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Certain classes of bi-univalent functions associated with the Horadam polynomials

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Abstract. In this paper we consider two subclasses of bi-univalent functions defined by the Horadam polynomials. Further, we obtain coefficient estimates for the defined classes.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in Δ.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both the function f and its inverse f^{-1} are univalent in Δ. Let Σ denote the class of bi-univalent functions in Δ given by (1).

In 2010, Srivastava et al. [28] revived the study of bi-univalent functions by their pioneering work on the study of coefficient problems. Various subclasses of the bi-univalent function class Σ were introduced and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1) were found in the very recent investigations (see, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30]) and including the references therein. The afore-cited all these papers on the subject were actually motivated by the work of Srivastava et al. [28]. However, the problem to find the coefficient bounds on $|a_n|$ $(n = 3, 4, \cdots)$ for functions $f \in \Sigma$ is still an open problem.

For analytic functions f and g in Δ, f is said to be subordinate to g if there exists an analytic function w such that

$$w(0) = 0, \qquad |w(z)| < 1 \quad \text{and} \quad f(z) = g(w(z)) \qquad (z \in \Delta).$$

This subordination will be denoted here by

$$f \prec g \qquad (z \in \Delta)$$

or, conventionally, by

$$f(z) \prec g(z) \qquad (z \in \Delta).$$

In particular, when g is univalent in Δ,

$$f \prec g \qquad (z \in \Delta) \;\Leftrightarrow\; f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

The Horadam polynomials $h_n(x,\ a,\ b;\ p,\ q)$, or briefly $h_n(x)$ are given by the following recurrence relation (see [14, 15]):

$$h_1(x) = a, \quad h_2(x) = bx \quad \text{and} \quad h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x) \quad (n \geq 3) \tag{2}$$

for some real constants a, b, p and q.

The generating function of the Horadam polynomials $h_n(x)$ (see [15]) is given by

$$\Pi(x,\ z) := \sum_{n=1}^{\infty} h_n(x) z^{n-1} = \frac{a + (b - ap)xz}{1 - pxz - qz^2}\ . \tag{3}$$

Here, and in what follows, the argument $x \in \mathbb{R}$ is independent of the argument $z \in \mathbb{C}$; that is, $x \neq \Re(z)$.

Note that for particular values of a, b, p and q, the Horadam polynomial $h_n(x)$ leads to various polynomials, among those, we list a few cases here (see, [14, 15] for more details):

1. For $a = b = p = q = 1$, we have the Fibonacci polynomials $F_n(x)$.
2. For $a = 2$ and $b = p = q = 1$, we obtain the Lucas polynomials $L_n(x)$.
3. For $a = q = 1$ and $b = p = 2$, we get the Pell polynomials $P_n(x)$.
4. For $a = b = p = 2$ and $q = 1$, we attain the Pell-Lucas polynomials $Q_n(x)$.
5. For $a = b = 1,\ p = 2$ and $q = -1$, we have the Chebyshev polynomials $T_n(x)$ of the first kind
6. For $a = 1,\ b = p = 2$ and $q = -1$, we obtain the Chebyshev polynomials $U_n(x)$ of the second kind.

Abirami et al. [1] considered bi- Mocanu - convex functions and bi-$\mu-$ starlike functions to discuss initial coefficient estimations of Taylor-Macularin series which is associated with Horadam polynomials, Abirami et al. [2] discussed coefficient estimates for the classes of $\lambda-$bi-pseudo-starlike and bi-Bazilevič functions using Horadam polynomial, Alamoush [3, 4] defined subclasses of bi-starlike and bi-convex functions involving the Poisson distribution series involving Horadam polynomials and a class of bi-univalent functions associated with Horadam polynomials respectively and obtained initial coefficient estimates, Altınkaya and Yalçın [7, 8] obtained coefficient estimates for Pascu-type bi-univalent functions and for the class of linear combinations of bi-univalent functions by means of (p, q)-Lucas polynomials respectively, Aouf et al. [10] discussed initial coefficient estimates for general class of pascu-type bi-univalent functions of complex order defined by $q-$Sălăgean operator and associated with Chebyshev polynomials, Awolere and Oladipo [11] found initial coefficients of bi-univalent functions defined by sigmoid functions involving pseudo-starlikeness associated with Chebyshev polynomials, Naeem et al. [18] considered a general class of bi-Bazilevič type functions associated with Faber polynomial to discuss n-th coefficients estimates, Magesh and Bulut [19] discussed Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions, Orhan et al. [21] discussed initial estimates and Fekete-Szegö bounds for bi-Bazilevič functions related to shell-like curves, Sakar and Aydogan [23] obtained initial bounds for the class of generalized Sălăgean type bi-$\alpha-$ convex functions of complex order associated with the Horadam polynomials, Singh et al. [24] found coefficient estimates for bi-α-convex functions defined by generalized Sãlãgean operator related to shell-like curves connected with Fibonacci numbers, Srivastava et al. [25] introduced a technique by defining a new class bi-univalent functions associated with the Horadam polynomials to discuss the coefficient estimates, Srivastava et al. [27] gave a direction to study the Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator, Srivastava et al. [29] obtained general coefficient $|a_n|$ for a general class analytic and bi-univalent functions defined by using differential subordination and a certain fractional derivative operator associated with Faber polynomial, Wanas and Alina [30] discussed applications of Horadam polynomials on Bazilevič bi-univalent functions by means of subordination and found initial bounds. Motivated in these lines, estimates on initial coefficients of the Taylor-Maclaurin series expansion (1) and Fekete-Szegő inequalities for certain classes of bi-univalent functions defined by means of Horadam polynomials are obtained. The classes introduced in this paper are motivated by the corresponding classes investigated in [16, 20].

2 Coefficient estimates and Fekete-Szegő inequalities

A function $f \in \mathcal{A}$ of the form (1) belongs to the class $\mathcal{G}^*_\Sigma(\alpha,\ x)$ for $0 \le \alpha \le 1$ and $z,\ w \in \Delta$, if the following conditions are satisfied:

$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)f'(z) \prec \Pi(x,\ z)+1-a$$

and for $g(w) = f^{-1}(w)$

$$\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\alpha)g'(w) \prec \Pi(x,\ w)+1-a,$$

where the real constant a is as in (2).

Remark 1 *The classes $\mathcal{K}_\Sigma(x)$ and $\mathcal{H}_\Sigma(x)$ are defined by $\mathcal{G}^*_\Sigma(1,\ x) := \mathcal{K}_\Sigma(x)$ and introduced by [1] and $\mathcal{G}^*_\Sigma(0,\ x) := \mathcal{H}_\Sigma(x)$ introduced by [4] respectively.*

For functions in the class $\mathcal{G}^*_\Sigma(\alpha,\ x)$, the following coefficient estimates and Fekete-Szegő inequality are obtained.

Theorem 1 *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{G}^*_\Sigma(\alpha,\ x)$. Then*

$$|a_2| \le \frac{|bx|\sqrt{|bx|}}{\sqrt{|(3-\alpha)\,b^2x^2-4\,(px^2b+qa)|}}, \qquad \text{and} \qquad |a_3| \le \frac{|bx|}{3\,(\alpha+1)}+\frac{b^2x^2}{4}$$

and for $\nu \in \mathbb{R}$

$$\left|a_3-\nu a_2^2\right| \le \begin{cases} \dfrac{|bx|}{3\alpha+3} & \text{if } |\nu-1| \le \dfrac{\left|(3-\alpha)\,b^2x^2-4\,(px^2b+qa)\right|}{b^2x^2\,(3\alpha+3)} \\[2ex] \dfrac{|bx|^3\,|\nu-1|}{|(3-\alpha)\,b^2x^2-4\,(px^2b+qa)|} & \text{if } |\nu-1| \ge \dfrac{\left|(3-\alpha)\,b^2x^2-4\,(px^2b+qa)\right|}{b^2x^2\,(3\alpha+3)}. \end{cases}$$

Proof. Let $f \in \mathcal{G}^*_\Sigma(\alpha,\ x)$ be given by the Taylor-Maclaurin expansion (1). Then, there are analytic functions r and s such that

$$r(0)=0; \quad s(0)=0, \quad |r(z)|<1 \quad \text{and} \quad |s(w)|<1 \quad (\forall\ z,\ w \in \Delta),$$

and we can write

$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)f'(z) = \Pi(x,\ r(z))+1-a \tag{4}$$

and

$$\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\alpha)g'(w)=\Pi(x,\ s(w))+1-a. \tag{5}$$

Equivalently,

$$\begin{aligned}\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)&+(1-\alpha)f'(z)\\ &=1+h_1(x)-a+h_2(x)r(z)+h_3(x)[r(z)]^2+\cdots\end{aligned} \tag{6}$$

and

$$\begin{aligned}\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)&+(1-\alpha)g'(w)\\ &=1+h_1(x)-a+h_2(x)s(w)+h_3(x)[s(w)]^2+\cdots.\end{aligned} \tag{7}$$

From (6) and (7) and in view of (3), we obtain

$$\begin{aligned}\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)&+(1-\alpha)f'(z)\\ &=1+h_2(x)r_1z+[h_2(x)r_2+h_3(x)r_1^2]z^2+\cdots\end{aligned} \tag{8}$$

and

$$\begin{aligned}\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)&+(1-\alpha)g'(w)\\ &=1+h_2(x)s_1w+[h_2(x)s_2+h_3(x)s_1^2]w^2+\cdots.\end{aligned} \tag{9}$$

If

$$r(z)=\sum_{n=1}^{\infty}r_nz^n \qquad \text{and} \qquad s(w)=\sum_{n=1}^{\infty}s_nw^n,$$

then it is well known that

$$|r_n|\le 1 \qquad \text{and} \qquad |s_n|\le 1 \qquad (n\in\mathbb{N}).$$

Thus upon comparing the corresponding coefficients in (8) and (9), we have

$$2a_2=h_2(x)r_1 \tag{10}$$

$$3\,(\alpha+1)\,a_3-4a_2^2\alpha=h_2(x)r_2+h_3(x)r_1^2 \tag{11}$$

$$-\,2a_2=h_2(x)s_1 \tag{12}$$

and

$$2\left(\alpha+3\right)a_2^2-3\left(\alpha+1\right)a_3=h_2(x)s_2+h_3(x)s_1^2. \tag{13}$$

From (10) and (12), we can easily see that

$$r_1=-s_1, \qquad \text{provided} \qquad h_2(x)=bx\neq 0 \tag{14}$$

and

$$\begin{aligned} 8\,a_2^2 &= \left(h_2\left(x\right)\right)^2\left(r_1{}^2+s_1{}^2\right) \\ a_2^2 &= \frac{1}{8}\left(h_2\left(x\right)\right)^2\left(r_1{}^2+s_1{}^2\right). \end{aligned} \tag{15}$$

If we add (11) to (13), we get

$$2\,a_2^2\left(3-\alpha\right) = \left(r_2+s_2\right)h_2\left(x\right)+h_3\left(x\right)\left(r_1{}^2+s_1{}^2\right). \tag{16}$$

By substituting (15) in (16), we obtain

$$a_2^2=\frac{\left(r_2+s_2\right)\left(h_2\left(x\right)\right)^3}{2\,\left(3-\alpha\right)\left(h_2\left(x\right)\right)^2-8\,h_3\left(x\right)} \tag{17}$$

and by taking $h_2(x)=bx$ and $h_3(x)=bpx^2+qa$ in (17), it further yields

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|(3-\alpha)\,b^2x^2-4\,\left(px^2b+qa\right)|}}. \tag{18}$$

By subtracting (13) from (11) we get

$$6\,\left(\alpha+1\right)\left(a_3-a_2{}^2\right)=\left(r_2-s_2\right)h_2\left(x\right)+\left(r_1{}^2-s_1{}^2\right)h_3\left(x\right).$$

In view of (14) , we obtain

$$a_3=\frac{\left(r_2-s_2\right)h_2\left(x\right)}{6(\alpha+1)}+a_2{}^2. \tag{19}$$

Then in view of (15), (19) becomes

$$a_3 = \frac{\left(r_2-s_2\right)h_2\left(x\right)}{6(\alpha+1)}+\frac{1}{8}\left(h_2\left(x\right)\right)^2\left(r_1{}^2+s_1{}^2\right).$$

Applying (2), we deduce that

$$|a_3| \leq \frac{|bx|}{3\left(\alpha+1\right)}+\frac{b^2x^2}{4}.$$

From (19), for $\nu \in \mathbb{R}$, we write

$$a_3 - \nu a_2^2 = \frac{h_2(x)\,(r_2 - s_2)}{6\,(\alpha + 1)} + (1 - \nu)\,a_2^2. \tag{20}$$

By substituting (17) in (20), we have

$$\begin{aligned} a_3 - \nu a_2^2 &= \frac{h_2(x)\,(r_2 - s_2)}{6\,(\alpha + 1)} + \left(\frac{(1 - \nu)\,(r_2 + s_2)\,(h_2\,(x))^3}{2\,(3 - \alpha)\,(h_2\,(x))^2 - 8\,h_3\,(x)} \right) \\ &= h_2(x) \left\{ \left(\Lambda_1(\nu,\ x) + \frac{1}{6\,(\alpha+1)} \right) r_2 + \left(\Lambda_1(\nu,\ x) - \frac{1}{6\,(\alpha+1)} \right) s_2 \right\}, \end{aligned} \tag{21}$$

where

$$\Lambda_1(\nu,\ x) = \frac{(1 - \nu)\,[h_2(x)]^2}{2\,(3 - \alpha)\,(h_2\,(x))^2 - 8\,h_3\,(x)}.$$

Hence, in view of (2) we conclude that

$$\left| a_3 - \nu a_2^2 \right| \leq \begin{cases} \dfrac{|h_2(x)|}{3(\alpha + 1)} & ; 0 \leq |\Lambda_1(\nu,\ x)| \leq \dfrac{1}{6\,(\alpha + 1)} \\ 2\,|h_2(x)|\,|\Lambda_1(\nu,\ x)| & ; |\Lambda_1(\nu,\ x)| \geq \dfrac{1}{6\,(\alpha + 1)} \end{cases}$$

and in view of (2), it evidently completes the proof of Theorem 1. □

Taking $\alpha = 1$ in Theorem 1, we have following corollary.

Corollary 1 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{K}_\Sigma(x)$*. Then*

$$|a_2| \leq \frac{|bx|\,\sqrt{|bx|}}{\sqrt{|2b^2x^2 - 4\,(px^2b + qa)|}}, \qquad \textit{and} \qquad |a_3| \leq \frac{|bx|}{6} + \frac{b^2x^2}{4}$$

and for $\nu \in \mathbb{R}$

$$\left| a_3 - \nu a_2^2 \right| \leq \begin{cases} \dfrac{|bx|}{6} & \textit{if } |\nu - 1| \leq \dfrac{\left|b^2x^2 - 2\,(px^2b + qa)\right|}{3b^2x^2} \\ \dfrac{|bx|^3\,|\nu - 1|}{|2b^2x^2 - 4\,(px^2b + qa)|} & \textit{if } |\nu - 1| \geq \dfrac{\left|b^2x^2 - 2\,(px^2b + qa)\right|}{3b^2x^2}. \end{cases}$$

Taking $\alpha = 0$ in Theorem 1, we have following corollary.

Corollary 2 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{H}_\Sigma(x)$. *Then*

$$|a_2| \leq \frac{|bx| \sqrt{|bx|}}{\sqrt{|3b^2x^2 - 4\ (px^2b + qa)|}}, \qquad \textit{and} \qquad |a_3| \leq \frac{|bx|}{3} + \frac{b^2x^2}{4}$$

and for $\nu \in \mathbb{R}$

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|bx|}{3} & \textit{if } |\nu - 1| \leq \dfrac{\left|3b^2x^2 - 4\ \left(px^2b + qa\right)\right|}{3b^2x^2} \\[2ex] \dfrac{|bx|^3\,|\nu - 1|}{|3b^2x^2 - 4\ (px^2b + qa)|} & \textit{if } |\nu - 1| \geq \dfrac{\left|3b^2x^2 - 4\ \left(px^2b + qa\right)\right|}{3b^2x^2}. \end{cases}$$

Next, a function $f \in \mathcal{A}$ of the form (1) belongs to the class $\mathcal{L}_\Sigma(x)$ and $z,\ w \in \Delta$, if the following conditions are satisfied:

$$\frac{1 + \dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} \prec \Pi(x,\ z) + 1 - a$$

and for $g(w) = f^{-1}(w)$

$$\frac{1 + \dfrac{wg''(w)}{g'(w)}}{\dfrac{wg'(w)}{g(w)}} \prec \Pi(x,\ w) + 1 - a,$$

where the real constant a is as in (2).

For functions in the class $\mathcal{L}_\Sigma(x)$, the following coefficient estimates and Fekete-Szegő inequality are obtained.

Theorem 2 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{L}_\Sigma(x)$. *Then*

$$|a_2| \leq \frac{|bx| \sqrt{|bx|}}{\sqrt{|px^2b + qa|}}, \qquad \textit{and} \qquad |a_3| \leq \frac{|bx|}{4} + b^2x^2$$

and for $\nu \in \mathbb{R}$

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|bx|}{4} & \text{if} \quad |\nu - 1| \leq \dfrac{\left|bpx^2 + aq\right|}{4b^2x^2} \\[2ex] \dfrac{|bx|^3 \, |\nu - 1|}{|bpx^2 + aq|} & \text{if} \quad |\nu - 1| \geq \dfrac{\left|bpx^2 + aq\right|}{4b^2x^2}. \end{cases}$$

Proof. Let $f \in \mathcal{L}_\Sigma(x)$ be given by the Taylor-Maclaurin expansion (1). Then, there are analytic functions r and s such that

$$r(0) = 0; \quad s(0) = 0, \quad |r(z)| < 1 \quad \text{and} \quad |s(w)| < 1 \quad (\forall\, z,\ w \in \Delta),$$

and we can write

$$\frac{1 + \dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} = \Pi(x,\ r(z)) + 1 - a \tag{22}$$

and

$$\frac{1 + \dfrac{wg''(w)}{g'(w)}}{\dfrac{wg'(w)}{g(w)}} = \Pi(x,\ s(w)) + 1 - a. \tag{23}$$

Equivalently,

$$\frac{1 + \dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} = 1 + h_1(x) - a + h_2(x)r(z) + h_3(x)[r(z)]^2 + \cdots \tag{24}$$

and

$$\frac{1 + \dfrac{wg''(w)}{g'(w)}}{\dfrac{wg'(w)}{g(w)}} = 1 + h_1(x) - a + h_2(x)s(w) + h_3(x)[s(w)]^2 + \cdots. \tag{25}$$

From (24) and (25) and in view of (3), we obtain

$$\frac{1 + \dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} = 1 + h_2(x)r_1 z + [h_2(x)r_2 + h_3(x)r_1^2]z^2 + \cdots \tag{26}$$

and

$$\frac{1+\frac{wg''(w)}{g'(w)}}{\frac{wg'(w)}{g(w)}} = 1 + h_2(x)s_1w + [h_2(x)s_2 + h_3(x)s_1^2]w^2 + \cdots . \tag{27}$$

If

$$r(z) = \sum_{n=1}^{\infty} r_n z^n \qquad \text{and} \qquad s(w) = \sum_{n=1}^{\infty} s_n w^n,$$

then it is well known that

$$|r_n| \leq 1 \qquad \text{and} \qquad |s_n| \leq 1 \qquad (n \in \mathbb{N}).$$

Thus upon comparing the corresponding coefficients in (26) and (27), we have

$$a_2 = h_2(x)r_1 \tag{28}$$

$$4\left(a_3 - a_2^2\right) = h_2(x)r_2 + h_3(x)r_1^2 \tag{29}$$

$$-a_2 = h_2(x)s_1 \tag{30}$$

and

$$4\left(a_2^2 - a_3\right) = h_2(x)s_2 + h_3(x)s_1^2. \tag{31}$$

From (28) and (30), we can easily see that

$$r_1 = -s_1, \qquad \text{provided} \qquad h_2(x) = bx \neq 0 \tag{32}$$

and

$$\begin{aligned} 2\,a_2^2 &= (h_2(x))^2\left({r_1}^2 + {s_1}^2\right) \\ a_2^2 &= \frac{1}{2}(h_2(x))^2\left({r_1}^2 + {s_1}^2\right) . \end{aligned} \tag{33}$$

If we add (29) to (31), we get

$$0 = (r_2 + s_2)\,h_2(x) + h_3(x)\left({r_1}^2 + {s_1}^2\right). \tag{34}$$

By substituting (33) in (34), we obtain

$$a_2^2 = -\frac{(r_2 + s_2)\,(h_2(x))^3}{2\,h_3(x)} \tag{35}$$

and by taking $h_2(x) = bx$ and $h_3(x) = bpx^2 + qa$ in (35), it further yields

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|px^2b + qa|}}. \tag{36}$$

By subtracting (31) from (29) we get

$$-8\left(a_2{}^2 - a_3\right) = (r_2 - s_2)\, h_2(x) + \left(r_1{}^2 - s_1{}^2\right) h_3(x)$$

In view of (32) , we obtain

$$a_3 = \frac{1}{8}(r_2 - s_2)\, h_2(x) + a_2{}^2. \tag{37}$$

Then in view of (33), (37) becomes

$$a_3 = \frac{1}{8}(r_2 - s_2)\, h_2(x) + \frac{1}{2}(h_2(x))^2\left(r_1{}^2 + s_1{}^2\right).$$

Applying (2), we deduce that

$$|a_3| \leq \frac{|bx|}{4} + b^2x^2.$$

From (37), for $\nu \in \mathbb{R}$, we write

$$a_3 - \nu a_2^2 = \frac{1}{8}h_2(x)\,(r_2 - s_2) + (1-\nu)\, a_2^2. \tag{38}$$

By substituting (35) in (38), we have

$$\begin{aligned} a_3 - \nu a_2^2 &= \frac{1}{8}h_2(x)\,(r_2 - s_2) + \left(\frac{(\nu-1)\,(r_2+s_2)\,(h_2(x))^3}{2\,h_3(x)}\right) \\ &= h_2(x)\left\{\left(\Lambda_2(\nu,\ x) + \frac{1}{8}\right) r_2 + \left(\Lambda_2(\nu,\ x) - \frac{1}{8}\right) s_2\right\}, \end{aligned} \tag{39}$$

where

$$\Lambda_2(\nu,\ x) = \frac{(\nu-1)\,(h_2(x))^2}{2\,h_3(x)}.$$

Hence, in view of (2) we conclude that

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|h_2(x)|}{4}; & 0 \leq |\Lambda_2(\nu,\ x)| \leq \dfrac{1}{8} \\ \\ 2\,|h_2(x)|\,|\Lambda_2(\nu,\ x)|; & |\Lambda_2(\nu,\ x)| \geq \dfrac{1}{8} \end{cases}$$

and in view of (2), it evidently completes the proof of Theorem 2. □

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