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Some new inequalities via s-convex functions on time scales

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Abstract. In this paper, we prove some new integral inequalities for s-convex function on time scale. We give results for the case when time scale is $\mathbb{R}$ and when time scale is $\mathbb{N}$.

1 Introduction

The study of various types of integral inequalities for convex functions has been the focus of great attention for well over a century by a number of scientists, interested both in pure and applied mathematics. Out of these inequalities Ostrowski inequality and Hermite-Hadamard inequality are the most common inequalities. These two inequalities have applications in numerical analysis, probability, optimization theory, stochastic, statistics, information and integral operator theory. Also these inequalities have various implementation in trapezoid, Simpson and quadrature rules, etc. The basic definitions of Ostrowski and Hermite-Hadamard inequalities are as follows.

The Ostrowski inequality [21] for a differetiable mapping Υ on the interior of an interval $\intercal$ with $|\Upsilon'(\mathsf{c})| \leq \mathsf{M}$, where Υ' implies first derivative of Υ, is

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defined as:

$$\left|\Upsilon(k) - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \Upsilon(c)dc\right| \leq M(b_2 - b_1)\left[\frac{1}{4} + \frac{\left(k - \frac{b_1+b_2}{2}\right)^2}{(b_2 - b_1)^2}\right], \quad (1)$$

for $b_1 < b_2 \in \intercal$. This inequality gives an upper bound for the approximation of the integral average $-\frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon(c)dc$ by the value $\Upsilon(c)$ at point $c \in [b_1, b_2]$. The above inequality is then further generalized by researchers. For instance see [2, 6, 19]. On the other hand, for a convex function $\Upsilon : \intercal \to \mathbb{R}$ on an interval $\intercal$, the Hermite-Hadamard inequality [10, 11] is defined as:

$$\Upsilon\left(\frac{b_1 + b_2}{2}\right) \leq \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \Upsilon(c)dc \leq \frac{\Upsilon(b_1) + \Upsilon(b_2)}{2}, \quad (2)$$

for all $b_1, b_2 \in \intercal$ with $b_1 < b_2$. The inequality (2) is the special case of Jensen inequality. For more generalizations and details see [9, 13, 14, 15, 16, 17, 18, 20].

During last few decades, the inequalities (1) and (2) have been proved on time scale, see [1, 3, 7, 8, 23] for more information. Of course the role of inequalities (1) and (2) on time scales are similar as in usual sense. Here we prove some Ostrowski and Hermite-Hadamard's type inequalities for s-convex functions on time scale. We also extend the results given in [22]. In [22], Tahir et. al. proved several useful identities for convex functions on time scales. By using some of these identities we find certain useful inequalities for s-convex functions. Our work has many mathematical applications and has flexibility to extend it for more useful results.

2 Preliminaries

A time scale is a nonepmty closed subset $\mathbb{T}$ of $\mathbb{R}$. Most common examples are $\mathbb{R}$ and $\mathbb{N}$.

The forward and the backward jump operators respectively, denoted by σ and ρ, are defined as:

$$\sigma(k) = \inf\{c \in \mathbb{T} : c > k\}, \quad \rho(k) = \sup\{c \in \mathbb{T} : c < k\},$$

for all $k \in \mathbb{T}$.

The number k is called right-scattered if $\sigma(k) > k$ and is called left scattered if $\rho(k) < k$. Moreover, k is called isolated if both the right-scattered and the left-scattered. Similarly, the number k is called right dense or left dense if

$\sigma(k) = k$ or $\rho(k) = k$, respectively. Furthermore, k is called dense if it is right dense and left dense simultaneously.

The mappings $\mu, \tau : \mathbb{T} \to [0, \infty)$ defined by

$$\mu(k) := \sigma(k) - k, \qquad \tau(k) := k - \rho(k),$$

are known as forward and backward graininess functions, respectively.

A function $\Upsilon : \mathbb{T} \to \mathbb{R}$ is called rd-continuous C_{rd} if it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points of $\mathbb{T}$.

If $\Upsilon \in C_{rd}$ and $k_1 \in \mathbb{T}$, then we have

$$F(k) = \int_{k_1}^{k} \Upsilon(c)\Delta c, \qquad k \in \mathbb{T}.$$

That is, for $\Upsilon \in C_{rd}$ implies $\int_{b_1}^{b_2} \Upsilon(c)\Delta c = F(b_1) - F(b_2)$, where $F^{\Delta} = \Upsilon$.

Theorem 1 ([4]) *Let* $b_1, b_2, b_3 \in \mathbb{T}$, $\Upsilon, \Upsilon_1, \Upsilon_2 \in C_{rd}$, $\varpi \in \mathbb{R}$ *and* σ *is forward jump operator, then*

(i). $\int_{b_1}^{b_2} (\Upsilon_1(c) + \Upsilon_2(c))\Delta c = \int_{b_1}^{b_2} \Upsilon_1(c)\Delta c + \int_{b_1}^{b_2} \Upsilon_2(c)\Delta c$;

(ii). $\int_{b_1}^{b_2} \varpi \Upsilon(c)\Delta c = \varpi \int_{b_1}^{b_2} \Upsilon(c)\Delta c$;

(iii). $\int_{b_2}^{b_1} \Upsilon(c)\Delta c = -\int_{b_1}^{b_2} \Upsilon(c)\Delta c$;

(iv). $\int_{b_1}^{b_2} \Upsilon(c)\Delta c = \int_{b_1}^{b_3} \Upsilon(c)\Delta c + \int_{b_3}^{b_2} \Upsilon(c)\Delta c$;

(v). $\int_{b_1}^{b_2} \Upsilon_1^{\sigma}(c) \Upsilon_2^{\Delta}(c)\Delta c = (\Upsilon_1\Upsilon_2)(b_2) - (\Upsilon_1\Upsilon_2)(b_1) - \int_{b_1}^{b_2} \Upsilon_1^{\Delta}(c) \Upsilon_2(c)\Delta c$;

(vi). $\int_{b_1}^{b_2} \Upsilon_1(c) \Upsilon_2^{\Delta}(c)\Delta c = (\Upsilon_1\Upsilon_2)(b_2) - (\Upsilon_1\Upsilon_2)(b_1) - \int_{b_1}^{b_2} \Upsilon_1^{\Delta}(c) \Upsilon_2^{\sigma}(c)\Delta c$;

(vii). $\int_{b_1}^{b_1} \Upsilon(c)\Delta c = 0$;

(viii). *If* $\Upsilon(c) \geq 0$ *for all* c, *then* $\int_{b_1}^{b_2} \Upsilon(c)\Delta c \geq 0$;

(ix). *If* $|\Upsilon_1(c)| \leq \Upsilon_2(c)$ *on* $[b_1, b_2]$, *then*

$$\left| \int_{b_1}^{b_2} \Upsilon_1(c)\Delta c \right| \leq \int_{b_1}^{b_2} \Upsilon_2(c)\Delta c.$$

From Theorem 1 (ix), for $\Upsilon_2(c) = |\Upsilon_1(c)|$ on $[b_1, b_2]$, we have

$$\left|\int_{b_1}^{b_2} \Upsilon(c)\Delta c\right| \leq \int_{b_1}^{b_2} |\Upsilon(c)|\Delta c.$$

Definition 1 ([12]) *Consider a time scale* $\mathbb{T}$ *and* $s \in (0,1]$. *A function* $\Upsilon : \mathsf{T} \subset \mathbb{T} \to \mathbb{R}_0$, *where* $\mathbb{R}_0 = [0,\infty)$, *is called* s*-convex function in second sense, if*

$$\Upsilon(tb_1 + (1-t)b_2) \leq t^s \Upsilon(b_1) + (1-t)^s \Upsilon(b_2), \tag{3}$$

for all $b_1, b_2 \in \mathsf{T}$ *and* $t \in [0,1]$.

3 Main results

First we prove the following identity.

Lemma 1 *Consider a time scale* $\mathbb{T}$ *and* $\mathsf{T} = [b_1, b_2] \subseteq \mathbb{T}$ *such that* $b_1 < b_2 \in \mathbb{T}$. *Let* $\Upsilon : \mathsf{T} \to \mathbb{R}$ *be a delta differentiable mapping on* T^o, *where* T^o *is the interior of* T. *If* $\Upsilon^\Delta \in C_{rd}$ *then following equality holds:*

$$\begin{aligned}&\frac{\Upsilon(b_1)+\Upsilon(b_2)}{2} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon^\sigma(c)\Delta c\\ &= \frac{1}{2(b_2-b_1)}\left[\int_{b_1}^{b_2}(c-b_1)\Upsilon^\Delta(c)\Delta c - \int_{b_1}^{b_2}(b_2-c)\Upsilon^\Delta(c)\Delta c\right].\end{aligned} \tag{4}$$

Proof. By using the formula

$$\int_{b_1}^{b_2} \Upsilon_1(c)\Upsilon_2^\Delta(c)\Delta c(\Upsilon_1\Upsilon_2)(b_2) - (\Upsilon_1\Upsilon_2)(b_1) - \int_{b_1}^{b_2} \Upsilon_1^\Delta(c)\Upsilon_2^\sigma(c)\Delta c,$$

with $\Upsilon_1(c) = \frac{c-b_1}{b_2-b_1}$, $\Upsilon_2(c) = \Upsilon(c)$ in first integral and $\Upsilon_1(c) = \frac{c-b_2}{b_1-b_2}$, $\Upsilon_2(c) = \Upsilon(c)$ in second integral , we have

$$\begin{aligned}&\int_{b_1}^{b_2} \frac{c-b_1}{b_2-b_1}\Upsilon^\Delta(c)\Delta c - \int_{b_1}^{b_2}\frac{c-b_2}{b_1-b_2}\Upsilon^\Delta(c)\Delta c\\ &= \left[\Upsilon(b_2) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^\sigma(c)\Delta c\right] - \left[-\Upsilon(b_1) + \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^\sigma(c)\Delta c\right] \qquad (5)\\ &= \Upsilon(b_1)+\Upsilon(b_2) - \frac{2}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^\sigma(c)\Delta c.\end{aligned}$$

Then by multiplying $\frac{1}{2}$ on both sides of equation (5), we get the required equality (4) (also see the proof of Lemma 3.1 in [5]). □

Corollary 1 *Let* $\mathbb{T} = \mathbb{R}$ *in Lemma 1, then we have*

$$\begin{aligned}&\frac{\Upsilon(b_1)+\Upsilon(b_2)}{2} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon(c)dc\\&= \frac{1}{2(b_2-b_1)}\left[\int_{b_1}^{b_2}(c-b_1)\,\Upsilon'(c)dc - \int_{b_1}^{b_2}(b_2-c)\,\Upsilon'(c)dc\right].\end{aligned} \tag{6}$$

Corollary 2 *Let* $\mathbb{T} = \mathbb{N}$ *in Lemma 1. Let* $b_1 = 0$, $b_2 = d$, $c = x$ *and* $\Upsilon(k) = c_k$, *then*

$$\frac{c_0+c_d}{2} - \frac{1}{d}\sum_{x=0}^{d} c_x = \frac{1}{2d}\left[\sum_{x=0}^{d-1} x\Delta c_x - \sum_{x=0}^{d-1}(d-x)\Delta c_x\right]. \tag{7}$$

Corollary 3 *Under the assumptions of Lemma 1, we have*

$$\begin{aligned}&\frac{\Upsilon(b_1)+\Upsilon(b_2)}{2} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\\&= \frac{b_2-b_1}{2}\left[\int_0^1 t\,\Upsilon^{\Delta}(tb_2+(1-t)b_1)\Delta t - \int_0^1 t\,\Upsilon^{\Delta}(tb_1+(1-t)b_2)\Delta t\right].\end{aligned} \tag{8}$$

Proof. In Lemma 1 using change of variable method, that is, by taking $t = \frac{c-b_1}{b_2-b_1}$, we find

$$\int_{b_1}^{b_2}\frac{c-b_1}{b_2-b_1}\,\Upsilon^{\Delta}(c)\Delta c = (b_2-b_1)\int_0^1 t\,\Upsilon^{\Delta}(tb_2+(1-t)b_1)\Delta t. \tag{9}$$

Similarly, by taking $t = \frac{c-b_2}{b_1-b_2}$, we get

$$\int_{b_1}^{b_2}\frac{c-b_2}{b_1-b_2}\,\Upsilon^{\Delta}(c)\Delta c = (b_2-b_1)\int_0^1 t\,\Upsilon^{\Delta}(tb_1+(1-t)b_2)\Delta t. \tag{10}$$

Hence by using (9) and (10), we get the required equality (8). □

Theorem 2 *Consider a time scale* $\mathbb{T}$ *and* $\mathsf{T} = [b_1, b_2] \subseteq \mathbb{T}$ *such that* $b_1 < b_2 \in \mathbb{T}$. *Let* $\Upsilon : \mathsf{T} \to \mathbb{R}$ *be a delta differentiable mapping on* T^o, *where* T^o *is the interior of* T. *If* $|\Upsilon^{\Delta}|$ *is* s*-convex then following inequality holds:*

$$\left|\frac{\Upsilon(b_1)+\Upsilon(b_2)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\right| \leq \frac{b_2-b_1}{2}\lambda_1\left(|\Upsilon^{\Delta}(b_2)|+|\Upsilon^{\Delta}(b_1)|\right), \tag{11}$$

where

$$\lambda_1=\int_0^1(t^{s+1}+t(1-t)^s)\Delta t.$$

Proof. Using Corollary 3, property of modulus and convexity of $|\Upsilon^{\Delta}|$, we find

$$\begin{aligned}
&\left|\frac{\Upsilon(b_2)+\Upsilon(b_1)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\right| \\
&\leq \frac{b_2-b_1}{2}\left[\int_0^1 t|\Upsilon^{\Delta}(tb_2+(1-t)b_1)|\Delta t\right. \\
&\quad\left.+\int_0^1 t|\Upsilon^{\Delta}(tb_1+(1-t)b_2)|\Delta t\right] \\
&\leq \frac{b_2-b_1}{2}\left[\int_0^1 t\{t^s|\Upsilon^{\Delta}(b_2)|+(1-t)^s|\Upsilon^{\Delta}(b_1)|\}\Delta t\right. \\
&\quad\left.+\int_0^1 t\{t^s|\Upsilon^{\Delta}(b_1)|+(1-t)^s|\Upsilon^{\Delta}(b_2)|\}\Delta t\right] \\
&= \frac{b_2-b_1}{2}\left[\left(|\Upsilon^{\Delta}(b_2)|+|\Upsilon^{\Delta}(b_1)|\right)\int_0^1(t^{s+1}+t(1-t)^s)\Delta t\right] \\
&= \frac{b_2-b_1}{2}\lambda_1\left(|\Upsilon^{\Delta}(b_2)|+|\Upsilon^{\Delta}(b_1)|\right).
\end{aligned} \tag{12}$$

Hence the proof. □

Remark 1 *If* $\mathbb{T}=\mathbb{R}$, *then inequality* (11) *becomes:*

$$\left|\frac{\Upsilon(b_1)+\Upsilon(b_2)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon(c)dc\right| \leq \frac{b_2-b_1}{2(s+1)}\left(|\Upsilon'(b_2)|+|\Upsilon'(b_1)|\right). \tag{13}$$

Theorem 3 *Consider a time scale* $\mathbb{T}$ *and* $\mathsf{T}=[b_1,b_2]\subseteq\mathbb{T}$ *such that* $b_1<b_2\in\mathbb{T}$. *Let* $\Upsilon:\mathsf{T}\to\mathbb{R}$ *be a delta differentiable mapping on* T^o, *where* T^o *is the interior of* T. *If* $|\Upsilon^{\Delta}|^q$ *is* s*-convex, for* $q>1$ *such that* $\frac{1}{r}+\frac{1}{q}=1$, *then following inequality holds:*

$$\begin{aligned}
&\left|\frac{\Upsilon(b_2)+\Upsilon(b_1)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\right| \\
&\leq \frac{b_2-b_1}{2}\left(\int_0^1 t^r \Delta t\right)^{\frac{1}{r}} \\
&\times\left[\left(\int_0^1\left(t^s|\Upsilon^{\Delta}(b_2)|^q+(1-t)^s|\Upsilon^{\Delta}(b_1)|^q\right)\Delta t\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left(t^s|\Upsilon^{\Delta}(b_1)|^q+(1-t)^s|\Upsilon^{\Delta}(b_2)|^q\right)\Delta t\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{14}$$

Proof. Using Corollary 3, property of modulus, Holder's integral inequality and convexity of $|\Upsilon^{\Delta}|^q$, we find

$$\begin{aligned}
&\left|\frac{\Upsilon(b_2)+\Upsilon(b_1)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\right| \\
&=\frac{b_2-b_1}{2}\left|\int_0^1 t\,\Upsilon^{\Delta}(tb_2+(1-t)b_1)\Delta t-\int_0^1 t\,\Upsilon^{\Delta}(tb_1+(1-t)b_2)\Delta t\right| \\
&\leq\frac{b_2-b_1}{2}\left[\left|\int_0^1 t\,\Upsilon^{\Delta}(tb_2+(1-t)b_1)\Delta t\right|+\left|\int_0^1 t\,\Upsilon^{\Delta}(tb_1+(1-t)b_2)\Delta t\right|\right] \\
&\leq\frac{b_2-b_1}{2}\left(\int_0^1 t^r\Delta t\right)^{\frac{1}{r}}\left[\left(\int_0^1\left|\Upsilon^{\Delta}(tb_2+(1-t)b_1)\right|\Delta t^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left|\Upsilon^{\Delta}(tb_1+(1-t)b_2)\right|^q\Delta t\right)^{\frac{1}{q}}\right] \\
&\leq\frac{b_2-b_1}{2}\left(\int_0^1 t^r\Delta t\right)^{\frac{1}{r}}\left[\left(\int_0^1\left(t^s|\Upsilon^{\Delta}(b_2)|^q+(1-t)^s|\Upsilon^{\Delta}(b_1)|^q\right)\Delta t\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left(t^s|\Upsilon^{\Delta}(b_1)|^q+(1-t)^s|\Upsilon^{\Delta}(b_2)|^q\right)\Delta t\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{15}$$

Hence the proof. □

Remark 2 *If $\mathbb{T}=\mathbb{R}$, then inequality (14) becomes*

$$\left|\frac{\Upsilon(b_1)+\Upsilon(b_2)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon(c)dc\right|\leq\frac{b_2-b_1}{(r+1)^{\frac{1}{r}}}\left(\frac{|\Upsilon'(b_1)|^q+|\Upsilon'(b_2)|^q}{s+1}\right)^{\frac{1}{q}}. \tag{16}$$

Lemma 2 *Consider a time scale* $\mathbb{T}$ *and* $\mathsf{T} = [b_1, b_2] \subseteq \mathbb{T}$ *such that* $b_1 < b_2 \in \mathbb{T}$. *Let* $\Upsilon : \mathsf{T} \to \mathbb{R}$ *be a delta differentiable mapping on* T^o, *where* T^o *is the interior of* T. *If* $\Upsilon^\Delta \in C_{rd}$ *then following equality holds:*

$$\begin{aligned} &\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon^\sigma(c)\Delta c \\ &= \int_{b_1}^{\frac{b_1+b_2}{2}} \frac{c-b_1}{b_2-b_1}\,\Upsilon^\Delta(c)\Delta c + \int_{\frac{b_1+b_2}{2}}^{b_2} \left(\frac{c-b_1}{b_2-b_1} - 1\right)\Upsilon^\Delta(c)\Delta c. \end{aligned} \tag{17}$$

Proof. By using the formula

$$\int_{b_1}^{b_2} \Upsilon_1(c)\,\Upsilon_2^\Delta(c)\Delta c = (\Upsilon_1\Upsilon_2)(b_2) - (\Upsilon_1\Upsilon_2)(b_1) - \int_{b_1}^{b_2} \Upsilon_1^\Delta(c)\,\Upsilon_2^\sigma(c)\Delta c,$$

with $\Upsilon_1(c) = \frac{c-b_1}{b_2-b_1}$, $\Upsilon_2(c) = \Upsilon(c)$ in first integral and $\Upsilon_1(c) = \frac{c-b_1}{b_2-b_1} - 1$, $\Upsilon_2(c) = \Upsilon(c)$ in second integral, we have

$$\begin{aligned} &\int_{b_1}^{\frac{b_1+b_2}{2}} \frac{c-b_1}{b_2-b_1}\,\Upsilon^\Delta(c)\Delta c + \int_{\frac{b_1+b_2}{2}}^{b_2} \frac{c-b_2}{b_2-b_1}\,\Upsilon^\Delta(c)\Delta c \\ &= \frac{1}{2}\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{\frac{b_1+b_2}{2}} \Upsilon^\sigma(c)\Delta c \\ &\quad + \frac{1}{2}\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{\frac{b_1+b_2}{2}}^{b_2} \Upsilon^\sigma(c)\Delta c \\ &= \Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon^\sigma(c)\Delta c. \end{aligned} \tag{18}$$

Hence the proof. □

Corollary 4 *Let* $\mathbb{T} = \mathbb{R}$ *in Lemma 2, then*

$$\begin{aligned} &\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon(c)dc \\ &= \int_{b_1}^{\frac{b_1+b_2}{2}} \frac{c-b_1}{b_2-b_1}\,\Upsilon'(c)dc + \int_{\frac{b_1+b_2}{2}}^{b_2} \left(\frac{c-b_1}{b_2-b_1} - 1\right)\Upsilon'(c)dc. \end{aligned} \tag{19}$$

Corollary 5 *Let $\mathbb{T} = \mathbb{N}$ in Lemma 2. Let $b_1 = 0$, $b_2 = d$(with d is even), $c = x$ and $\Upsilon(k) = c_k$, then*

$$c_{\frac{d}{2}} - \frac{1}{d}\sum_{x=0}^{d} c_x = \frac{1}{d}\sum_{x=0}^{\frac{d}{2}-1} x\Delta c + \frac{1}{d}\sum_{x=\frac{d}{2}}^{d-1} (x-d)\Delta c. \tag{20}$$

Corollary 6 *Under the assumptions of Lemma 2, we have*

$$\begin{aligned} &\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon^{\sigma}(c)\Delta c \\ &= (b_2-b_1)\left[\int_0^{1/2} t\,\Upsilon^{\Delta}(tb_2+(1-t)b_1)\Delta t \right. \\ &\left. + \int_{1/2}^{1} (t-1)\,\Upsilon^{\Delta}(tb_2+(1-t)b_1)\Delta t\right]. \end{aligned} \tag{21}$$

Proof. In Lemma 2 using change of variable method, that is, by taking $t = \frac{c-b_1}{b_2-b_1}$, we find

$$\int_{b_1}^{\frac{b_1+b_2}{2}} \frac{c-b_1}{b_2-b_1}\Upsilon^{\Delta}(c)\Delta c = (b_2-b_1)\int_0^{1/2} t\,\Upsilon^{\Delta}(tb_2+(1-t)b_1)\Delta t, \tag{22}$$

and

$$\int_{\frac{b_1+b_2}{2}}^{b_2}\left(\frac{c-b_1}{b_2-b_1}-1\right)\Upsilon^{\Delta}(c)\Delta c = (b_2-b_1)\int_{1/2}^{1}(t-1)\,\Upsilon^{\Delta}(tb_2+(1-t)b_1)\Delta t. \tag{23}$$

Hence by using (22) and (23), we get the required equality (21). □

Theorem 4 *Consider a time scale $\mathbb{T}$ and $\mathsf{T} = [b_1, b_2] \subseteq \mathbb{T}$ such that $b_1 < b_2 \in \mathbb{T}$. Let $\Upsilon : \mathsf{T} \to \mathbb{R}$ be a delta differentiable mapping on T^o, where T^o is the interior of T. If $|\Upsilon^{\Delta}|$ is s-convex then following inequality holds:*

$$\left|\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\right| \le (b_2-b_1)\left(H_1|\Upsilon^{\Delta}(b_2)| + H_2|\Upsilon^{\Delta}(b_1)|\right), \tag{24}$$

where

$$H_1 = \int_0^{\frac{1}{2}} t^{s+1}\Delta t + \int_{\frac{1}{2}}^{1} t^s(1-t)\Delta t, \text{ and } H_2 = \int_0^{\frac{1}{2}} t(1-t)^s\Delta t + \int_{\frac{1}{2}}^{1}(1-t)^{s+1}\Delta t.$$

Proof. Using Corollary 6, property of modulus and s-convexity of $|\Upsilon^{\Delta}|$, we find

$$\begin{aligned}
&\left|\Upsilon\left(\frac{b_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\right| \\
&\leq (b_2-b_1)\left[\int_0^{1/2} t|\Upsilon^{\Delta}(tb_2+(1-t)b_1)|\Delta t\right. \\
&\quad\left.+\int_{1/2}^{1}|t-1||\Upsilon^{\Delta}(tb_2+(1-t)b_1)|\Delta t\right] \\
&\leq (b_2-b_1)\left[\int_0^{1/2} t\left(t^s|\Upsilon^{\Delta}(b_2)|+(1-t)^s|\Upsilon^{\Delta}(b_1)|\right)\Delta t\right. \\
&\quad\left.+\int_{1/2}^{1}(1-t)|\left(t^s|\Upsilon^{\Delta}(b_2)|+(1-t)^s|\Upsilon^{\Delta}(b_1)|\right)\Delta t\right] \\
&\leq (b_2-b_1)\left(H_1|\Upsilon^{\Delta}(b_2)|+H_2|\Upsilon^{\Delta}(b_1)|\right),
\end{aligned} \tag{25}$$

where

$$H_1=\int_0^{\frac{1}{2}} t^{s+1}\Delta t+\int_{\frac{1}{2}}^{1} t^s(1-t)\Delta t, \text{ and } H_2=\int_0^{\frac{1}{2}} t(1-t)^s\Delta t+\int_{\frac{1}{2}}^{1}(1-t)^{s+1}\Delta t.$$

Hence the proof is completed. □

Corollary 7 *If $\mathbb{T}=\mathbb{R}$ in Theorem 4, we get*

$$H_1=\int_0^{\frac{1}{2}} t^{s+1}dt+\int_{\frac{1}{2}}^{1} t^s(1-t)dt=\frac{1}{(s+1)(s+2)}\left[1-\frac{1}{2^{s+1}}\right],$$

and

$$H_2=\int_0^{\frac{1}{2}} t(1-t)^s dt+\int_{\frac{1}{2}}^{1}(1-t)^{s+1}dt=\frac{1}{(s+1)(s+2)}\left[1-\frac{1}{2^{s+1}}\right].$$

Hence inequality (24) becomes

$$\begin{aligned}
&\left|\Upsilon\left(\frac{b_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon(c)dc\right| \\
&\leq \frac{b_2-b_1}{(s+1)(s+2)}\left(1-\frac{1}{2^{s+1}}\right)\left(|\Upsilon'(b_1)|+|\Upsilon'(b_2)|\right).
\end{aligned} \tag{26}$$

Theorem 5 *Consider a time scale* $\mathbb{T}$ *and* $\mathsf{T} = [b_1, b_2] \subseteq \mathbb{T}$ *such that* $b_1 < b_2 \in \mathbb{T}$. *Let* $\Upsilon : \mathsf{T} \to \mathbb{R}$ *be a delta differentiable mapping on* T^o, *where* T^o *is the interior of* T. *If* $|\Upsilon^{\Delta}|^q$ *is* s*-convex, for* $q > 1$ *such that* $\frac{1}{r} + \frac{1}{q} = 1$, *then following inequality holds:*

$$\begin{aligned} \left|\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon^{\sigma}(c)\Delta c\right| &\leq (b_2-b_1)\left[\left(\int_0^{1/2} t^r \Delta t\right)^{\frac{1}{r}}\right. \\ &\times \left(|\Upsilon^{\Delta}(b_2)|^q \int_0^{1/2} t^s \Delta t + |\Upsilon^{\Delta}(b_1)|^q \int_0^{1/2} (1-t)^s \Delta t\right)^{\frac{1}{q}} \\ &+ \left(\int_{1/2}^{1} (1-t)^r \Delta t\right)^{\frac{1}{r}} \\ &\times \left.\left(|\Upsilon^{\Delta}(b_2)|^q \int_{1/2}^{1} t^s \Delta t + |\Upsilon^{\Delta}(b_1)|^q \int_{1/2}^{1} (1-t)^s \Delta t\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{27}$$

Proof. Using Corollary 6, property of modulus, Holder's integral inequality and s-convexity of $|\Upsilon^{\Delta}|^q$, we find

$$\begin{aligned} &\left|\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon^{\sigma}(c)\Delta c\right| \\ &\leq (b_2-b_1)\left[\left|\int_0^{1/2} t\,\Upsilon^{\Delta}(tb_2+(1-t)b_1)\Delta t\right|\right. \\ &\quad \left.+ \left|\int_{1/2}^{1} (t-1)\,\Upsilon^{\Delta}(tb_2+(1-t)b_1)\Delta t\right|\right] \\ &\leq (b_2-b_1)\left[\left(\int_0^{1/2} t^r \Delta t\right)^{\frac{1}{r}} \left(\int_0^{1/2} |\Upsilon^{\Delta}(tb_2+(1-t)b_1)|^q \Delta t\right)^{\frac{1}{q}}\right. \\ &\quad \left.+ \left(\int_{1/2}^{1} |1-t|^r \Delta t\right)^{\frac{1}{r}} \left(\int_{1/2}^{1} |\Upsilon^{\Delta}(tb_2+(1-t)b_1)|^q \Delta t\right)^{\frac{1}{q}}\right] \\ &\leq (b_2-b_1)\left[\left(\int_0^{1/2} t^r \Delta t\right)^{\frac{1}{r}}\left(\int_0^{1/2} (t^s|\Upsilon^{\Delta}(b_2)|^q + (1-t)^s|\Upsilon^{\Delta}(b_1)|^q)\Delta t\right)^{\frac{1}{q}}\right. \\ &\quad \left.+ \left(\int_{1/2}^{1} (1-t)^r \Delta t\right)^{\frac{1}{r}}\left(\int_{1/2}^{1} (t^s|\Upsilon^{\Delta}(b_2)|^q + (1-t)^s|\Upsilon^{\Delta}(b_1)|^q)\Delta t\right)^{\frac{1}{q}}\right] \end{aligned} \tag{28}$$

$$= (b_2 - b_1)\left[\left(\int_0^{1/2} t^r \Delta t\right)^{\frac{1}{r}} \left(|\Upsilon^{\Delta}(b_2)|^q \int_0^{1/2} t^s \Delta t + |\Upsilon^{\Delta}(b_1)|^q \int_0^{1/2} (1-t)^s \Delta t\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_{1/2}^1 (1-t)^r \Delta t\right)^{\frac{1}{r}} \left(|\Upsilon^{\Delta}(b_2)|^q \int_{1/2}^1 t^s \Delta t + |\Upsilon^{\Delta}(b_1)|^q \int_{1/2}^1 (1-t)^s \Delta t\right)^{\frac{1}{q}}\right].$$

Hence the proof. □

Corollary 8 *If $\mathbb{T} = \mathbb{R}$ in Theorem 5, then we have*

$$\int_0^{\frac{1}{2}} t^r dt = \int_{\frac{1}{2}}^1 (1-t)^r dt = \frac{1}{(r+1)2^{r+1}},$$
$$\int_0^{\frac{1}{2}} t^s dt = \int_{\frac{1}{2}}^1 (1-t)^s dt = \frac{1}{(s+1)2^{s+1}},$$

and

$$\int_0^{\frac{1}{2}} (1-t)^s dt = \int_{\frac{1}{2}}^1 t^s dt = \frac{1}{s+1} - \frac{1}{(s+1)2^{s+1}}.$$

Hence the inequality (27) becomes

$$\begin{aligned}
&\left|\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon(c)dc\right| \\
&\leq (b_2-b_1)\left(\frac{1}{2^{r+1}(r+1)}\right)^{\frac{1}{r}} \left[\left\{\frac{1}{2^{s+1}(s+1)}|\Upsilon'(b_2)|^q\right.\right. \\
&\quad \left. + \left(\frac{1}{s+1} - \frac{1}{2^{s+1}(s+1)}\right)|\Upsilon'(b_1)|^q\right\}^{\frac{1}{q}} \\
&\quad \left. + \left\{\frac{1}{2^{s+1}(s+1)}|\Upsilon'(b_1)|^q + \left(\frac{1}{s+1} - \frac{1}{2^{s+1}(s+1)}\right)|\Upsilon'(b_2)|^q\right\}^{\frac{1}{q}}\right].
\end{aligned} \tag{29}$$

Definition 2 ([5]) *Let $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by*

$$h_0(t,r) = 1 \text{ for all } r,t \in \mathbb{T}$$

and then recursively by

$$h_{k+1}(t,r) = \int_r^t h_k(\tau,r)\Delta\tau$$

for all $r,t \in \mathbb{T}$.

For next result we need following lemma.

Lemma 3 ([22]) *Let $\Upsilon : \mathbb{T} \to \mathbb{R}$ be a differentaible mapping and $b_1 < b_2 \in \mathbb{T}$. Let $\Upsilon^{\Delta} \in C_{rd}$ then following holds:*

$$\begin{aligned}&\Upsilon\,(b_1)\{1 - h_2(1,0)\} + \Upsilon(b_2)h_2(1,0) - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \Upsilon^{\sigma}(c)\Delta c \\ &= \frac{b_2 - b_1}{2}\int_0^1\int_0^1 [\Upsilon^{\Delta}(tb_1 + (1-t)b_2) - \Upsilon^{\Delta}(rb_1 + (1-r)b_2)](r - t)\Delta t\Delta r.\end{aligned} \tag{30}$$

Theorem 6 *Let $\Upsilon : \mathbb{T} \to \mathbb{R}$ be a differentaible mapping and $b_1 < b_2 \in \mathbb{T}$. Let $|\,\Upsilon^{\Delta}\,|$ be s-convex function, then following inequality holds:*

$$\begin{aligned}&\left|\Upsilon\,(b_1)\{1 - h_2(1,0)\} + \Upsilon(b_2)h_2(1,0) - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \Upsilon^{\sigma}(c)\Delta c\right| \\ &\leq \frac{b_2 - b_1}{2}[A_1|\,\Upsilon^{\Delta}\,(b_1)| + A_2|\,\Upsilon^{\Delta}\,(b_2)|],\end{aligned} \tag{31}$$

where

$$A_1 = \int_0^1\int_0^1 (t^s + r^s)(r + t)\Delta t\Delta r,$$

$$A_2 = \int_0^1\int_0^1 ((1 - t)^s + (1 - r)^s)(r + t)\Delta t\Delta r.$$

Proof. Using Lemma 3, modulus property and s-convexity of $|\,\Upsilon^{\Delta}\,|$, we have

$$\begin{aligned}&\left|\Upsilon\,(b_1)\{1 - h_2(1,0)\} + \Upsilon(b_2)h_2(1,0) - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \Upsilon^{\sigma}(c)\Delta c\right| \\ &\leq \frac{b_2 - b_1}{2}\int_0^1\int_0^1 |\,\Upsilon^{\Delta}\,(tb_1 + (1 - t)b_2) - \Upsilon^{\Delta}(rb_1 + (1 - r)b_2)||r - t|\Delta t\Delta r \\ &\leq \frac{b_2-b_1}{2}\int_0^1\int_0^1 [|\,\Upsilon^{\Delta}\,(tb_1+(1-t)b_2)|+|\,\Upsilon^{\Delta}\,(rb_1+(1-r)b_2)|](r+t)\Delta t\Delta r \\ &\leq \frac{b_2 - b_1}{2}\int_0^1\int_0^1 [(t^s|\,\Upsilon^{\Delta}(b_1)| + (1 - t)^s|\,\Upsilon^{\Delta}\,(b_2)|) \\ &\quad + (r^s|\,\Upsilon^{\Delta}\,(b_1)| + (1 - r)^s|\,\Upsilon^{\Delta}\,(b_2)|)](r + t)\Delta t\Delta r \\ &= \frac{b_2 - b_1}{2}\int_0^1\int_0^1 [(t^s+r^s)|\Upsilon^{\Delta}(b_1)|+((1-t)^s+(1-r)^s)|\Upsilon^{\Delta}(b_2)|](r+t)\Delta t\Delta r \\ &= \frac{b_2 - b_1}{2}[A_1|\,\Upsilon^{\Delta}\,(b_1)| + A_2|\,\Upsilon^{\Delta}\,(b_2)|],\end{aligned} \tag{32}$$

where

$$A_1 = \int_0^1 \int_0^1 (t^s + r^s)(r+t)\Delta t \Delta r,$$

$$A_2 = \int_0^1 \int_0^1 ((1-t)^s + (1-r)^s)(r+t)\Delta t \Delta r.$$

Hence the proof. □

Corollary 9 *Let $\mathbb{T} = \mathbb{R}$ in Theorem 6, then we have $\sigma(b) = b$ and*

$$h_2(1,0) = \int_0^1 (\tau - 1) d\tau = \frac{1}{2}.$$

Also,

$$\begin{aligned} A_1 &= \int_0^1 \int_0^1 (t^s + r^s)(r+t) dt dr = \frac{3s+4}{(s+1)(s+2)}, \\ A_2 &= \int_0^1 \int_0^1 ((1-t)^s + (1-r)^s)(r+t) dt dr = 2\beta(2, s+1) + \frac{1}{s+1}, \end{aligned} \tag{33}$$

and hence inequality (31) becomes,

$$\left| \frac{\Upsilon(b_1) + \Upsilon(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \Upsilon(c) dc \right|$$

$$\leq \frac{b_2 - b_1}{2} \left[\left(\frac{3s+4}{(s+1)(s+2)} \right) | \Upsilon'(b_1)| + \left(2\beta(2, s+1) + \frac{1}{s+1} \right) | \Upsilon'(b_2)| \right], \tag{34}$$

where β is Beta function.

Lemma 4 ([22]) *Let $\Upsilon : \mathsf{T} \subseteq \mathbb{T} \to \mathbb{R}$ be a delta differentaible mapping on T^o and $b_1 < b_2 \in \mathsf{T}$. Let $\Upsilon^\Delta \in C_{rd}$ then following equality holds:*

$$\begin{aligned} &\Upsilon\left(\frac{b_1 + b_2}{2}\right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \Upsilon^\sigma(c) \Delta c \\ &= \frac{b_2 - b_1}{2} \int_0^1 \int_0^1 [\Upsilon^\Delta(t b_1 + (1-t) b_2) \\ &- \Upsilon^\Delta(r b_1 + (1-r) b_2)](m(r) - m(t) \Delta t \Delta r, \end{aligned} \tag{35}$$

where

$$m(c) = \begin{cases} c, & c \in \left[0, \frac{1}{2}\right] \\ c-1, & c \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Theorem 7 *Let $\Upsilon : \mathsf{T} \subseteq \mathbb{T} \to \mathbb{R}$ be a delta differentaible mapping on T^o and $b_1 < b_2 \in \mathsf{T}$. Let $|\Upsilon^\Delta|$ be s-convex function, then following inequality holds:*

$$\left|\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon^\sigma(c)\Delta c\right| \leq \frac{b_2-b_1}{2}[B_1|\Upsilon^\Delta(b_1)| + B_2|\Upsilon^\Delta(b_2)|], \tag{36}$$

where

$$B_1 = \int_0^1\int_0^1 (t^s + r^s)(m(r) + m(t))\Delta t\Delta r,$$

$$B_2 = \int_0^1\int_0^1 ((1-t)^s + (1-r)^s)(m(r) + m(t))\Delta t\Delta r.$$

Proof. Using Lemma 4, modulus property and s-convexity of $|\Upsilon^\Delta|$, we have

$$\begin{aligned}
&\left|\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon^\sigma(c)\Delta c\right| \\
&\leq \frac{b_2-b_1}{2}\int_0^1\int_0^1 |\Upsilon^\Delta(tb_1 + (1-t)b_2) \\
&\quad - \Upsilon^\Delta(rb_1 + (1-r)b_2)||m(r) - m(t)|\Delta t\Delta r \\
&\leq \frac{b_2-b_1}{2}\int_0^1\int_0^1 [|\Upsilon^\Delta(tb_1 + (1-t)b_2)| \\
&\quad + |\Upsilon^\Delta(rb_1 + (1-r)b_2)|](m(r) + m(t))\Delta t\Delta r \\
&\leq \frac{b_2-b_1}{2}\int_0^1\int_0^1 [(t^s|\Upsilon^\Delta(b_1)| + (1-t)^s|\Upsilon^\Delta(b_2)|) \\
&\quad + (r^s|\Upsilon^\Delta(p_1)| + (1-r)^s|\Upsilon^\Delta(p_2)|)](m(r) + m(t))\Delta t\Delta r \\
&= \frac{b_2-b_1}{2}\int_0^1\int_0^1 [(t^s + r^s)|\Upsilon^\Delta(b_1)| \\
&\quad + ((1-t)^s + (1-r)^s)|\Upsilon^\Delta(b_2)|](m(r) + m(t))\Delta t\Delta r \\
&= \frac{b_2-b_1}{2}[B_1|\Upsilon^\Delta(b_1)| + B_2|\Upsilon^\Delta(b_2)|],
\end{aligned} \tag{37}$$

where

$$B_1 = \int_0^1 \int_0^1 (t^s + r^s)(m(r) + m(t))\Delta t \Delta r,$$

$$B_2 = \int_0^1 \int_0^1 ((1-t)^s + (1-r)^s)(m(r) + m(t))\Delta t \Delta r.$$

Hence the proof. □

Corollary 10 *Let* $\mathbb{T} = \mathbb{R}$ *in Theorem* 7*, then we have* $\sigma(b) = b$ *and*

$$B_1 = \int_0^1 \int_0^1 (t^s + r^s)(m(r) + m(t)dtdr = \frac{1}{s+1}\left[\frac{1}{2^s} - \frac{2}{s+2}\right], \tag{38}$$

$$\begin{aligned} B_2 &= \int_0^1 \int_0^1 ((1-t)^s + (1-r)^s)(m(r) + m(t)dtdr \\ &= 2\beta_{\frac{1}{2}}(2, s+1) - \frac{1}{s^{s+1}(s+2)}, \end{aligned} \tag{39}$$

and hence inequality (36) *becomes,*

$$\begin{aligned} &\left| \frac{\Upsilon(b_1) + \Upsilon(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \Upsilon(c)dc \right| \\ &\leq \frac{b_2 - b_1}{2}\left[\left(\frac{1}{s+1}\left[\frac{1}{2^s} - \frac{2}{s+2}\right]\right)|\Upsilon'(b_1)| \right. \\ &\quad \left. + \left(2\beta_{\frac{1}{2}}(2, s+1) - \frac{1}{s^{s+1}(s+2)}\right)|\Upsilon'(b_2)|\right], \end{aligned} \tag{40}$$

where β_u *is incomplete Beta function defined by*

$$\beta_u(b_1, b_2) = \int_0^u x^{b_1 - 1}(1-x)^{b_2 - 1}dx, \quad u \in (0, 1).$$

4 Conclusion

This research investigation includes some inequalities for s-convex function on time scales such as Hermite-Hadamard type inequalities. Some special cases are discussed, that is, when the time scale is $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$.

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References

[1] R. P. Agarwal, M. Bohner, A. Peterson, Inequalities on Time Scales: A Survey, *Math. Inequal. Appl.*, **4**(4) (2001), 537–557.

[2] M. Alomari, M. Darus, S. S. Dragomir, et al. Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, *Appl. Math. Lett.*, **23** (2010), 1071–1076.

[3] M. R. S. Ammi, R. A. C. Ferreira, D. F. M. Torres, Diamond-α Jensen's inequality on time scales, *J. Inequal. Appl.*, Hindawi Publishing Corporation, **2008** (2008), 13 pages.

[4] M. Bohner, A. Peterson, Dynamic Equation on Time Scale: An Introduction with Application, Birkhauser, Boston, Mess, USA, 2001.

[5] M. Bohner, T. Matthews, Ostrowski inequalities on time scales, *J. Inequal. Pure Appl. Math.*, **9**(1) (2008).

[6] P. Cerone, S. S. Dragomir, Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions, *Demonstratio Math.*, **37** (2004), 299–308.

[7] C. Dinu, Hermite-Hadamard inequality on time scale, *J. Inequal. Appl.*, Hindawi Publishing Corporation, **2008** (2018), p. 24.

[8] C. Dinu, Ostrowski type inequalities on time scales, *An. Univ. Craiova Ser. Mat. Inform.*, **34**(1) (2007), 43–58.

[9] S. S. Dragomir, R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. Lett.*, **11**(5) (1998), 91–95.

[10] J. Hadamard, "Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann", *J. Math. Pures Appl.*, (1893), 171–215.

[11] Ch. Hermite, Sur deux limites d'une intégrale denie, Mathesis., **3** (1883), 82.

[12] H. Hudzik and L. Maligranda, Some remarks on s-convex functions, A*equationes Math.* **48**(1) (1994), 100–111.

[13] N. Mehreen, M. Anwar, Integral inequalities for some convex functions via generalized fractional integrals, *J. Inequal. Appl.*, **2018** (2018), p. 208.

[14] N. Mehreen, M. Anwar, Hermite-Hadamard type inequalities via exponentially $\mathfrak{p}$-convex functions and exponentially $\mathfrak{s}$-convex functions in second sense with applications, *J. Inequal. Appl.*, **2019** (2019), p. 92.

[15] N. Mehreen, M. Anwar, Hermite-Hadamard type inequalities via exponentially $(\mathfrak{p}, \mathfrak{h})$-convex functions, *IEEE Access*, **8** (2020), 37589–37595.

[16] N. Mehreen, M. Anwar, On some Hermite-Hadamard type inequalities for $\mathfrak{tgs}$-convex functions via generalized fractional integrals, *Adv. Difference Equ.*, **2020** (2020), p. 6.

[17] N. Mehreen, M. Anwar, Hermite-Hadamard and Hermite-Hadamard-Fejer type inequalities for $\mathfrak{p}$-convex functions via conformable fractional integrals, *J. Inequal. Appl.*, **2020** (2020), p. 107.

[18] N. Mehreen, M. Anwar, Hermite-Hadamard and Hermite-Hadamard-Fejer type inequalities for $\mathfrak{p}$-convex functions via new fractional conformable integral operators, *J. Math. Compt. Sci.*, **19** (2019), 230–240.

[19] N. Mehreen, M. Anwar, Ostrowski type inequalities via some exponentially convex functions with applications, *AIMS Mathematics*, **5(2)** (2020), 1476–1483.

[20] N. Mehreen, M. Anwar, Some inequalities via ψ-Riemann—Liouville fractional integrals, *AIMS Mathematics*, **4(5)** (2019), 1403–1415.

[21] A. M. Ostrowski, Uber die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert, *Comment. Math. Helv.*, **10** (1938), 226–227.

[22] S. F. Tahir, M. Mushtaq, M. Muddassar, A new interpretation of Hermite-Hadamard's inequalities by the way of time scales, *J. Comput. Anal. Appl.*, **26**(2) (2019), 223–233.

[23] H. Yaldiz, P. Agarwal, s-convex functions on discrete time domains, *Analysis*, (2017).

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