say that two meromorphic functions f and g share a small function $a(z)$ CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share $a(z)$ IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM, and we say that f and g share ∞

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IM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM. A finite value z_0 is a fixed point of $f(z)$ if $f(z_0) = z_0$ and we define

$$E_f = \{z \in \mathbb{C} : f(z) = z, \text{counting multiplicities}\}.$$

In 1995, W. Bergweiler and A. Eremenko, H. H. Chen and M. L. Fang, L. Zalcman respectively proved the following result.

Theorem A (see ([3], Theorem 2), ([5], Theorem 1) and [17]) *Let f be a transcendental meromorphic function and $n(\geq 1)$ is an integer. Then $f^n f' = 1$ has infinitely many solutions.*

In 1997, C. C. Yang and X. H. Hua proved the following result, which corresponded to Theorem A.

Theorem B (see [14], Theorem 1) *Let f and g be two nonconstant meromorphic functions, $n \geq 11$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

In 2000, M. L. Fang proved the following result.

Theorem C (see [6], Theorem 2) *Let f be a transcendental meromorphic function, and let n be a positive integer. Then $f^n f' - z = 0$ has infinitely many solutions.*

In 2002, M. L. Fang and H. L. Qiu proved the following result, which corresponded to Theorem C.

Theorem D (see [7], Theorem 1) *Let f and g be two nonconstant meromorphic functions, and let $n \geq 11$ be a positive integer. If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a complex number t such that $t^{n+1} = 1$.*

In 2009, J. F. Xu, H. X. Yi and Z. L. Zhang proved the following result.

Theorem E (see [12]) *Let f be a transcendental meromorphic function, $n(\geq 2)$, k be two positive integers. Then $f^n f^{(k)}$ takes every finite nonzero value infinitely many times or has infinitely many fixed points.*

Regarding Theorem E, it is natural to ask the following question:

Question 1 *Is there a corresponding uniqueness theorem to Theorem E?*

Recently, Y. H. Cao and X. B. Zhang proved the following results which deal with Question 1.

Theorem F (see [4], Theorem 1.1) *Let* f *and* g *be two transcendental meromorphic functions, whose zeros are of multiplicities at least* k*, where* k *is a positive integer. Let* $n > \max\{2k-1, k+4/k+4\}$ *be a positive integer. If* $f^n f^{(k)}$ *and* $g^n g^{(k)}$ *share* z *CM,* f *and* g *share* ∞ *IM, then one of the following two conclusions hold:*

(i) $f^n f^{(k)} = g^n g^{(k)}$;

(ii) $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, *where* c_1, c_2 *and* c *are constants satisfying* $4(c_1c_2)^{n+1}c^2 = -1$.

Theorem G (see [4], Theorem 1.2) *Let* f *and* g *be two nonconstant meromorphic functions, whose zeros are of multiplicities at least* k*, where* k *is a positive integer. Let* $n > \max\{2k-1, k+4/k+4\}$ *be a positive integer. If* $f^n f^{(k)}$ *and* $g^n g^{(k)}$ *share* 1 *CM,* f *and* g *share* ∞ *IM, then one of the following two conclusions hold:*

(i) $f^n f^{(k)} = g^n g^{(k)}$;

(ii) $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, *where* c_3, c_4 *and* d *are constants satisfying* $(-1)^k (c_3c_4)^{n+1} d^{2k} = 1$.

Regarding Theorem F and Theorem G, one may ask the following questions which are the motive of the author.

Question 2 *Is it really possible in any way to relax the nature of sharing the fixed point (*1*-point) in Theorem F (Theorem G) without increasing the lower bound of* n*?*

Question 3 *What will be the IM-analogous of Theorems F and G?*

In the paper, we will prove two theorems first one of which improves Theorem F and second one improves Theorem G and dealt with Question 2 and Question 3. To state the main results of the paper we need the following notion of weighted sharing of values introduced by I. Lahiri [9, 10] which measure how close a shared value is to being shared CM or to being shared IM.

Definition 1 *Let* k *be a nonnegative integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $E_k(a; f)$ *the set of all* a*-points of* f *where an* a*-point of multiplicity* *m is counted m times if* $m \leq k$ *and k+1 times if* $m > k$. *If* $E_k(a; f) = E_k(a; g)$, *we say that* f, g *share the value* a *with weight k.*

The definition implies that if f, g *share a value* a *with weight* k, *then* z_0 *is an* a*-point of* f *with multiplicity* $m(\leq k)$ *if and only if it is an* a*-point of* g *with multiplicity* $m(\leq k)$ *and* z_0 *is an a-point of* f *with multiplicity* $m(> k)$ *if and only if it is an a-point of* g *with multiplicity* $n(> k)$, *where m is not necessarily equal to n.*

We write f, g *share* (a, k) *to mean that* f, g *share the value* a *with weight k. Clearly if* f, g *share* (a, k) *then* f, g *share* (a, p) *for any integer* p, $0 \leq p < k$. *Also we note that* f, g *share a value* a *IM or CM if and only if* f, g *share* $(a, 0)$ *or* (a, ∞) *respectively.*

We now state the main results of the paper.

Theorem 1 *Let* f *and* g *be two transcendental meromorphic functions, whose zeros are of multiplicities at least* k, *where* k *is a positive integer. If* $f^n f^{(k)}$ *and* $g^n g^{(k)}$ *share* (z, l), *where* l, n *are positive integers;* f *and* g *share* ∞ *IM, then conclusions of Theorem F hold provided one of the following holds:*

(i) $l \geq 2$ *and* $n > \max\{2k-1, k+4/k+4\}$;

(ii) $l = 1$ *and* $n > \max\{2k-1, 3k/2+5/k+5\}$;

(iii) $l = 0$ *and* $n > \max\{2k-1, 4k+10/k+10\}$.

Theorem 2 *Let* f *and* g *be two nonconstant meromorphic functions, whose zeros are of multiplicities at least* k, *where* k *is a positive integer. If* $f^n f^{(k)}$ *and* $g^n g^{(k)}$ *share* $(1, l)$, *where* l, n *are positive integers;* f *and* g *share* ∞ *IM, then conclusions of Theorem G hold provided one of the following holds:*

(i) $l \geq 2$ *and* $n > \max\{2k-1, k+4/k+4\}$;

(ii) $l = 1$ *and* $n > \max\{2k-1, 3k/2+5/k+5\}$;

(iii) $l = 0$ *and* $n > \max\{2k-1, 4k+10/k+10\}$.

We now explain some definitions and notations which are used in the paper.

Definition 2 [8] *For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $N(r, a; f \mid= 1)$ *the counting functions of simple* a*-points of* f. *For a positive integer* p *we denote by* $N(r, a; f \mid\geq p)$ *the counting function of those* a*-points of* f *(counted with proper multiplicities) whose multiplicities are not less than* p. *By* $\overline{N}(r, a; f \mid\geq p)$ *we denote the corresponding reduced counting function.*

Analogously we can define $N(r, a; f \mid\leq p)$ *and* $\overline{N}(r, a; f \mid\leq p)$.

Definition 3 [10] *Let* k *be a positive integer or infinity. We denote by* $N_k(r, a; f)$ *the counting function of* a*-points of* f*, where an* a*-point of multiplicity* m *is counted* m *times if* $m \leq k$ *and* k *times if* $m > k$*. Then*

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \cdots + \overline{N}(r, a; f \mid\geq k).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 4 [1] *Let* f *and* g *be two nonconstant meromorphic functions such that* f *and* g *share the value* 1 *IM. Let* z_0 *be a* 1*-point of* f *with multiplicity* p *and also a* 1*-point of* g *with multiplicity* q*. We denote by* $\overline{N}_L(r, 1; f)$ *the counting function of those* 1*-points of* f *and* g*, where* $p > q$*, by* $N_E^{(k}(r, 1; f)$ ($k \geq 2$ *is an integer) the counting function of those* 1*-points of* f *and* g*, where* $p = q \geq k$*, where each point in these counting functions is counted only once. In the same manner we can define* $\overline{N}_L(r, 1; g)$ *and* $N_E^{(k}(r, 1; g)$.

Definition 5 [9, 10] *Let* f *and* g *be two nonconstant meromorphic functions such that* f *and* g *share the value* a *IM. We denote by* $\overline{N}_*(r, a; f, g)$ *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the multiplicities of the corresponding* a*-points of* g*. Clearly* $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ *and* $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two nonconstant meromorphic functions defined in $\mathbb{C}$. We shall denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 1 [13] *Let* f *be a nonconstant meromorphic function and let* $a_n(z)(\not\equiv 0)$, $a_{n-1}(z), \ldots, a_0(z)$ *be meromorphic functions such that* $T(r, a_i(z)) = S(r, f)$ *for* $i = 0, 1, 2, \ldots, n$*. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2 [16] *Let* f *be a nonconstant meromorphic function, and let* k *be positive integer. Suppose that* $f^{(k)} \not\equiv 0$*. Then*

$$N\left(r, 0; f^{(k)}\right) \leq N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f). \tag{1}$$

Lemma 3 [18] *Let* f *be a nonconstant meromorphic function, and* p, k *be positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right) \le k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f). \tag{2}$$

Lemma 4 [2] *Let* F, G *be two nonconstant meromorphic functions sharing* $(1,2)$, $(\infty,0)$ *and* $H \not\equiv 0$. *Then*

(i) $T(r,F) \le N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)- m(r,1;G) - N_E^{(3}(r,1;F) - \overline{N}_L(r,1;G) + S(r,F) + S(r,G)$;

(ii) $T(r,G) \le N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)- m(r,1;F) - N_E^{(3}(r,1;G) - \overline{N}_L(r,1;F) + S(r,F) + S(r,G)$.

Lemma 5 [11] *Let* F, G *be two nonconstant meromorphic functions sharing* $(1,1)$, $(\infty,0)$ *and* $H \not\equiv 0$. *Then*

(i) $T(r,F) \le N_2(r,0;F)+N_2(r,0;G)+\frac{3}{2}\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\frac{1}{2}\overline{N}(r,0;F)+ \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G)$;

(ii) $T(r,G) \le N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\frac{3}{2}\overline{N}(r,\infty;G)+\frac{1}{2}\overline{N}(r,0;G)+ \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G)$.

Lemma 6 [11] *Let* F, G *be two nonconstant meromorphic functions sharing* $(1,0)$, $(\infty,0)$ *and* $H \not\equiv 0$. *Then*

(i) $T(r,F) \le N_2(r,0;F)+N_2(r,0;G)+3\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+2\overline{N}(r,0;F)+ \overline{N}(r,0;G) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G)$;

(ii) $T(r,G) \le N_2(r,0;F)+N_2(r,0;G)+2\overline{N}(r,\infty;F)+3\overline{N}(r,\infty;G)+\overline{N}(r,0;F)+ 2\overline{N}(r,0;G) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G)$.

Lemma 7 [4] *Let* f *and* g *be nonconstant meromorphic functions, whose zeros are of multiplicities at least* k, *where* k *is a positive integer. Let* $n > 2k-1$ *be a positive integer. If* f, g *share* ∞ *IM and if* $f^n f^{(k)} g^n g^{(k)} = z^2$, *then* $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, *where* c_1, c_2 *and* c *are three constants satisfying* $4(c_1c_2)^{n+1}c^2 = -1$.

Lemma 8 [4] *Let* f *and* g *be nonconstant meromorphic functions, whose zeros are of multiplicities at least* k, *where* k *is a positive integer. Let* $n > 2k-1$ *be a positive integer. If* f, g *share* ∞ *IM and if* $f^n f^{(k)} g^n g^{(k)} = 1$, *then* $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, *where* c_3, c_4 *and* d *are three constants satisfying* $(-1)^k (c_3c_4)^{n+1} d^{2k} = 1$.

3 Proof of the theorems

Proof of Theorem 1. We consider $F(z) = f^n f^{(k)}$, $G(z) = g^n g^{(k)}$, $F_1(z) = F(z)/z$ and $G_1(z) = G(z)/z$. Then F_1, G_1 are transcendental meromorphic functions that share $(1, l)$ and f, g share $(\infty, 0)$. Since f and g are transcendental, z is a small function with respect to both F and G. We now discuss the following two cases separately.

Case 1 *We assume that* $H \not\equiv 0$. *Now we consider the following three subcases.*

Subcase 1 *Suppose that* $l \geq 2$. *Then using Lemma 4 we obtain*

$$\begin{aligned} T(r,F) &\leq T(r,F_1) + S(r,F) \\ &\leq N_2(r,0;F_1) + N_2(r,0;G_1) + \overline{N}(r,\infty;F_1) + \overline{N}(r,\infty;G_1) \\ &\quad + \overline{N}_*(r,\infty;F_1,G_1) - m(r,1;G_1) - N_E^{(3}(r,1;F_1) \\ &\quad - \overline{N}_L(r,1;G_1) + S(r,F_1) + S(r,G_1) \\ &\leq N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \\ &\quad + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G). \end{aligned} \tag{3}$$

Noting that

$$\begin{aligned} \overline{N}_*(r,\infty;F,G) &= \overline{N}_L(r,\infty;F) + \overline{N}_L(r,\infty;G) \\ &\leq \overline{N}(r,\infty;F) = \overline{N}(r,\infty;G), \end{aligned} \tag{4}$$

we obtain from (3) that

$$\begin{aligned} T(r,F) \leq\ & N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \\ & + S(r,F) + S(r,G). \end{aligned} \tag{5}$$

Obviously,

$$N(r,\infty;F) = (n+1)N(r,\infty;f) + k\overline{N}(r,\infty;f) + S(r,f). \tag{6}$$

Again

$$\begin{aligned} nm(r,f) &= m(r,F/f^{(k)}) \leq m(r,F) + m(r,1/f^{(k)}) + S(r,f) \\ &= m(r,F) + T(r,f^{(k)}) - N(r,0;f^{(k)}) + S(r,f) \\ &\leq m(r,F) + T(r,f) + k\overline{N}(r,\infty;f) - N(r,0;f^{(k)}) + S(r,f). \end{aligned} \tag{7}$$

From (6) and (7) we obtain

$$(n-1)T(r,f) \quad \le \quad T(r,F) - N(r,\infty;f) - N(r,0;f^{(k)}) + S(r,f). \tag{8}$$

Similarly,

$$(n-1)T(r,g) \quad \le \quad T(r,G) - N(r,\infty;g) - N(r,0;g^{(k)}) + S(r,g). \tag{9}$$

Using (6), Lemma 2 we obtain from (8)

$$\begin{aligned}(n-1)T(r,f) &\le N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G)\\ &\quad - N(r,\infty;f) - N(r,0;f^{(k)}) + S(r,f) + S(r,g)\\ &\le N_2(r,0;f) + N_2(r,0;g) + N_2(r,0;f^{(k)}) + N_2(r,0;g^{(k)})\\ &\quad + 2\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) - N(r,\infty;f)\\ &\quad - N(r,0;f^{(k)}) + S(r,f) + S(r,g)\\ &\le 2\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + N(r,0;f^{(k)}) + N(r,0;g^{(k)})\\ &\quad + 2N(r,\infty;f) + \overline{N}(r,\infty;g) - N(r,\infty;f)\\ &\quad - N(r,0;f^{(k)}) + S(r,f) + S(r,g)\\ &\le 2\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + N(r,0;g) + N(r,\infty;f)\\ &\quad + (k+1)\overline{N}(r,\infty;g) + S(r,f) + S(r,g)\\ &\le \frac{2}{k}N(r,0;f) + \frac{2}{k}N(r,0;g) + N(r,0;g) + N(r,\infty;g)\\ &\quad + (k+1)\overline{N}(r,\infty;g) + S(r,f) + S(r,g)\\ &\le \frac{2}{k}(T(r,f) + T(r,g)) + (k+3)T(r,g)\\ &\quad + S(r,f) + S(r,g).\end{aligned} \tag{10}$$

Similarly,

$$\begin{aligned}(n-1)T(r,g) \quad \le \quad & \frac{2}{k}(T(r,f) + T(r,g)) + (k+3)T(r,f)\\ & + S(r,f) + S(r,g).\end{aligned} \tag{11}$$

Combining (10) and (11) we get

$$(n - k - 4/k - 4)(T(r,f) + T(r,g)) \le S(r,f) + S(r,g),$$

a contradiction with the fact that $n > k + 4/k + 4$.

Subcase 2 *Let* $l = 1$*. Then using (4) and Lemma 5 we obtain*

$$\begin{aligned}
T(r,F) &\leq T(r,F_1) + S(r,F) \\
&\leq N_2(r,0;F_1) + N_2(r,0;G_1) + \frac{3}{2}\overline{N}(r,\infty;F_1) + \overline{N}(r,\infty;G_1) \\
&\quad + \overline{N}_*(r,\infty;F_1,G_1) + \frac{1}{2}\overline{N}(r,0;F_1) + S(r,F_1) + S(r,G_1) \\
&\leq N_2(r,0;F) + N_2(r,0;G) + \frac{3}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \qquad (12) \\
&\quad + \overline{N}_*(r,\infty;F,G) + \frac{1}{2}\overline{N}(r,0;F) + S(r,F) + S(r,G) \\
&\leq N_2(r,0;F) + N_2(r,0;G) + \frac{5}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \\
&\quad + \frac{1}{2}\overline{N}(r,0;F) + S(r,F) + S(r,G).
\end{aligned}$$

Using (12), Lemma 2 and Lemma 3 we obtain from (8)

$$\begin{aligned}
(n-1)T(r,f) \quad &\leq \quad N_2(r,0;F) + N_2(r,0;G) + \frac{5}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \\
&\quad +\frac{1}{2}\overline{N}(r,0;F) - N(r,\infty;f) - N(r,0;f^{(k)}) \\
&\quad +S(r,f) + S(r,g) \\
&\leq \quad N_2(r,0;f) + N_2(r,0;g) + N_2(r,0;f^{(k)}) + N_2(r,0;g^{(k)}) \\
&\quad +\frac{5}{2}\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \frac{1}{2}\overline{N}(r,0;f) \\
&\quad +\frac{1}{2}\overline{N}(r,0;f^{(k)}) - N(r,\infty;f) - N(r,0;f^{(k)}) \\
&\quad +S(r,f) + S(r,g) \\
&\leq \quad \frac{5}{2}\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + N(r,0;f^{(k)}) + N(r,0;g^{(k)}) \\
&\quad +\frac{5}{2}N(r,\infty;f) + \overline{N}(r,\infty;g) + \frac{1}{2}N_{k+1}(r,0;f) \\
&\quad +\frac{k}{2}\overline{N}(r,\infty;f) - N(r,\infty;f) - N(r,0;f^{(k)}) \\
&\quad +S(r,f) + S(r,g) \\
&\leq \quad \frac{k+6}{2}\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + (k+1)\overline{N}(r,\infty;g) \\
&\quad +N(r,0;g) + \frac{k+3}{2}N(r,\infty;f) + S(r,f) + S(r,g)
\end{aligned}$$

$$\leq \frac{k+6}{2k}N(r,0;f)+\frac{k+2}{k}N(r,0;g)+\frac{3k+5}{2}N(r,\infty;g)$$
$$+S(r,f)+S(r,g)$$
$$\leq \left(\frac{2}{k}+\frac{1}{2}\right)(T(r,f)+T(r,g))+\frac{1}{k}T(r,f)+\frac{3k+6}{2}T(r,g)$$
$$+S(r,f)+S(r,g).$$

This implies

$$\left(n-1-\frac{1}{k}\right)T(r,f)\leq \left(\frac{2}{k}+\frac{1}{2}\right)(T(r,f)+T(r,g)) + \frac{3k+6}{2}T(r,g)+S(r,f)+S(r,g). \tag{13}$$

Similarly,

$$\left(n-1-\frac{1}{k}\right)T(r,g)\leq \left(\frac{2}{k}+\frac{1}{2}\right)(T(r,f)+T(r,g)) + \frac{3k+6}{2}T(r,f)+S(r,f)+S(r,g). \tag{14}$$

From (13) and (14) we obtain

$$\left(n-\frac{3k}{2}-\frac{5}{k}-5\right)(T(r,f)+T(r,g))\leq S(r,f)+S(r,g),$$

a contradiction with our assumption that $n > 3k/2+5/k+5$.

Subcase 3 *Let* $l=0$. *Then using (4) and Lemma 6 we obtain*

$$\begin{aligned} T(r,F) &\leq T(r,F_1)+S(r,F)\\ &\leq N_2(r,0;F_1)+N_2(r,0;G_1)+3\overline{N}(r,\infty;F_1)+2\overline{N}(r,\infty;G_1)\\ &\quad +\overline{N}_*(r,\infty;F_1,G_1)+2\overline{N}(r,0;F_1)+\overline{N}(r,0;G_1)\\ &\quad +S(r,F_1)+S(r,G_1)\\ &\leq N_2(r,0;F)+N_2(r,0;G)+3\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)\\ &\quad +\overline{N}_*(r,\infty;F,G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G)\\ &\leq N_2(r,0;F)+N_2(r,0;G)+4\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)\\ &\quad +2\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G). \end{aligned} \tag{15}$$

Using (15), Lemma 2 and Lemma 3 we obtain from (8)

$$\begin{aligned}
(n-1)T(r,f) &\leq N_2(r,0;F)+N_2(r,0;G)+4\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)\\
&\quad +2\overline{N}(r,0;F)+\overline{N}(r,0;G)-N(r,\infty;f)-N(r,0;f^{(k)})\\
&\quad +S(r,f)+S(r,g)\\
&\leq 4\overline{N}(r,0;f)+3\overline{N}(r,0;g)+N_2(r,0;f^{(k)})+N_2(r,0;g^{(k)})\\
&\quad +4\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+2\overline{N}(r,0;f^{(k)})+\overline{N}(r,0;g^{(k)})\\
&\quad -N(r,\infty;f)-N(r,0;f^{(k)})+S(r,f)+S(r,g)\\
&\leq 4\overline{N}(r,0;f)+3\overline{N}(r,0;g)+N(r,0;g^{(k)})+2\overline{N}(r,0;f^{(k)})\\
&\quad +\overline{N}(r,0;g^{(k)})+3N(r,\infty;f)+2\overline{N}(r,\infty;g)\\
&\quad +S(r,f)+S(r,g)\\
&\leq 4\overline{N}(r,0;f)+3\overline{N}(r,0;g)+N(r,0;g)+2N_{k+1}(r,0;f)\\
&\quad +N_{k+1}(r,0;g)+(2k+3)N(r,\infty;f)+(2k+2)\overline{N}(r,\infty;g)\\
&\quad +S(r,f)+S(r,g)\\
&\leq (2k+6)\overline{N}(r,0;f)+(k+4)\overline{N}(r,0;g)+N(r,0;g)\\
&\quad +(2k+3)N(r,\infty;f)+(2k+2)\overline{N}(r,\infty;g)\\
&\quad +S(r,f)+S(r,g)\\
&\leq \frac{k+4}{k}(N(r,0;f)+N(r,0;g))+\frac{k+2}{k}N(r,0;f)\\
&\quad +N(r,0;g)+(2k+3)N(r,\infty;f)+(2k+2)\overline{N}(r,\infty;g)\\
&\quad +S(r,f)+S(r,g)\\
&\leq \left(1+\frac{4}{k}\right)(T(r,f)+T(r,g))+\left(2k+\frac{2}{k}+4\right)T(r,f)\\
&\quad +(2k+3)T(r,g)+S(r,f)+S(r,g).
\end{aligned}$$

This gives

$$\begin{aligned}
\left(n-2k-\frac{2}{k}-5\right)T(r,f) \leq \left(1+\frac{4}{k}\right)(T(r,f)+T(r,g))\\
+(2k+3)T(r,g)+S(r,f)+S(r,g).
\end{aligned} \tag{16}$$

Similarly,

$$\begin{aligned}
\left(n-2k-\frac{2}{k}-5\right)T(r,g) \leq \left(1+\frac{4}{k}\right)(T(r,f)+T(r,g))\\
+(2k+3)T(r,f)+S(r,f)+S(r,g).
\end{aligned} \tag{17}$$

In view of (16) and (17) we obtain

$$\left(n-4k-\frac{10}{k}-10\right)(T(r,f)+T(r,g)) \quad \leq \quad S(r,f)+S(r,g),$$

which contradicts our assumption that $n > 4k + 10/k + 10$.

Case 2 *We now assume that* $H = 0$. *That is*

$$\left(\frac{F_1''}{F_1'}-\frac{2F_1'}{F_1-1}\right)-\left(\frac{G_1''}{G_1'}-\frac{2G_1'}{G_1-1}\right)=0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F_1-1}=\frac{A}{G_1-1}+B, \tag{18}$$

where $A(\neq 0)$ *and* B *are constants. From (18) it is clear that* F_1 *and* G_1 *share* 1 *CM and hence they share the value* 1 *with weight* 2, *and therefore,* $n > k + 4/k + 4$. *Now we consider the following three subcases.*

Subcase 4 *Let* $B \neq 0$ *and* $A = B$. *Then from (18) we get*

$$\frac{1}{F_1-1}=\frac{BG_1}{G_1-1}. \tag{19}$$

If $B = -1$, *then from (19) we obtain*

$$F_1G_1 = 1,$$

i.e.,

$$f^n f^{(k)} g^n g^{(k)} = z^2.$$

Therefore by Lemma 7 we obtain $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, *where* c_1, c_2 *and* c *are three constants satisfying* $4(c_1c_2)^{n+1}c^2 = -1$. *If* $B \neq -1$, *from (19) we have* $\frac{1}{F_1} = \frac{BG_1}{(1+B)G_1-1}$, *and therefore,* $\overline{N}(r, \frac{1}{1+B}; G_1) = \overline{N}(r, 0; F_1)$. *Now using the second fundamental theorem of Nevanlinna, we get*

$$\begin{aligned}
T(r,G) &\leq T(r,G_1)+S(r,G)\\
&\leq \overline{N}(r,0;G_1)+\overline{N}\left(r,\frac{1}{1+B};G_1\right)+\overline{N}(r,\infty;G_1)+S(r,G)\\
&\leq \overline{N}(r,0;F_1)+\overline{N}(r,0;G_1)+\overline{N}(r,\infty;G_1)+S(r,G)\\
&\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+S(r,G).
\end{aligned} \tag{20}$$

Using (20), Lemma 2 and Lemma 3 we obtain from (9)

$$\begin{aligned}(n-1)T(r,g) &\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)-N(r,\infty;g)\\ &\quad -N(r,0;g^{(k)})+S(r,g)\\ &\leq \overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f^{(k)})\\ &\quad +\overline{N}(r,0;g^{(k)})-N(r,\infty;g)-N(r,0;g^{(k)})\\ &\quad +S(r,f)+S(r,g)\\ &\leq \overline{N}(r,0;f)+\overline{N}(r,0;g)+N_{k+1}(r,0;f)+k\overline{N}(r,\infty;f)\\ &\quad +S(r,f)+S(r,g)\\ &\leq \frac{k+2}{k}N(r,0;f)+\frac{1}{k}N(r,0;g)+k\overline{N}(r,\infty;f)\\ &\quad +S(r,f)+S(r,g)\\ &\leq \frac{1}{k}(T(r,f)+T(r,g))+(k+\frac{1}{k}+1)T(r,f)\\ &\quad +S(r,f)+S(r,g).\end{aligned}$$

Thus we obtain

$$\left(n-k-\frac{3}{k}-2\right)(T(r,f)+T(r,g))\leq S(r,f)+S(r,g),$$

a contradiction as $n>k+4/k+4$.

Subcase 5 *Let* $B\neq 0$ *and* $A\neq B$. *Then from (18) we get* $F_1=\frac{(B+1)G_1-(B-A+1)}{BG_1+(A-B)}$, *and so,* $\overline{N}(r,\frac{B-A+1}{B+1};G_1)=\overline{N}(r,0;F_1)$. *Proceeding as in Subcase 4 we obtain a contradiction.*

Subcase 6 *Let* $B=0$ *and* $A\neq 0$. *Then from (18) we get* $F_1=\frac{G_1+A-1}{A}$ *and* $G_1=AF_1-(A-1)$. *If* $A\neq 1$, *we have* $\overline{N}(r,\frac{A-1}{A};F_1)=\overline{N}(r,0;G_1)$ *and* $\overline{N}(r,1-A;G_1)=\overline{N}(r,0;F_1)$. *Using the similar arguments as in Subcase 4 we obtain a contradiction. Thus* $A=1$ *which implies* $F_1=G_1$, *and therefore,* $f^nf^{(k)}=g^ng^{(k)}$.

This completes the proof of Theorem 1. □

Proof of Theorem 2. Using Lemma 8 and proceeding similarly as in the proof of Theorem 1, we can prove Theorem 2. □

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