Acta Univ. Sapientiae, Mathematica, **13**, 1 (2021) 209–216

DOI: 10.2478/ausm-2021-0012

Induced star-triangle factors of graphs

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Abstract. An induced star-triangle factor of a graph G is a spanning subgraph F of G such that each component of F is an induced subgraph on the vertex set of that component and each component of F is a star (here star means either $K_{1,n}$, $n \geq 2$ or K_2) or a triangle (cycle of length 3) in G. In this paper, we establish that every graph without isolated vertices admits an induced star-triangle factor in which any two leaves from different stars $K_{1,n}$ $(n \geq 2)$ are non-adjacent.

1 Introduction

A simple *graph* is denoted by $G(V(G), E(G))$, where $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G)$ are respectively the vertex set and edge set of G.. The *order* and *size* of G are $|V(G)|$ and $|E(G)|$, respectively. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, refers to the *neighborhood* of v. A cycle of order n is denoted by C_n and a *triangle* is denoted by C_3. A complete bipartite graph $K_{1,n}$ is called a *star*. In $K_{1,n}$, the vertex of degree n is its *center* and all other vertices are *leaves.*

2010 Mathematics Subject Classification: 05C60, 05C62, 05C70
Key words and phrases: Matching, factor, star-triangle factor

A *matching* in a graph is a set of independent edges. That is, a subset M of the edge set E of G is a matching if no two edges of M have a common vertex. A matching M is said to be *maximal* if there is no matching N strictly containing M, that is, M is maximal if it cannot be enlarged. A matching M is said to be *maximum* if it has the largest possible cardinality, that is, M is maximum if there is no matching N such that $|N| > |M|$. A vertex v is said to be M-*saturated* (or saturated by M) if there is an edge $e \in M$ incident with v. A vertex which is not incident with any edge of M is said to be M-*unsaturated.* An M-*alternating path* in G is a path whose edges are alternately in $E(G) - M$ and M. That is, in an M-alternating path, the edges alternate between M-edges and non-M-edges. An M-alternating path whose end vertices are M-unsaturated is said to be an M-*augmenting path.*

For $S \subset V(G)$, the *induced graph* on S is a subgraph of G with vertex set S and the edge set consisting of all the edges of G which have both end vertices in S. An *induced star* of G is an induced subgraph of G which itself is a star.

For a set S of connected graphs, a spanning subgraph F of a graph G is called an S-*factor* of G if each component of F is isomorphic to an element of S. A spanning subgraph F of a graph G is a *star-cycle* factor of G if each component of F is a star or a cycle. A spanning subgraph S of a graph G will be called an *induced star-triangle factor* of G if each component of S is an induced star ($K_{1,n}$, $n \geq 2$, or K_2) or a triangle of G.

For a vertex subset S of $V(G)$, let $G[S]$ and $G - S$, respectively, denote the subgraph of G induced by S and $V(G) - S$. Further, let $\mathrm{iso}(G)$ mean the number of isolated vertices in G and $\mathrm{Iso}(G)$ be the set of isolated vertices of G. Clearly $|\mathrm{Iso}(G)| = \mathrm{iso}(G)$. For more definitions and notations, we refer to [7].

Tutte [8] characterized graphs having $\{K_2,\ C_n : n \geq 3\}$-factor. An elementary and short proof of Tutte's characterization can be seen in [1]. Las Vergnas [6] and Amahashi and Kano [2] showed that, for an integer $n \geq 2$, a graph has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}\}$-factor if and only if $\mathrm{iso}(G - S) \leq n|S|$ for all $S \subset V(G)$. Berge and Las Vergnas [3] showed the existence of $\{K_{1,n},\ C_m : n \geq 1,\ m \geq 3\}$-factor in graphs. A short proof of this theorem can be seen in [4].

2 Main results

In [5], we established Boyer's conjecture on the dimension of sphere of influence of graphs having perfect matchings, by obtaining a factor of a given graph and then embedding that into a suitable finite dimensional Euclidean space. While working on the main conjecture, we encountered the following result, which we believe would of interest to a general reader.

Theorem 1 *Every graph without isolated vertices,*
admits an induced star-triangle factor in which any two leaves from different stars $K_{1,n}$ *(*$n \geq 2$*) are non adjacent.*

To prove the result, let G be any graph without isolated vertices.

Let $V(G)$ and $E(G)$, respectively, denote the vertex set and the edge set of G. Let M be the maximum matching in G, M' be the set of M-saturated vertices and I be the set of M-unsaturated vertices.

We adopt the following algorithm, which contains the gist of the proof of Theorem 1.

Algorithm 1

1. Let $M_1 = M$.
2. If $I \neq \emptyset$, then pick a vertex v from I, otherwise go to step 10.
3. Pick $u \in N(v)$ and call the edge uv as the *neighborhood edge* of v. (As v is not isolated, there exists an edge $uv \in E(G)$.) Then $u \in M'$. Otherwise $M \cup \{uv\}$ will be a larger matching than M, which is impossible.
4. Let $w \in M'$ such that $uw \in M$.
5. If S_u is not defined, define $S_u := \{w, v\}$, otherwise go to step 7.
6. Remove uw from M_1, go to step 8.
7. If S_u is defined then add v to S_u.
8. Set $J = I \setminus \{v\}$.
9. With $I = J$, go to step 2.
10. Stop.

At the end of this algorithm, we obtain a matching M_1, finitely many vertices $u_1, \ldots, u_k$ and the corresponding sets $S_{u_1}, \ldots, S_{u_k}$. Before we analyze these sets, let us consider an example to see how the algorithm works.

Example 1 *Consider a graph* G *on* 17 *vertices, given by Figure 1.*

Here

$$M = \{\{1,2\},\{7,8\},\{9,10\},\{13,14\},\{15,16\}\}$$

is a maximum matching and the corresponding set I *is given by* $\{3,4,5,6,11,12\}$. *Applying Algorithm 1, we obtain a factor of* G, *given by Figure 2.*

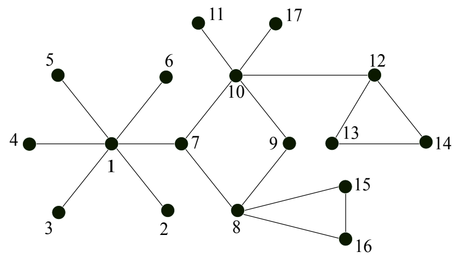

Figure 1: Graph G

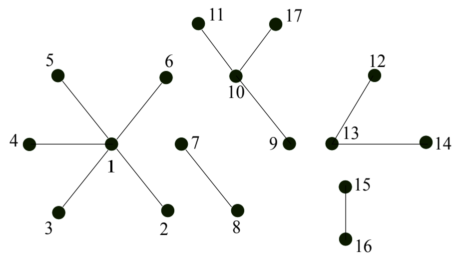

Figure 2: Output of Algorithm 1 on G

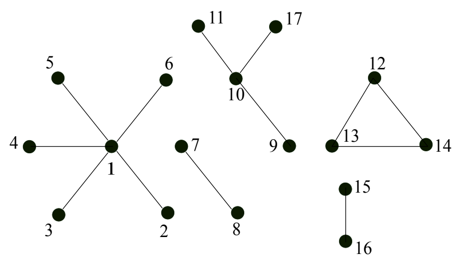

Figure 3: Star-triangle factor of G

After applying the procedure specified in the proof of Theorem 1 we will obtain the graph given by Figure 3, which is a required star-triangle factor of G. □

To prove Theorem 1, we need a series of lemmas. The first one is immediate.

Lemma 1 *1. Each* $v \in I$ *has exactly one neighborhood edge.*

2. *Each* S_{u_i} *has at least two vertices, exactly one vertex from* M'*, and* u_i *has a matching edge with that vertex.*

Using this lemma, we obtain the following result.

Lemma 2 *For each* $1 \leq i < j \leq k$*, we have*

$$(\{u_i\} \cup S_{u_i}) \cap (\{u_j\} \cup S_{u_j}) = \emptyset.$$

Proof. It is enough to prove the result for $i = 1$ and $j = 2$. Assume that there exists some $x \in (\{u_1\} \cup S_{u_1}) \cap (\{u_2\} \cup S_{u_2})$.

If $x \in I$, then $x \in S_{u_1}$ and $x \in S_{u_2}$. Therefore, xu_1 and xu_2 are the neighborhood edges of x. By Lemma 1, x has only one neighborhood edge, a contradiction. Therefore $x \notin I$ and thus $x \in M'$.

If $x \in \{u_1, u_2\}$, without loss of generality, let $x = u_1$. Then $u_1 \in S_{u_2}$. By Lemma 1, S_{u_2} has only one vertex from M', and u_2 has a matching edge with that vertex. Therefore, u_1u_2 is a matching edge, that is, $u_1u_2 \in M$. This implies that $u_2 \in S_{u_1}$.

Also, by Lemma 1, we have $|S_{u_1}| \geq 2$ and $|S_{u_2}| \geq 2$. Choose $x_1 \in S_{u_1}$ and $x_2 \in S_{u_2}$ such that $\{x_1, x_2\} \cap \{u_1, u_2\} = \emptyset$. Then $\{x_1, x_2\} \subseteq I$ and thus $x_1 \neq x_2$.

Therefore, $x_1u_1u_2x_2$ is an augmenting path of M, which implies that M is not a maximum matching, a contradiction. Hence $x \notin \{u_1, u_2\}$.

Consequently $x \in M'$ such that $x \in S_{u_1}$ and $x \in S_{u_2}$. Again, Lemma 1 ensures that xu_1 and xu_2 are matching edges. Hence xu_1 and xu_2 are not independent edges, a contradiction. □

Lemma 3 *The residual set* M_1 *is a matching. Further, if* M_1' *is the set of vertices of* M_1*, then* $V(G)$ *can be partitioned as*

$$V(G) = \left(\dot\cup_{i=1}^{k}(\{u_i\}\dot\cup S_{u_i})\right)\dot\cup M_1'.$$

Proof. Since M_1 embeds inside the matching M, it is a matching in G.

Pick any $y \in V(G)$. Then, either $y \in M_1'$ or $y \notin M_1'$. If $y \notin M_1'$, then by our construction $y \in \{u_i\} \cup S_{u_i}$, for some i.

Therefore, $y \in \cup_{i=1}^{k}(\{u_i\} \cup S_{u_i})$.

Thence, $V(G) \subset \left(\cup_{i=1}^{k}(\{u_i\} \cup S_{u_i})\right) \cup M_1'$. The other inclusion is trivial.

To prove that the union is disjoint, let $x \in \{u\} \cup S_u$, for some $u \in V(G)$. Then, either $x \in I$ or $x \in M'$. If $x \in I$, then $x \notin M'$ and thus $x \notin M_1'$. If $x \in M'$, then either S_x is defined or $x \in S_{x'}$ where xx' is a matching edge removed from M_1. Therefore, $x \notin M_1'$ and thence

$$M_1' \cap (\cup_{S_u}(\{u\} \cup S_u)) = \emptyset.$$

This along with Lemma 2, establishes the result. □

Lemma 4 *If* $u \in \{u_1, \ldots, u_k\}$ *and if there are* $v, w \in S_u$ *such that* $vw \in E(G)$, *then*

$$S_u = \{w, v\}.$$

Proof. If possible, choose $v' \in S_u \setminus \{w, v\}$. By our construction, there exists some $v'' \in S_u$ such that $v''u \in M$. We have the following cases to consider.

1. If $v'' \notin \{w, v\}$, then by construction, S_u has exactly one vertex from M' and all other vertices from I. Therefore $vw \notin M$ and thus $\{vw\} \cup M$ is a matching in G, larger than M.

2. If $v'' = w$, then $vwuv'$ is an M-augmented path.

3. If $v'' = v$, then $wvuv'$ is an M-augmented path.

Therefore, in each case, the augmented paths contradict the choice of M as a maximum matching. This proves our assertion. □

Proof of Theorem 1: First, we make a small change in our notations from Algorithm 1.

For each $S_u = \{v_1, v_2\}$, if $v_1v_2 \in E(G)$, then destroy (remove) S_u means from now onwards this S_u does not exist. Instead, if such an S_u exists, we do the following.

If T is not defined, then define $T := \{\{u, v_1, v_2\}\}$, otherwise add $\{u, v_1, v_2\}$ to T.

Basically, we are separating out the class of triangles from stars. Thus, we have found mutually exclusive stars $\{u\} \cup S_u$, triangles and a matching M_1 in G covering all the vertices.

Now, we establish that the remaining sets $\{u\} \cup S_u$ are stars.

Claim 1. Each S_u is an independent set.

To see this, note that we first defined S_u as having one vertex from M' and other from I. Then we added some vertices from I to S_u. Therefore, each S_u has one vertex from M' and remaining vertices from I.

Let $\{v_1, v_2\} \subseteq S_u$. If $\{v_1, v_2\} \subseteq I$, then clearly $v_1v_2 \notin E(G)$. Otherwise, without loss of generality, assume that $v_1 \in M'$ and $v_2 \in I$.

If $|S_u| = 2$, then by our construction, we have
$v_1v_2 \notin E(G)$. If $|S_u| > 2$, then there exists some $v_3 \in S_u \setminus \{v_1, v_2\}$. Therefore, $v_3 \in I$ and $v_1u \in M$.

If $v_1v_2 \in E(G)$, then $v_2v_1uv_3$ is an M-augmenting path. Therefore, M is not a maximum matching, a contradiction. Hence, $v_1v_2 \notin E(G)$. This establishes claim 1.

So we obtain a matching M_1, finitely many induced stars and triangles, all of which span our given graph G. Note that the matching M_1 can also be treated as a finite collection of induced stars K_2. Consequently, we obtain an induced star-triangle factor of G.

To conclude our main result, we claim the following.

Claim 2. The set $\cup S_u$ is independent.

To see this, let $\{v_1, v_2\} \subseteq \cup S_u$. We have to prove that $v_1v_2 \notin E(G)$. If $\{v_1, v_2\} \subseteq S_u$, for some u, then this follows by Claim 1. Without loss of generality, it is enough to assume that $v_1 \in S_{u_1}$ and $v_2 \in S_{u_2}$.

We have the following cases to consider.

1. $\{v_1, v_2\} \subseteq M'$. To prove by contradiction, assume that $v_1v_2 \in E(G)$.

   By our construction, $|S_{u_1}| \geq 2$ and $|S_{u_2}| \geq 2$. Therefore, we can choose $x_1 \in S_{u_1}$ and $x_2 \in S_{u_2}$ such that $x_1 \neq v_1$ and $x_2 \neq v_2$. Then $\{x_1, x_2\} \subseteq I$ and $\{u_1v_1, u_2v_2\} \subseteq M$. Therefore, $x_1u_1v_1v_2u_2x_2$ is an M-augmenting path, concluding that M is not the maximum matching, a contradiction.

2. $\{v_1, v_2\} \subseteq I$. Clearly, $v_1v_2 \notin E(G)$, as I is an independent set.

3. $v_1 \in M'$ and $v_2 \in I$. (The other case $v_1 \in I$ and $v_2 \in M'$ is similar.) To prove by contradiction, assume that $v_1v_2 \in E(G)$.

   Since $|S_{u_1}| \geq 2$, there exists some $x_1 \in S_{u_1} \setminus \{v_1\}$. As S_u has only one vertex from M' and $v_1 \in M'$, we have $x_1 \in I$. Also, $u_1v_1 \in M$ ensures that $x_1u_1v_1v_2$ is an M-augmenting path. Thus, M is not the maximum matching, a contradiction.

Therefore, in every case $v_1v_2 \notin E(G)$. This establishes claim 2. Hence, Theorem 1 is proved. □

Acknowledgements

We are grateful to the anonymous referee for his useful suggestions.

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Received: August 22, 2020