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Contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold

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Abstract. In this paper we prove some properties of the indefinite Lorentzian para-Sasakian manifolds. Section 1 is introductory. In Section 2 we define D-totally geodesic and $D^{\perp}$-totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold and deduce some results concerning such a manifold. In Section 3 we state and prove some results on mixed totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold. Finally, in Section 4 we obtain a result on the anti-invariant distribution of totally umbilic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold.

1 Introduction

Many valuable and essential results were given on differential geometry with contact and almost contact structure. In 1970 the geometry of cosymplectic

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manifold was studied by G. D. Ludden [14]. After them, in 1973 and 1974, B. Y. Chen and K. Ogive introduced the geometry of submanifolds and totally real submanifolds in [8], [17], [7]. Then K. Ogive expressed the differential geometry of Kaehler submanifolds in [17]. In 1976 contact manifolds in Riemannian geometry were discussed by D. E. Blair [5]. Later on, A. Bejancu discussed CR-submanifolds of a Kaehler manifold [1], [2], [4], and then, K. Yano and M. Kon gave the notion of invariant and anti invariant submanifold in [13] and [21]. M. Kobayashi studied CR-submanifolds of a Sasakian manifold in 1981 [12]. New classes of almost contact metric structures and normal contact manifold in [18], [6] were studied by J. A. Oubina, C. Calin and I. Mihai. A. Bejancu and K. L. Duggal introduced (ϵ)-Sasakian manifolds. Lightlike submanifold of semi Riemannian manifolds was introduced by K. L. Duggal and A. Bejancu [10], [9]. In 2003 and 2007, lightlike submanifolds and hypersurfaces of indefinite Sasakian manifolds were introduced [11]. Lastly, LP-Sasakian manifolds were studied by many authors in [15], [16], [19], [20].

In this paper we define D-totally and $D^{\perp}$- totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold and prove some interesting results.

An n-dimensional differentiable manifold is called indefinite Lorentzian para-Sasakian manifold if the following conditions hold

$$\phi^2 X = X + \eta(X)\xi, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \tag{1}$$

$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2}$$

$$\tilde{g}(X, \xi) = \epsilon\eta(X), \tag{3}$$

for all vector fields X, Y on $\tilde{M}$ [5] and where ϵ is 1 or -1 according to ξ is space-like or time-like vector field.

An indefinite almost metric structure $(\phi, \xi, \eta, \tilde{g})$ is called an indefinite Lorentzian para-Sasakian manifold if

$$(\tilde{\nabla}_X\phi)Y = g(X, Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi, \tag{4}$$

where $\tilde{\nabla}$ is the Levi-Civita $(L - C)$ connection for a semi-Riemannian metric $\tilde{g}$. Also we have

$$\tilde{\nabla}_X\xi = \epsilon\phi X, \tag{5}$$

where $X \in T\tilde{M}$.

From the definition of contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold we have

Definition 1 *An n-dimensional Riemannian submanifold M of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$ is called a contact CR-submanifold if*

i) ξ *is tangent to* M,

ii) *there exists on* M *a differentiable distribution* $D : x \longrightarrow D_x \subset T_x(M)$, *such that* D_x *is invariant under* ϕ*; i.e.,* $\phi D_x \subset D_x$*, for each* $x \in M$ *and the orthogonal complementary distribution* $D^\perp : x \longrightarrow D_x^\perp \subset T_x{}^\perp(M)$ *of the distribution* D *on* M *is totally real; i.e.,* $\phi D_x^\perp \subset T_x^\perp(M)$*, where* $T_x(M)$ *and* $T_x^\perp(M)$ *are the tangent space and the normal space of* M *at* x.

D (*resp.* $D^\perp$) *is the horizontal* (*resp. vertical*) *distribution. The contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold is called* ξ*-horizontal* (*resp.* ξ*-vertical*) *if* $\xi_x \in D_x$ (*resp.* $\xi_x \in D_x^\perp$) *for each* $x \in M$ *by* [12].

The Gauss and Weingarten formulae are as follows

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{6}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N, \tag{7}$$

for any $X, Y \in TM$ and $N \in T^\perp M$, where $\nabla^\perp$ is the connection on the normal bundle $T^\perp M$, h is the second fundamental form and A_N is the Weingarten map associated with N via

$$g(A_N X, Y) = g(h(X, Y), N). \tag{8}$$

The equation of Gauss is given by

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)), \tag{9}$$

where $\tilde{R}$ (resp. R) is the curvature tensor of $\tilde{M}$ (resp. M).

For any $x \in M, X \in T_x M$ and $N \in T_x^\perp M$, we write

$$X = PX + QX \tag{10}$$

$$\phi N = BN + CN, \tag{11}$$

where PX (resp. BN) denotes the tangential part of X (resp. ϕN) and QX (resp. CN) denotes the normal part of X (resp. ϕN) respectively.

Using (6), (7), (10), (11) in (4) after a brief calculation we obtain on comparing the horizontal, vertical and normal parts

$$P\nabla_X\phi PY - PA_{\phi QY}X = \phi P\nabla_X Y + g(PX,Y)\xi + \epsilon\eta(Y)PX + 2\epsilon\eta(Y)\eta(X), \quad (12)$$
$$Q\nabla_X\phi PY + QA_{\phi QY}X = Bh(X,Y) + g(QX,Y)\xi + \epsilon\eta(Y)QX, \quad (13)$$
$$h(X,\phi PY) + \nabla^{\perp}_X\phi QY = \phi Q\nabla_X Y + Ch(X,Y). \quad (14)$$

From (5) we have

$$\nabla_X\xi = \epsilon\phi PX, \quad (15)$$
$$h(X,\xi) = \epsilon\phi QX. \quad (16)$$

Also we have

$$h(X,\xi) = 0 \quad \text{if} \quad X \in D, \quad (17)$$
$$\nabla_X\xi = 0, \quad (18)$$
$$h(\xi,\xi) = 0, \quad (19)$$
$$A_N\xi \in D^{\perp}. \quad (20)$$

2 D-totally geodesic and $D^{\perp}$-totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold

First we define the D-totally (resp. $D^{\perp}$-totally) geodesic contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold.

Definition 2 *A contact CR-submanifold M of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$ is called D-totally geodesic (resp. $D^{\perp}$-totally geodesic) if $h(X,Y) = 0$, $\forall\ X, Y \in D$ (resp. $X, Y \in D^{\perp}$).*

From the above definition, the following propositions follow immediately.

Proposition 1 *Let M be a contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold. Then M is a D-totally geodesic if and only if $A_N X \in D^{\perp}$ for each $X \in D$ and N a normal vector field to M.*

Proof. Let M be D-totally geodesic. Then from (8) we get

$$g(h(X,Y),N) = g(A_N X, Y) = 0.$$

So if

$$h(X,Y) = 0, \quad \forall \; X, \; Y \in \; D$$

i.e.,

$$A_N X \in \; D^{\perp}.$$

Conversely, let $A_N X \in \; D^{\perp}$. Then for $X, \; Y \in \; D$ we can obtain

$$g(A_N X, Y) = 0 = g(h(X,Y), N)$$

i.e.,

$$h(X,Y) = 0$$

$\forall \; X, \; Y \in \; D$, which implies that M is D-totally geodesic. Thus our proof is complete. □

Proposition 2 *Let M be a contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$. Then M is $D^{\perp}$-totally geodesic if and only if $A_N X \in D$ for each $X \in D^{\perp}$ and N a normal vector field to M.*

Proof. The proof follows immediately from the above proposition. □

Concerning the integrability of the horizontal distribution D and vertical distribution $D^{\perp}$ on M, we can state the following theorem:

Theorem 1 *Let M be a contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold. If M is ξ-horizontal, then the distribution D is integrable iff*

$$h(X, \phi Y) = h(\phi X, Y) \tag{21}$$

$\forall \; X, \; Y \in \; D$. *If M is ξ-vertical then the distribution $D^{\perp}$ is integrable iff*

$$A_{\phi X} Y - A_{\phi Y} X = \epsilon[\eta(Y)X - \eta(X)Y] \tag{22}$$

$\forall \; X, \; Y \in \; D^{\perp}$.

Proof. If M is ξ-horizontal, then using (14) we get

$$h(X, \phi PY) = \phi Q \nabla_X Y + Ch(X,Y)$$

$\forall \; X, \; Y \in \; D$. Therefore $[X,Y] \in D \quad \text{iff} \;\; h(X, \phi Y) = h(Y, \phi X)$

Hence, if M is ξ-horizontal, $[X,Y] \in \; D$ iff $h(X, \phi Y) = h(\phi X, Y)$.

Again using (14) we get

$$\nabla^{\perp}_{X}\phi Y = Ch(X,Y) + \phi Q \nabla_X Y \tag{23}$$

for $X,\ Y\ \in\ D^{\perp}$.

After some calculations we see that

$$\begin{aligned}\tilde{\nabla}_X \phi Y = g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(Y)\eta(X)\xi + \phi P\nabla_X Y \\ + \phi Q \nabla_X Y + Bh(X,Y) + Ch(X,Y).\end{aligned} \tag{24}$$

Again from (7) and (24) we get

$$\begin{aligned}\nabla^{\perp}_{X}\phi Y = A_{\phi Y}X + g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(Y)\eta(X)\xi \\ + \phi P\nabla_X Y + \phi Q\nabla_X Y + Bh(X,Y) + Ch(X,Y)\end{aligned} \tag{25}$$

for $X, Y \in D^{\perp}$. From (24) and (25) we can write

$$\phi P \nabla_X Y = -A_{\phi Y}X - g(X,Y)\xi - \epsilon\eta(Y)X - 2\epsilon\eta(Y)\eta(X)\xi - Bh(X,Y). \tag{26}$$

Interchanging X and Y in (26) we get

$$\phi P \nabla_Y X = -A_{\phi X}Y - g(X,Y)\xi - \epsilon\eta(X)Y - 2\epsilon\eta(Y)\eta(X)\xi - Bh(X,Y). \tag{27}$$

Substracting (27) from (26) we have

$$\phi P[X,Y] = -A_{\phi Y}X + A_{\phi X}Y - \epsilon\eta(Y)X + \epsilon\eta(X)Y. \tag{28}$$

Now since M is ξ-vertical, $[X,Y] \in\ D^{\perp}$ iff

$$A_{\phi X}Y - A_{\phi Y}X = \epsilon[\eta(Y)X - \eta(X)Y].$$

So the proof is complete. □

D-umbilic (resp. $D^{\perp}$-umbilic) contact CR-submanifold of indefinite Lorentzian para-Sasakian manifold is defined as follows:

Definition 3 *A contact CR-submanifold* M *of an indefinite Lorentzian para-Sasakian manifold is said to be* D*-umbilic (resp.* $D^{\perp}$*-umbilic) if* $h(X,Y) = g(X,Y)L$ *holds for all* $X,Y \in D$ *(resp.* $X,Y \in D^{\perp}$*),* L *being some normal vector field.*

In view of the above definition we state and prove the following proposition:

Proposition 3 *Suppose* M *is a* D*-umbilic contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold* $\tilde{M}$*. If* M *is* ξ*-horizontal (resp.* ξ*-vertical) then* M *is* D*-totally geodesic (resp.* $D^{\perp}$*-totally geodesic).*

Proof. Consider M as D-umbilic ξ-horizontal contact CR-submanifold. Then we have from Definition 3

$$h(X, Y) = g(X, Y)L \qquad \forall \ X, Y \in \ D,$$

L being some normal vector field on M. By putting $X = Y = \xi$ and using (19) we have

$$h(\xi, \xi) = g(\xi, \xi)L$$

i.e.
$$L = 0,$$
and consequently we get $h(X, Y) = 0$, which proves that M is D-totally geodesic.

Similarly, it can be easily shown that if M is $D^{\perp}$-umbilic ξ-vertical contact CR-submanifold then it is $D^{\perp}$-totally geodesic. □

3 Mixed totally geodesic contact CR-submanifolds of indefinite Lorentzian para-Sasakian manifold

In this section we define mixed totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold (followed [12]).

Definition 4 *A contact CR-submanifold* M *of an indefinite Lorentzian para-Sasakian manifold* $\tilde{M}$ *is said to be mixed totaly geodesic if* $h(X, Y) = 0 \ \forall \ X \in D$ *and* $Y \in D^{\perp}$.

Then we extract the following lemma and theorem

Lemma 1 *Let* M *be a contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold. Then* M *is mixed totally geodesic iff*

$$A_N X \in D, \qquad \forall \ X \in D, \quad \text{and} \ \forall \ \text{normal vector field } N, \tag{29}$$

$$A_N X \in D^{\perp}, \qquad \forall \ X \in D^{\perp} \quad \text{and} \ \forall \ \text{normal vector field } N. \tag{30}$$

Proof. If M is mixed totally geodesic, then from (8), we get

$$h(X, Y) = 0,$$

i.e., iff $A_N X \in D$, $\forall\ X \in\ D$ and $\forall$ normal vector field N. Conversely, if M is mixed totally geodesic, then using (8) we easily observe that $A_N X \in D^{\perp}$, $\forall\ X \in D^{\perp}$ and $\forall$ normal vector field N.
Hence the lemma is proved. □

Using condition (29) we obtain the following theorem

Theorem 2 *If M is a mixed totally geodesic contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold, then*

$$A_{\phi N} X = -\phi A_N X, \tag{31}$$

$$\nabla^{\perp}_X \phi N = \phi \nabla^{\perp}_X N \tag{32}$$

$\forall\ X \in\ D$ *and* $\forall$ *normal vector field* N.

Proof. We get from (29), (6), (7) and after having some calculations we derive

$$\nabla_X \phi N = \phi \nabla^{\perp}_X N - \phi A_N X, \tag{33}$$

$$\nabla_X \phi N = -A_{\phi N} X + \nabla^{\perp}_X \phi N. \tag{34}$$

Comparing the above two equations we have the required theorem. Hence the proof follows. □

Again we have the following definition

Definition 5 *A contact CR-submanifold M of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$ is called foliate contact CR-submanifold $\tilde{M}$ if D is involute. If M is a foliate ξ-horizontal contact CR-submanifold, we know from* [3]

$$h(\phi X, \phi Y) = h(\phi^2 X, Y) = -h(X, Y). \tag{35}$$

Considering the above definition we give the following proposition.

Proposition 4 *If M is a foliate ξ-horizontal mixed totally geodesic contact CR-submanifold M of an indefinite Lorentzian para-Sasakian manifold, then*

$$\phi A_N X = A_N \phi X \tag{36}$$

for all $X \in D$ and normal vector field N.

Proof. From (21) and (8) we compute the following:

$$g(h(X,\phi Y),N) = g(\phi A_N X, Y),$$

i.e.

$$g(h(\phi X,Y),N) = g(A_N \phi X, Y).$$

Therefore

$$\phi A_N X = A_N \phi X.$$

Hence the proof follows. □

4 Anti-invariant distribution $D^\perp$ on totally umbilical contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold

Here we consider a contact CR-submanifold M of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$. Then we establish the following theorem.

Theorem 3 *Let M be a totally umbilical contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$. Then the anti invariant distribution $D^\perp$ is one dimensional, i.e.* $\dim D^\perp = 1$.

Proof. For an indefinite Lorentzian para-Sasakian structure we have

$$(\tilde{\nabla}_Z \phi)W = g(Z,W)\xi + \epsilon\eta(W)Z + 2\epsilon\eta(W)\eta(Z)\xi. \tag{37}$$

Also by the covariant derivative of tensor fields (for any $Z, W \in \Gamma(D^\perp)$) we know

$$\tilde{\nabla}_Z \phi W = (\tilde{\nabla}_Z \phi)W + \phi\tilde{\nabla}_Z W. \tag{38}$$

Using (37), (38), (6), (7) and (4) we obtain

$$\begin{aligned} \nabla^\perp_Z \phi W - g(H,\phi W)Z = \phi[\nabla_Z W + g(Z,W)H] + g(Z,W)\xi \\ + \epsilon\eta(W)Z + 2\epsilon\eta(W)\eta(Z)\xi \end{aligned} \tag{39}$$

for any $Z, W \in \Gamma(D^\perp)$.

Taking the inner product with $Z \in \Gamma(D^\perp)$ in (39) we obtain

$$\begin{aligned} -g(H,\phi W)\|Z\|^2 = g(Z,W)g(\phi H, Z) + \epsilon\eta(W)\|Z\|^2 + g(Z,W)g(\xi,Z) \\ + 2\eta(W)\eta(Z)g(Z,\xi). \end{aligned} \tag{40}$$

Using (2) after a brief calculation we have

$$\begin{aligned} g(H, \phi W) = &-\frac{g(Z,W)g(\phi H, Z)}{||Z||^2} - \frac{g(Z,W)g(\xi, Z)}{||Z||^2} \\ &- \epsilon g(W,\xi) - 2\frac{g(Z,\xi)^2 g(W,\xi)}{||Z||^2}. \end{aligned} \tag{41}$$

Interchanging Z and W we have

$$\begin{aligned} g(H, \phi Z) = &-\frac{g(Z,W)g(\phi H, W)}{||W||^2} - \frac{g(Z,W)g(\xi, W)}{||W||^2} \\ &- \epsilon g(Z,\xi) - 2\frac{g(W,\xi)^2 g(Z,\xi)}{||W||^2}. \end{aligned} \tag{42}$$

Substituting (41) in (40) and simplifying we get

$$\begin{aligned} &g(H, \phi W)\left[1 - \frac{g(Z,W)^2}{||Z||^2||W||^2}\right] - \frac{g(Z,W)}{||Z||^2}\left[\frac{g(Z,W)g(\xi,W)}{||W||^2} - g(Z,\xi)\right] \\ &- \epsilon\left[\frac{g(Z,W)g(\xi,Z)}{||Z||^2} - g(W,\xi)\right] \\ &- 2g(z,\xi)g(W,\xi)\left[\frac{g(Z,W)g(W,\xi)}{||W||^2||Z||^2} - \frac{g(Z,W)}{||Z||^2}\right] = 0. \end{aligned} \tag{43}$$

The equation (43) has a solution if $Z \parallel W$, i.e. dim $D^{\perp}$=1.
Hence the theorem is proved. □

Example 1 *Let* $\mathbf{R}^3$ *be a* 3*-dimensional Euclidean space with rectangular coordinates* (x, y, z)*. In* $\mathbf{R}^3$ *we define*

$$\eta = -dz - y\,dx \qquad \xi = \frac{\partial}{\partial z}$$

$$\phi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}, \qquad \phi\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \qquad \phi\left(\frac{\partial}{\partial z}\right) = 0.$$

The Lorentzian metric g *is defined by the matrix:*

$$\begin{pmatrix} -\epsilon y^2 & 0 & \epsilon y \\ 0 & 0 & 0 \\ \epsilon y & 0 & -\epsilon \end{pmatrix}.$$

Then it can be easily seen that (ϕ, ξ, η, g) *forms an indefinite Lorentzian para-Sasakian structure in* $\mathbf{R}^3$ *and the above results can be verified for this example.*

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