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Related fixed point theorem for four complete metric spaces

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Abstract. In the present paper, we obtain a new result on fixed point theorem for four metric spaces. Here we choose continuous mappings. In fact our result is the generalization of many results of fixed point theorem on two and three metric spaces. We also give some illustrative examples to justify our result.

1 Introduction

Related fixed point theorems on two metric spaces have been studied by B. Fisher [2]. Also some fixed point theorems on three metric spaces have been studied by B. Fisher *et al* [3], R. K. Jain *et al.* [4], R. K. Namdeo and B. Fisher [7], K. Kikina *et al.* [6], Z. Ansari *et al.* [1], and V. Gupta [8]. Also, the fixed point theorems on four metric spaces have been studied by L. Kikina *et al.* [5]. In the present paper a generalization is given for four complete metric space. Our theorem improves Theorem (2.1) of R. K. Jain *et al.* [4].

The following fixed point theorem was proved by R. K. Jain, H. K. Sahu, B. Fisher [4].

Theorem 1 *Let* (X, d)*,* (Y, ρ) *and* (Z, σ) *be complete metric spaces. If* T *is continuous mapping of* $X \mapsto Y$*,* S *is a continuous mapping of* $Y \mapsto Z$ *and* R *is mapping of* $Z \mapsto X$ *satisfying the inequalities*

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$$d(RSTx, RSTx') \leqslant c\max\{d(x,x'), d(x, RSTx), \\ d(x', RSTx'), \rho(Tx, Tx'), \sigma(STx, STx')\}, \tag{1}$$

$$\rho(TRSy, TRSy') \leqslant c\max\{\rho(y,y'), \rho(y, TRSy), \\ \rho(y', TRSy'), \sigma(Sy, Sy'), d(RSy, RSy')\}, \tag{2}$$

$$\sigma(STRz, STRz') \leqslant c\max\{\sigma(z,z'), \sigma(z, STRz) \\ \sigma(z', STRz'), d(Rz, Rz'), \rho(TRz, TRz')\}, \tag{3}$$

$\forall\ x, x' \in X$, $y, y' \in Y$ and $z, z' \in Z$, where $0 \leqslant c < 1$, then RST has a unique fixed point $u \in X$, TRS has a unique fixed point $v \in Y$ and STR has a unique fixed point $w \in Z$. Further $Tu = v$, $Sv = w$ and $Rw = u$.

2 Main result

Theorem 2 *Let $(Z_1, d_1), (Z_2, d_2), (Z_3, d_3)$, and (Z_4, d_4) be complete metric spaces. If A_1 is a continuous mapping of $Z_1 \mapsto Z_2$, A_2 is continuous mapping of $Z_2 \mapsto Z_3$, A_3 is continuous mapping of $Z_3 \mapsto Z_4$ and A_4 is a mappings of $Z_4 \mapsto Z_1$, satisfying the inequalities*

$$\begin{aligned} & d_1(A_4A_3A_2A_1z_1, A_4A_3A_2A_1z_1') \\ & \quad \leqslant c\max\{d_1(z_1, z_1'), d_1(z_1, A_4A_3A_2A_1z_1), \\ & \qquad d_1(z_1', A_4A_3A_2A_1z_1'), d_2(A_1z_1, A_1z') \\ & \qquad d_3(A_2A_1z_1, A_2A_1z_1'), d_4(A_3A_2A_1z_1, A_3A_2A_1z_1')\}, \end{aligned} \tag{4}$$

$$\begin{aligned} & d_2(A_1A_4A_3A_2z_2, A_1A_4A_3A_2z_2') \\ & \quad \leqslant\ c\max\{d_2(z_2, z_2'), d_2(z_2, A_1A_4A_3A_2z_2), \\ & \qquad d_2(z_2', A_1A_4A_3A_2z_2'), d_3(A_2z_2, A_2z_2'), \\ & \qquad d_4(A_3A_2z_2, A_3A_2z_2'), d_1(A_4A_3A_2z_2, A_4A_3A_2z_2')\}, \end{aligned} \tag{5}$$

$$\begin{aligned} & d_3(A_2A_1A_4A_3z_3, A_2A_1A_4A_3z_3') \\ & \quad \leqslant\ c\max\{d_3(z_3, z_3'), d_3(z_3, A_2A_1A_4A_3z_3), \\ & \qquad d_3(z_3', A_2A_1A_4A_3z_3'), d_4(A_3z_3, A_3z_3'), \\ & \qquad d_1(A_4A_3z_3, A_4A_3z_3'), d_2(A_1A_4A_3z_3, A_1A_4A_3z_3')\}, \end{aligned} \tag{6}$$

$$\begin{aligned} & d_4(A_3A_2A_1A_4z_4, A_3A_2A_1A_4z_4') \\ & \quad \leqslant\ c\max\{d_4(z_4, z_4'), d_4(z_4, A_3A_2A_1A_4z_4), \\ & \qquad d_4(z_4', A_3A_2A_1A_4z_4'), d_1(A_4z_4, A_4z_4'), \\ & \qquad d_2(A_1A_4z_4, A_1A_4z_4'), d_3(A_2A_1A_4z_4, A_2A_1A_4z_4')\}, \end{aligned} \tag{7}$$

$\forall\, z_1, z_1' \in Z_1,\ z_2, z_2' \in Z_2,\ z_3, z_3' \in Z_3$ *and* $z_4, z_4' \in Z_4$, *where* $0 \leqslant c < 1$, *then* $A_4A_3A_2A_1$ *has a unique fixed point* $\alpha_1 \in Z_1$, $A_1A_4A_3A_2$ *has a unique fixed point* $\alpha_2 \in Z_2$, $A_2A_1A_4A_3$ *has a unique fixed point* $\alpha_3 \in Z_3$ *and* $A_3A_2A_1A_4$ *has a unique fixed point* $\alpha_4 \in Z_4$.

Further $A_1\alpha_1 = \alpha_2,\ A_2\alpha_2 = \alpha_3,\ A_3\alpha_3 = \alpha_4,\ A_4\alpha_4 = \alpha_1$.

Proof. Let z_1^0 be an arbitrary point in Z_1.

Define the sequence $\{z_n^1\}, \{z_n^2\}, \{z_n^3\}$ and $\{z_n^4\}$ in Z_1, Z_2, Z_3 and Z_4 respectively by

$$\begin{aligned}
(A_4A_3A_2A_1)^n z_1^0 &= z_n^1 \\
A_1 z_{n-1}^1 &= z_n^2 \\
A_2 z_n^2 &= z_n^3 \\
A_3 z_n^3 &= z_n^4 \\
A_4 z_n^4 &= z_n^1 \qquad \text{for } n = 1, 2, \ldots
\end{aligned}$$

Applying inequality (5), we get,

$$\begin{aligned}
d_2(z_n^2, z_{n+1}^2) &= d_2(A_1A_4A_3A_2 z_{n-1}^2, A_1A_4A_3A_2 z_n^2) \\
&\leqslant c \max\{d_2(z_{n-1}^2, z_n^2), d_2(z_{n-1}^2, A_1A_4A_3A_2 z_{n-1}^2), \\
&\quad d_2(z_n^2, A_1A_4A_3A_2 z_n^2), d_3(A_2 z_{n-1}^2, A_2 z_n^2), \\
&\quad d_4(A_3A_2 z_{n-1}^2, A_3A_2 z_n^2), d_1(A_4A_3A_2 z_{n-1}^2, A_4A_3A_2 z_n^2)\} \\
d_2(z_n^2, z_{n+1}^2) &\leqslant c \max\{d_2(z_{n-1}^2, z_n^2), d_2(z_{n-1}^2, z_n^2), d_2(z_n^2, z_{n+1}^2), \\
&\quad d_3(z_{n-1}^3, z_n^3), d_4(z_{n-1}^4, z_n^4), d_1(z_{n-1}^1, z_n^1)\} \\
d_2(z_n^2, z_{n+1}^2) &\leqslant c \max\{d_1(z_{n-1}^1, z_n^1), d_2(z_{n-1}^2, z_n^2), d_3(z_{n-1}^3, z_n^3), d_4(z_{n-1}^4, z_n^4)\}
\end{aligned} \tag{8}$$

Using inequality (6), we get,

$$\begin{aligned}
d_3(z_n^3, z_{n+1}^3) &= d_3(A_2A_1A_4A_3 z_{n-1}^3, A_2A_1A_4A_3 z_n^3) \\
&\leqslant \max\{d_3(z_{n-1}^3, z_n^3), d_3(z_{n-1}^3, A_2A_1A_4A_3 z_{n-1}^3), \\
&\quad d_3(z_n^3, A_2A_1A_4A_3 z_n^3), d_4(A_3 z_{n-1}^3, A_3 z_n^3), \\
&\quad d_1(A_4A_3 z_{n-1}^3, A_4A_3 z_n^3), d_2(A_1A_4A_3 z_{n-1}^3, A_1A_4A_3 z_n^3)\} \\
d_3(z_n^3, z_{n+1}^3) &\leqslant c \max\{d_3(z_{n-1}^3, z_n^3), d_3(z_{n-1}^3, z_n^3), d_3(z_n^3, z_{n+1}^3), \\
&\quad d_4(z_{n-1}^4, z_n^4), d_1(z_{n-1}^1, z_n^1), d_2(z_n^2, z_{n+1}^2)\} \\
d_3(z_n^3, z_{n+1}^3) &\leqslant c \max\{d_1(z_{n-1}^1, z_n^1), d_2(z_{n-1}^2, z_n^2), \\
&\quad d_3(z_{n-1}^3, z_n^3), d_4(z_{n-1}^4, z_n^4)\}
\end{aligned} \tag{9}$$

Using inequality (7), we have,

$$\begin{aligned}
d_4(z^4_n, z^4_{n+1}) &= d_4(A_3A_2A_1A_4z^4_{n-1}, A_3A_2A_1A_4z^4_n) \\
&\leqslant c\max\{d_4(z^4_{n-1}, z^4_n), d_4(z^4_{n-1}, A_3A_2A_1A_4z^4_{n-1}), \\
&\quad d_4(z^4_n, A_3A_2A_1A_4z^4_n), d_1(A_4z^4_{n-1}, A_4z^4_n), \\
&\quad d_2(A_1A_4z^4_{n-1}, A_1A_4z^4_n), d_3(A_2A_1A_4z^4_{n-1}, A_2A_1A_4z^4_n)\} \\
d_4(z^4_n, z^4_{n+1}) &\leqslant c\max\{d_4(z^4_{n-1}, z^4_n), d_4(z^4_{n-1}, z^4_n), d_4(z^4_n, z^4_{n+1}), \\
&\quad d_1(z^1_{n-1}, z^1_n), d_2(z^2_n, z^2_{n+1}), d_3(z^3_n, z^3_{n+1})\} \\
d_4(z^4_n, z^4_{n+1}) &\leqslant c\max\{d_1(z^1_{n-1}, z^1_n), d_2(z^2_{n-1}, z^2_n), \\
&\quad d_3(z^3_{n-1}, z^3_n), d_4(z^4_{n-1}, z^4_n)\}
\end{aligned} \tag{10}$$

Using inequality (4), we have,

$$\begin{aligned}
d_1(z^1_n, z^1_{n+1}) &= d_1(A_4A_3A_2A_1z^1_{n-1}, A_4A_3A_2A_1z^1_n) \\
&\leqslant c\max\{d_1(z^1_{n-1}, z^1_n), d_1(z^1_{n-1}, A_4A_3A_2A_1z^1_{n-1}), \\
&\quad d_1(z^1_n, A_4A_3A_2A_1z^1_n), d_2(A_1z^1_{n-1}, A_1z^1_n), \\
&\quad d_3(A_2A_1z^1_{n-1}, A_2A_1z^1_n), d_4(A_3A_2A_1z^1_{n-1}, A_3A_2A_1z^1_n)\} \\
d_1(z^1_n, z^1_{n+1}) &\leqslant c\max\{d_1(z^1_{n-1}, z^1_n), d_1(z^1_{n-1}, z^1_n), d_1(z^1_n, z^1_{n+1}), \\
&\quad d_2(z^2_n, z^2_{n+1}), d_3(z^3_n, z^3_{n+1}), d_4(z^4_n, z^4_{n+1})\} \\
d_1(z^1_n, z^1_{n+1}) &\leqslant c\max\{d_1(z^1_{n-1}, z^1_n), d_2(z^2_{n-1}, z^2_n), \\
&\quad d_3(z^3_{n-1}, z^3_n), d_4(z^4_{n-1}, z^4_n)\}
\end{aligned} \tag{11}$$

By induction on using inequalities (8), (9), (10) and (11), we have,

$$\begin{aligned}
d_1(z^1_n, z^1_{n+1}) &\leqslant c^{n-1}\max\{d_1(z^1_1, z^1_2), d_2(z^2_1, z^2_2), \\
&\quad d_3(z^3_1, z^3_2), d_4(z^4_1, z^4_2)\} \\
d_2(z^2_n, z^2_{n+1}) &\leqslant c^{n-1}\max\{d_1(z^1_1, z^1_2), d_2(z^2_1, z^2_2), \\
&\quad d_3(z^3_1, z^3_2), d_4(z^4_1, z^4_2)\} \\
d_3(z^3_n, z^3_{n+1}) &\leqslant c^{n-1}\max\{d_1(z^1_1, z^1_2), d_2(z^2_1, z^2_2), \\
&\quad d_3(z^3_1, z^3_2), d_4(z^4_1, z^4_2)\} \\
d_4(z^4_n, z^4_{n+1}) &\leqslant c^{n-1}\max\{d_1(z^1_1, z^1_2), d_2(z^2_1, z^2_2), \\
&\quad d_3(z^3_1, z^3_2), d_4(z^4_1, z^4_2)\}
\end{aligned}$$

Since $c < 1$, it follows that $\{z_n^1\}$,$\{z_n^2\}$,$\{z_n^3\}$ and $\{z_n^4\}$ are Cauchy sequences with limit α_1, α_2, α_3 and α_4 in Z_1, Z_2, Z_3 and Z_4 respectively.

Since A_1, A_2 and A_3 are continuous, we have,

$$\lim_{n\to\infty} z_n^2 = \lim_{n\to\infty} A_1 z_n^1 = A_1\alpha_1 = \alpha_2$$
$$\lim_{n\to\infty} z_n^3 = \lim_{n\to\infty} A_2 z_n^2 = A_2\alpha_2 = \alpha_3$$
$$\lim_{n\to\infty} z_n^4 = \lim_{n\to\infty} A_3 z_n^3 = A_3\alpha_3 = \alpha_4$$

Using inequality (4), again, we get,

$$\begin{aligned} d_1(A_4A_3A_2A_1\alpha_1, z_n^1) &= d_1(A_4A_3A_2A_1\alpha_1, A_4A_3A_2A_1 z_{n-1}^1) \\ &\leqslant c\max\{d_1(\alpha_1, z_{n-1}^1), d_1(\alpha_1, A_4A_3A_2A_1\alpha_1), \\ &\quad d_1(z_{n-1}^1, A_4A_3A_2A_1 z_{n-1}^1), d_2(A_1\alpha_1, A_1 z_{n-1}^1), \\ &\quad d_3(A_2A_1\alpha_1, A_2A_1 z_{n-1}^1), d_4(A_3A_2A_1\alpha_1, A_3A_2A_1 z_{n-1}^1)\} \end{aligned}$$

Since A_1, A_2 and A_3 are continuous, it follows on letting $n \to \infty$ that

$$d_1(A_4A_3A_2A_1\alpha_1, \alpha_1) \leqslant c\max\{d_1(\alpha_1, A_4A_3A_2A_1\alpha_1)\}$$

Thus, we have, $A_4A_3A_2A_1\alpha_1 = \alpha_1$. Since $c < 1$ and α_1 is the fixed point of $A_4A_3A_2A_1$

$$A_1A_4A_3A_2\alpha_2 = A_1A_4A_3A_2A_1\alpha_1 = A_1\alpha_1 = \alpha_2$$

and $$A_2A_1A_4A_3\alpha_3 = A_2A_1A_4A_3A_2\alpha_2 = A_2\alpha_2 = \alpha_3$$

and $$A_3A_2A_1A_4\alpha_4 = A_3A_2A_1A_4A_3\alpha_3 = A_3\alpha_3 = \alpha_4$$

Hence α_2, α_3 and α_4 are fixed points of $A_1A_4A_3A_2$, $A_2A_1A_4A_3$ and $A_3A_2A_1A_4$ respectively.

2.1 Uniqueness

Suppose that $A_4A_3A_2A_1$ has a second fixed point α_1'. Then, using inequality (4), we have,

$$\begin{aligned} d_1(\alpha_1, \alpha_1') &= d_1(A_4A_3A_2A_1\alpha_1, A_4A_3A_2A_1\alpha_1') \\ &\leqslant c\max\{d_1(\alpha_1, \alpha_1'), d_1(\alpha_1, A_4A_3A_2A_1\alpha_1), \\ &\quad d_1(\alpha_1', A_4A_3A_2A_1\alpha_1'), d_2(A_1\alpha_1, A_1\alpha_1'), \\ &\quad d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'), \\ &\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')\} \end{aligned}$$

$$\begin{aligned} d_1(\alpha_1,\alpha_1') \leqslant c\max\{&d_1(\alpha_1,\alpha_1'), d_1(\alpha_1,\alpha_1), d_1(\alpha_1',\alpha_1'),\\ &d_2(A_1\alpha_1, A_1\alpha_1'), d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),\\ &d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')\} \qquad (12)\\ d_1(\alpha_1,\alpha_1') \leqslant c\max\{&d_2(A_1\alpha_1, A_1\alpha_1'), d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),\\ &d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')\} \end{aligned}$$

Using inequality (5), we have

$$\begin{aligned} d_2(A_1\alpha_1, A_1\alpha_1') &= d_2(A_1A_4A_3A_2A_1\alpha_1, A_1A_4A_3A_2A_1\alpha_1')\\ &\leqslant c\max\{d_2(A_1\alpha_1, A_1\alpha_1'),\\ &\quad d_2(A_1\alpha_1, A_1A_4A_3A_2A_1\alpha_1),\\ &\quad d_2(A_1\alpha_1', A_1A_4A_3A_2A_1\alpha_1'),\\ &\quad d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),\\ &\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'),\\ &\quad d_4(A_4A_3A_2A_1\alpha_1, A_4A_3A_2A_1\alpha_1')\}\\ d_2(A_1\alpha_1, A_1\alpha_1') &\leqslant c\max\{d_2(A_1\alpha_1, A_1\alpha_1'), d_2(A_1\alpha_1, A_1\alpha_1'),\\ &\quad d_1(A_1\alpha_1', A_1\alpha_1'),\\ &\quad d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),\\ &\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'),\\ &\quad d_1(A_4A_3A_2A_1\alpha_1, A_4A_3A_2A_1\alpha_1')\}\\ d_2(A_1\alpha_1, A_1\alpha_1') &\leqslant c\max\{d_2(A_1\alpha_1, A_1\alpha_1'), d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),\\ &\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'), d_1(\alpha_1,\alpha_1')\}\\ d_2(A_1\alpha_1, A_1\alpha_1') &\leqslant c\max\{d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),\\ &\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'), d_1(\alpha_1,\alpha_1')\} \end{aligned}$$

Now, we have

$$\begin{aligned} d_2(A_1\alpha_1, A_1\alpha_1') \leqslant c\max\{&d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),\\ &d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'),\\ &cd_2(A_1\alpha_1, A_1\alpha_1'), cd_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),\\ &cd_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')\} \qquad (13)\\ d_2(A_1\alpha_1, A_1\alpha_1') \leqslant c\max\{&d_3(A_2A_1\alpha_1, A_2A_1\alpha_1')\\ &d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')\} \end{aligned}$$

Similarly on using inequality (6), we get

$$\begin{aligned}
d_3(A_2A_1\alpha_1, A_2A_1\alpha_1') \leqslant c\max\{&d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),\\
&d_3(A_2A_1\alpha_1, A_2A_1A_4A_3A_2A_1\alpha_1),\\
&d_3(A_2A_1\alpha_1', A_2A_1A_4A_3A_2A_1\alpha_1'),\\
&d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'),\\
&d_1(A_4A_3A_2A_1\alpha_1, A_4A_3A_2A_1\alpha_1'),\\
&d_2(A_1A_4A_3A_2A_1\alpha_1, A_1A_4A_3A_2A_1\alpha_1')\}\\
d_3(A_2A_1\alpha_1, A_2A_1\alpha_1') \leqslant c\max\{&d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),\\
&d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),\\
&d_3(A_2A_1\alpha_1', A_2A_1\alpha_1'),\\
&d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'),\\
&d_1(\alpha_1, \alpha_1'), d_2(A_1\alpha_1, A_1\alpha_1')\}\\
d_3(A_2A_1\alpha_1, A_2A_1\alpha_1') \leqslant c\max\{&d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'),\\
&d_1(\alpha_1, \alpha_1'), d_2(A_1\alpha_1, A_1\alpha_1')\}\\
d_3(A_2A_1\alpha_1, A_2A_1\alpha_1') \leqslant c\max\{&d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'),\\
&cd_2(A_1\alpha_1, A_1\alpha_1'), cd_3(A_2A_1\alpha_1, A_2A_1\alpha_1')\\
&cd_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')\}
\end{aligned} \tag{14}$$

Using inequality (13) and (14), we have

$$d_3(A_2A_1\alpha_1, A_2A_1\alpha_1') \leqslant c\max\{d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')\} \tag{15}$$

Similarly on using inequality (7), (13) and (15), we have,

$$d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')) \leqslant cd_1(\alpha_1, \alpha_1') \tag{16}$$

Using inequality (12), (13), (15) and (16), we have

$$\begin{aligned}
d_1(\alpha_1, \alpha_1') &\leqslant cd_2(A_1\alpha_1, A_1\alpha_1')\\
&\leqslant c^2d_3(A_2A_1\alpha_1, A_2A_1\alpha_1')\\
&\leqslant c^3d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')\\
&\leqslant c^4d_1(\alpha_1, \alpha_1')
\end{aligned}$$

Now we have

$$d_1(\alpha_1, \alpha_1') \leqslant c^4d_1(\alpha_1, \alpha_1')$$

Since $0 \leqslant c < 1$, we have

$$d_1(\alpha_1, \alpha_1') = 0$$

$\Rightarrow \quad \alpha_1 = \alpha_1'$, proving the uniqueness of α_1.

We can similarly prove that $A_1A_4A_3A_2$ has a unique fixed point $d_2 \in Z_2$ and $A_2A_1A_4A_3$ has a unique fixed point $\alpha_3 \in Z_3$ and $A_3A_2A_1A_4$ has unique fixed point $\alpha_4 \in Z_4$. □

Now, in support of our result, we give some examples.

Example 1 *Let suppose* $X = [0,1], Y = [1,2], Z = [2,3]$ *and* $L = [3,4]$ *be complete metric spaces with usual metric. If* $T : [0,1] \to [1,2]$, $S : [1,2] \to [2,3]$ *and* $R : [2,3] \to [3,4]$ *are continuous mappings and* $U : [3,4] \to [0,1]$ *is a mapping satisfying given conditions (in Theorem 2.1), where*

$$T(x) = \begin{cases} 1, & \text{if} \quad 0 \le x \le \frac{3}{4} \\ \frac{4}{3}x, & \text{if} \quad \frac{3}{4} < x \le 1 \end{cases}, \quad S(y) = \begin{cases} 2, & \text{if} \quad 1 \le y \le \frac{3}{2} \\ \frac{4}{3}y, & \text{if} \quad \frac{3}{2} < y \le 2 \end{cases}$$

$$R(z) = \begin{cases} 3, & \text{if} \quad 2 \le z \le \frac{5}{2} \\ \frac{6}{5}z, & \text{if} \quad \frac{5}{2} < z \le 3 \end{cases}, \quad U(u) = \begin{cases} 1, & \text{if} \quad 3 \le u \le \frac{7}{2} \\ \frac{3}{5}, & \text{if} \quad \frac{7}{2} < u \le 4 \end{cases}$$

then $URST$ *has fixed point 1 such that* $URST(1) = 1$, $TURS$ *has fixed point 4/3 such that* $TURS(4/3) = 4/3$, $STUR$ *has fixed point 2 such that* $STUR(2) = 2$ *and* $RSTU$ *has fixed point 3 such that* $RSTU(3) = 3$. *Also* $T(1) = 4/3$, $S(4/3) = 2$, $R(2) = 3$ *and* $U(3) = 1$.

Remark 1 *Below we give an example which satisfies all the condition of Theorem 2.1 but does not satisfies the condition of Theorem 1.1.*

Example 2 *Let* $X = [0,1], Y = [1,2], Z = [2,3]$ *and* $L = [3,4]$ *be complete metric space with usual metric. If* $T : [0,1] \to [1,2]$, $S : [1,2] \to [2,3]$ *and* $R : [2,3] \to [3,4]$ *are continuous mappings and* $U : [3,4] \to [0,1]$ *is a mapping satisfying given conditions (in Theorem 2.1), where*

$$T(x) = \begin{cases} 1, & \text{if} \quad 0 \le x \le \frac{1}{4} \\ 2x + \frac{1}{2}, & \text{if} \quad \frac{1}{4} \le x \le \frac{3}{4} \\ 2, & \text{if} \quad \frac{3}{4} \le x \le 1 \end{cases}, \quad S(y) = \begin{cases} 2, & \text{if} \quad 1 \le y \le \frac{5}{4} \\ \frac{4}{5}y + 1, & \text{if} \quad \frac{5}{4} \le y \le \frac{7}{4} \\ 2.4, & \text{if} \quad \frac{7}{4} \le y \le 2 \end{cases}$$

$$R(z) = \begin{cases} 3, & \text{if} \quad 2 \le z \le \frac{9}{4} \\ z + \frac{3}{4}, & \text{if} \quad \frac{9}{4} \le z \le \frac{11}{4} \\ 3.5, & \text{if} \quad \frac{11}{4} \le z \le 3 \end{cases}, \quad U(u) = \begin{cases} 0, & \text{if} \quad 3 \le u \le \frac{7}{2} \\ \frac{2}{7}u - 1, & \text{if} \quad \frac{7}{2} \le u \le 4 \end{cases}$$

then $URST$ *has fixed point* 0 *such that* $URST(0) = 0$*,* $TURS$ *has fixed point* 1 *such that* $TURS(1) = 1$*,* $STUR$ *has fixed point 2 such that* $STUR(2) = 2$ *and* $RSTU$ *has fixed point 3 such that* $RSTU(3) = 3$*. Also* $T(0)$*=1,* $S(1)$*=2,* $R(2)$*=3 and* $U(3)$*=0.*

Example 3 *Let suppose* $X = [0, 2], Y = [1, 5], Z = [0, 10]$ *and* $L = [1, 12]$ *be complete metric spaces with usual metric. If* $T : [0, 2] \to [1, 5]$*,* $S : [1, 5] \to [0, 10]$ *and* $R : [0, 10] \to [1, 12]$ *are continuous mappings and* $U : [1, 12] \to [0, 2]$ *is a mapping satisfying given conditions (in Theorem 2.1), where*

$$T(x) = [1 + x, 2], \qquad S(y) = [2y + 1, 5]$$

$$R(z) = [1 + z, 10], \qquad U(u) = \begin{cases} \left[\frac{u}{6}, 5\right] & \text{if} \quad 1 \leq z \leq 6 \\ \left[1, \frac{u}{6}\right] & \text{if} \quad 6 < z \leq 12 \end{cases}$$

then $URST$ *has fixed point* 1 *such that* $URST(1) = 1$*,* $TURS$ *has fixed point* 2 *such that* $TURS(2) = 2$*,* $STUR$ *has fixed point 5 such that* $STUR(5) = 5$ *and* $RSTU$ *has fixed point 6 such that* $RSTU(6) = 6$*. Also* $T(1) = 2$*,* $S(2) = 5$*,* $R(5) = 6$ *and* $U(6) = 1$*.*

Example 4 *Let suppose* $X = [0, 3], Y = [1, 4], Z = [4, 7]$ *and* $L = [3, 10]$ *be complete metric spaces with usual metric. If* $T : [0, 3] \to [1, 4]$*,* $S : [1, 4] \to [4, 7]$ *and* $R : [4, 7] \to [3, 10]$ *be a continuous mappings and* $U : [3, 10] \to [0, 3]$ *be a mapping satisfying given conditions (in Theorem 2.1), where*

$$T(x) = 1 + x, \qquad S(y) = y + 2,$$

$$R(z) = z + 3, \qquad U(u) = \begin{cases} \frac{u}{7} & \text{if} \quad 3 \leq z \leq 5 \\ \frac{u+1}{8} & \text{if} \quad 5 < z \leq 10 \end{cases}$$

then $URST$ *has fixed point* 1 *such that* $URST(1) = 1$*,* $TURS$ *has fixed point* 2 *such that* $TURS(2) = 2$*,* $STUR$ *has fixed point 4 such that* $STUR(4) = 4$ *and* $RSTU$ *has fixed point 7 such that* $RSTU(7) = 7$*. Also* $T(1) = 2$*,* $S(2) = 4$*,* $R(5) = 7$ *and* $U(7) = 1$*.*

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