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Generalizations of graded second submodules

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Abstract. Let G be a group with identity e. Let R be a graded ring, I a graded ideal of R and M be a G-graded R-module. Let $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be a function, where $S^{gr}(M)$ denote the set of all graded submodules of M. In this article, we introduce and study the concepts of graded ψ-second submodules and graded I-second submodules of a graded R-module which are generalizations of graded second submodules of M and investigate some properties of this class of graded modules.

1 Introduction

The study of graded rings arises naturally out of the study of affine schemes and allows them to formalize and unify arguments by induction [13]. However, this is not just an algebraic trick. The concept of grading in algebra, in particular graded modules is essential in the study of homological aspect of rings. Much of the modern development of the commutative algebra emphasizes graded rings. Graded rings play a central role in algebraic geometry and commutative algebra. Gradings appear in many circumstances, both in elementary and advanced level. In recent years, rings with a group-graded

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structure have become increasingly important and consequently, the graded analogues of different concepts are widely studied (see [1, 4, 10, 11, 12, 14]). Throughout this work, all graded rings are assumed to be commutative graded rings with identity, and all graded modules are unitary graded R-modules. We will denote the set of graded ideals of R by $S^{gr}(R)$ and the set of all graded submodules of M by $S^{gr}(M)$. Let G be a group with identity e and R be a ring. Then R is said to be a G-graded if $R = \bigoplus_{g\in G} R_g$ such that $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are homogeneous of degree g. Consider $supp(R, G) = \{g \in G \mid R_g \neq 0\}$. For simplicity, we will denote the graded ring (R, G) by R. If $r \in R$, then r can be written as $\sum_{g\in G} r_g$, where r_g is the component r in R_g. Moreover, R_e is a subring of R and if R contains a unitary 1, then $1 \in R_e$. Furthermore, $h(R) = \bigcup_{g\in G} R_g$.

Let I be a left ideal of a graded ring R. Then I is said to be a graded ideal of R, if $I = \bigoplus_{g\in G}(I \cap R_g)$, i. e., for $x \in I$, $x = \sum_{g\in G} x_g$, where $x_g \in I$ for all $g \in G$. A proper graded ideal I of a graded ring R is said to be graded prime if whenever $r_g s_h \in I$ for some $r_g, s_h \in h(R)$, then $r_g \in I$ or $s_h \in I$. Graded primary (prime) ideals over commutative graded rings have been studied by [14].

Assume that M is an R-module. Then M is said to be G-graded if $M = \bigoplus_{g\in G} M_g$ with $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$, where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. Also, we consider $supp(M, G) = \{g \in G \mid M_g \neq 0\}$. It is clear that M_g is an R_e-submodule of M for all $g \in G$. Moreover $h(M) = \bigcup_{g\in G} M_g$. Let N be an R-submodule of a graded R-module M. Then N is said to be a graded R-submodule if $N = \bigoplus_{g\in G}(N \cap M_g)$, i. e., for $m \in N$, $m = \sum_{g\in G} m_g$, where $m_g \in N$ for all $g \in G$. Moreover, M/N becomes a G-graded module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$.

A proper graded submodule N of a graded R-module M is said to be graded prime, if $r_g m_h \in N$ where $r_g \in h(R)$ and $m_h \in h(M)$, then $m_h \in N$ or $r_g \in (N : M)$. A graded R-module M is called graded prime, if the zero graded submodule is graded prime in M. For more information about graded prime submodules over commutative graded rings see [3, 7, 9]. A graded R-module M is called graded finitely generated if $M = Rm_{g_1} + Rm_{g_2} + \cdots + Rm_{g_n}$ for some $m_{g_1}, \cdots, m_{g_n} \in h(M)$. Farshadifar and Ansari-Toroghy in [5, 6] introduced the concepts of I-second submodules of M and ψ-second submodules of M which are two generalizations of second submodules of M. In the first section of this paper, we introduce and study the notion of graded ψ-second submodules of a graded R-module M and we investigate some properties of such graded submodules. For example, in Theorem 7, we characterize graded ψ-second submodules of a graded R-module M. In the second section, we introduce the notion of graded I-second submodules of a graded R-module M and obtain some related results. For example, we prove when a graded submodule of a graded R-module is a graded I-second submodule.

2 Graded ψ-second submodules

In this section, we define and study graded ψ-second submodules of a graded module over a commutative graded ring.

The following Lemma is known, but we write it here for the sake of references.

Lemma 1 *Let M be a graded module over a graded ring R. Then the following hold:*

(i) *If I and J are graded ideals of R, then $I+J$ and $I \bigcap J$ are graded ideals of R.*

(ii) *If I is a graded ideal of R, N is a graded submodule of M, $r \in h(R)$ and $x \in h(M)$, then Rx, IN, rN and $(0:_M I)$ are graded submodules of M.*

(iii) *If N and K are graded submodules of M, then $N+K$ and $N \bigcap K$ are also graded submodules of M and $(N:_R M)$ is a graded ideal of R. Also, $Ann_R(M) = (0:_R M)$ is a graded ideal of R.*

(iv) *Let $\{N_\lambda\}_{\lambda \in \Lambda}$ be a collection of graded submodules of M. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodues of M.*

Definition 1 *Let M be a graded R-module and let $g \in G$.*

(a) *A non-zero submodule N_g of R_e- module M_g is said to be g-second submodule of M_g, if for each $r_e \in R_e$, either $r_e N_g = 0$ or $r_e N_g = N_g$.*

(b) *A non-zero graded submodule N of M is said to be a graded second submodule of M if for each $r_g \in h(R)$, either $r_g N = 0$ or $r_g N = N$.*

Definition 2 *Let M be a graded R-module and let $g \in G$. Let $\psi : S(M_g) \to S(M_g) \cup \{\emptyset\}$ be a function, where $S(M_g)$ is the set of all submodules of M_g. We say that a non-zero submodule N_g of R_e-module M_g is a g-ψ-second submodule, if $r_e \in R_e$, K a submodule of M_g, $r_e N_g \subseteq K$, and $r_e \psi(N_g) \nsubseteq K$, then $N_g \subseteq K$ or $r_e N_g = 0$.*

Definition 3 *Let M be a graded R-module, $S^{gr}(M)$ be the set of all graded submodules of M, and let $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be a function. We say that a non-zero graded submodule N of M is a graded ψ-second submodule of M if $r_g \in h(R)$, K a graded submodule of M, $r_gN \subseteq K$, and $r_g\psi(N) \nsubseteq K$, then $N \subseteq K$ or $r_gN = 0$*

We use the following functions $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$.

$$\begin{aligned} \psi_M(N) &= M, \ \forall N \in S^{gr}(M), \\ \psi_i(N) &= (N :_M Ann_R^i(N)), \ \forall N \in S^{gr}(M), \ \forall i \in \mathbb{N}, \\ \psi_\sigma(N) &= \sum_{i=1}^{\infty} \psi_i(N), \ \forall N \in S^{gr}(M). \end{aligned}$$

Then it is clear that for any graded submodule and every positive integer n, we have the following implications:

$$\begin{aligned} \text{graded second} \Rightarrow \text{graded } \psi_{n-1} - \text{second} \Rightarrow \text{graded } \psi_n - \text{second} \\ \Rightarrow \text{graded } \psi_\sigma - \text{second} \end{aligned}$$

For functions $\psi, \theta : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$, we write $\psi \leq \theta$ if $\psi(N) \subseteq \theta(N)$ for each $N \in S^{gr}(M)$. So whenever $\psi \leq \theta$, any graded ψ-second submodule is graded θ-second.

Theorem 1 *Let M be a graded R-module and N be a graded submodule of R. Then the following statements are equivalent:*

(i) *N is a graded second submodule of M.*

(ii) *$N \neq 0$ and $r_gN \subseteq K$, where $r_g \in h(R)$ and K is a graded submodule of M, implies either $r_gN = 0$ or $N \subseteq K$.*

Proof. (i) $\Rightarrow$ (ii) is obvious.
(ii) $\Rightarrow$ (i) Let $r_g \in h(R)$ and $r_gN \neq 0$. Since $r_gN \subseteq r_gN$, so $N \subseteq r_gN$ by assumption. Therefore $r_gN = N$, as needed. □

Theorem 2 *Let M be a graded R-module, N a graded submodule of M and let $g \in G$. Let $\psi : S(M_g) \to S(M_g) \cup \{\emptyset\}$ be a function and N_g be a g-ψ-second submodule of R_e-module M_g such that $Ann_{R_e}(N_g)\psi(N_g) \nsubseteq N_g$. Then N_g is a g-second submodule of M_g.*

Proof. Let $r_e \in R_e$ and K be a submodule of M_g such that $r_eN_g \subseteq K$. If $r_e\psi(N_g) \nsubseteq K$, then we are done because N_g is a g-ψ-second submodule of R_e-module M_g. Thus suppose that $r_e\psi(N_g) \subseteq K$. If $r_e\psi(N_g) \nsubseteq N_g$, then $r_e\psi(N_g) \nsubseteq N_g \cap K$. Since $r_eN_g \subseteq N_g \cap K$, then $N_g \subseteq N_g \cap K \subseteq K$ or $r_eN_g = 0$, as required. So let $r_e\psi(N_g) \subseteq N_g$. If $Ann_{R_e}(N_g)\psi(N_g) \nsubseteq K$, then $(r_e + Ann_{R_e}(N_g))\psi(N_g) \nsubseteq K$. Thus $(r_e + Ann_{R_e}(N_g))N_g \subseteq K$ implies that $N_g \subseteq K$ or $r_eN_g = (r_e + Ann_{R_e}(N_g))N_g = 0$, as needed. Hence let $Ann_{R_e}(N_g)\psi(N_g) \subseteq K$. Since $Ann_{R_e}(N_g)\psi(N_g) \nsubseteq N_g$, there exists $s_e \in Ann_{R_e}(N_g)$ such that $(s_e\psi(N_g) \nsubseteq N_g$. Thus $s_e\psi(N_g) \nsubseteq N_g \cap K$. Hence we have $(r_e + s_e)\psi(N_g) \nsubseteq N_g \cap K$. Therefore, $(r_e + s_e)N_g \subseteq N_g \cap K$ implies that $N_g \subseteq N_g \cap K \subseteq K$ or $(r_e + s_e)N_g = r_eN_g = 0$, as needed. □

Corollary 1 *Let M be a graded R-module, N a graded submodule of M and $g \in G$. Let $\psi : S(M_g) \to S(M_g) \cup \{\emptyset\}$ be a function and N_g be a g-ψ-second submodule of R_e-module M_g such that $(N_g :_{M_g} Ann^2_{R_e}(N_g)\psi(N_g) \subseteq \psi(N_g)$. Then N_g is a g-ψ_σ-second submodule of M_g.*

Proof. If N_g is a g-second submodule of M_g, then the result is clear. So assume that N_g is not g-second submodule of M_g. Then by Theorem 2, we have $Ann_{R_e}(N_g)\psi(N_g) \subseteq N_g$. Therefore, by assumption,

$$(N_g :_{M_g} Ann^2_{R_e}(N_g)) \subseteq \psi(N_g) \subseteq (N_g :_{M_g} Ann_{R_e}(N_g)).$$

We conclude that $\psi(N_g) = (N_g :_{M_g} Ann^2_{R_e}(N_g)) = (N_g :_{M_g} Ann_{R_e}(N_g))$, because $(N_g :_{M_g} Ann_{R_e}(N_g)) \subseteq (N_g :_{M_g} Ann^2_{R_e}(N_g))$. So we get

$$(N_g :_{M_g} Ann^3_{R_e}(N_g)) = (((N_g :_{M_g} Ann^2_{R_e}(N_g)) :_{M_g} Ann_{R_e}(\psi(N_g))) =$$

$$((N_g :_{M_g} Ann_{R_e}(N_g)) :_{M_g} Ann_{R_e}(N_g)) = (N_g :_{M_g} Ann^2_{R_e}(N_g)) = \psi(N_g).$$

By continuing, we get that $\psi(N_g) = (N_g :_{M_g} Ann^i_{R_e}(N_g))$ for all $i \geq 1$. Hence $\psi(N_g) = \psi_\sigma(N_g)$, as needed. □

Theorem 3 *Let M be a graded R-module and $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be a function. Let N be a graded submodule of M such that for all graded ideals I and J of R, $(N :_M I) \subseteq (N :_M J)$ implies that $J \subseteq I$. If N is not a graded second submodule of M, then N is not a graded ψ_1-second submodule of M.*

Proof. Since N is not a graded second submodule of M, there exists $r_g \in h(R)$ and a graded submodule K of M such that $r_gN \neq 0$ and $N \nsubseteq K$, but $r_gN \subseteq K$

by Theorem 1. We have $N \nsubseteq N \cap K$ and $r_g N \subseteq N \cap K$. If $r_g(N :_M Ann_R(N)) \nsubseteq N \cap K$, then N is not a graded ψ_1-second submodule of M. Hence let $r_g(N :_M Ann_R(N)) \subseteq N \cap K$. Thus $r_g(N :_M Ann_R(N)) \subseteq N \cap K \subseteq N$. Therefore, $(N :_M Ann_R(N)) \nsubseteq (N :_M r_g)$ and so by assumption, $r_g \in Ann_R(N)$, which is a contradiction. □

Corollary 2 *Let M be a graded R-module and $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be a function. Let N be a graded submodule of M such that for all graded ideals I and J of R, $(N :_M I) \subseteq (N :_M J)$ implies that $J \subseteq I$. Then N is a graded second submodule of M if and only if N is a graded ψ_1-second submodule of M.*

A graded R-module M is said to be a graded multiplication module if for every graded submodule N of M, there exists a graded ideal I of R such that $N = IM$. It is easy to see that M is a graded multiplication module if and only if $N = (N : M)M$ for each graded submodule N of M [8].

A graded R-module M is said to be a graded comultiplication module if for every graded submodule N of M, there exists a graded ideal I of R such that $N = (0 :_M I)$ [2].

Definition 4 *Let R be a graded ring and $\varphi : S^{gr}(R) \to S^{gr}(R) \cup \{\emptyset\}$ be a function. A proper graded ideal P of R is called graded φ-prime, if for $a_g, b_h \in h(R)$, $a_g b_h \in P - \varphi(P)$, then $a_g \in P$ or $b_h \in P$.*

Definition 5 *Let M be a graded R-module and $\varphi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be a function. A proper graded submodule N of M is said to be graded φ-prime, if for each $r_g \in h(R)$ and $m_g \in h(M)$, $r_g m_h \in N \setminus \varphi(N)$, then $m_h \in N$ or $r_g \in (N :_R M)$.*

Theorem 4 *Let M be a graded R-module, $\varphi : S^{gr}(R) \to S^{gr}(R) \cup \{\emptyset\}$, and $\theta : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be functions such that $\theta(P) = \varphi((P :_R M))M$. The following statements hold:*

(i) *If P is a graded θ-prime submodule of M such that $(\theta(P) :_R M) \subseteq \varphi((P :_R M))$, then $(P :_R M)$ is a graded φ-prime ideal of R.*

(ii) *If M is a graded multiplication R-module and $(P :_R M)$ is a graded φ-prime ideal of R, then P is a graded θ-prime submodule of M.*

Proof. (i) Let $a_g b_h \in (P :_R M) \setminus \varphi((P :_R M))$ for some $a_g, b_h \in h(R)$. If $a_g b_h M \subseteq \theta(P)$, then $a_g b_h \in \varphi((P :_R M))$, a contradiction. Thus $a_g b_h M \nsubseteq$

$\theta(P)$. Therefore, $a_gM \subseteq P$ or $b_hM \subseteq P$ because P is a graded θ-prime submodule of M. Thus $a_g \in (P :_R M)$ or $b_h \in (P :_R M)$, as needed.
(ii) Let $a_gm_h \in P \setminus \theta(P) = P \setminus \varphi((P :_R M))M$. Then $a_g((Rm_h :_R M))M \subseteq P$. If $a_g((P :_R M)) \subseteq \varphi((Rm_h :_R M))$, then $a_g((Rm_h :_R M))M \subseteq \varphi((P :_R M))M$. As M is a graded multiplication R-module, we have $a_gm_h \in Rm_h = (Rm_h :_R M)M$. Therefore, $a_gm_h \in \varphi((P :_R M))M$ which is a contradiction. Thus $a_g((Rm_h :_R M)) \nsubseteq \varphi((P :_R M))$ and so by assumption, $a_g \in (P :_R M)$ or $(Rm_h :_M M) \subseteq (P :_R M)$, as needed. □

Theorem 5 *Let M be a graded R-module, $\varphi : S^{gr}(R) \to S^{gr}(R) \cup \{\emptyset\}$, and $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be functions. Then the following hold:*

(i) *If S is a graded ψ-second submodule of M such that $Ann_R(\psi(S)) \subseteq \varphi(Ann_R(S))$, then $Ann_R(S)$ is a graded φ-prime ideal of R.*

(ii) *If M is a graded comultiplication R-module, S is a graded submodule of M such that $\psi(S) = (0 :_M \varphi(Ann_R(S)))$, and $Ann_R(S)$ is a graded φ-prime ideal of R, then S is a graded ψ-second submodule of M.*

Proof. (i) Let $a_gb_h \in Ann_R(S) \setminus \varphi(Ann_R(S))$ for some $a_g, b_h \in h(R)$. Then $a_gb_h\psi(S) \neq 0$ by assumption. If $a_g\psi(S) \subseteq (0 :_M b_h)$, then $a_gb_h\psi(S) = 0$, a contradiction. Thus $a_g\psi(S) \nsubseteq (0 :_M b_h)$. Therefore, $S \subseteq (0 :_M b_h)$ or $a_gS = 0$ because S is a graded ψ-second submodule of M. Hence $a_g \in Ann_R(S)$ or $b_h \in Ann_R(S)$, as required.
(ii) Let $a_g \in h(R)$ and K be a graded submodule of M such that $a_gS \subseteq K$ and $a_g\psi(S) \nsubseteq K$. As $a_gS \subseteq K$, we have $S \subseteq (K :_M a_g)$. It follows that

$$S \subseteq ((0 :_M Ann_R(K)) :_M a_g) = (0 :_M a_gAnn_R(K)).$$

This implies that $a_gAnn_R(K) \subseteq Ann_R((0 :_M a_gAnn_R(K))) \subseteq Ann_R(S)$. Hence $a_gAnn_R(K) \subseteq Ann_R(S)$. If $a_gAnn_R(K) \subseteq \varphi(Ann_R(S))$, then

$$\psi(S) = ((0 :_M \varphi(Ann_R(S)) = ((0 :_M Ann_R(K)) :_M a_g).$$

As M is a graded comultiplication R-module, we have $a_g\psi(S) \subseteq K$, a contradiction. Thus $a_gAnn_R(K) \nsubseteq \varphi(Ann_R(S))$ and so as $Ann_R(S)$ is a graded φ-prime ideal of R, we conclude that $a_gS = 0$ or

$$S = (0 :_M Ann_R(S)) \subseteq (0 :_M Ann_R(K)) = K$$

as needed. □

Example 1 *Let* $G = \mathbb{Z}_2$ *and* $R = \mathbb{Z}$ *be a* G*-graded ring with* $R_0 = \mathbb{Z}$ *and* $R_1 = \{0\}$*. Let* $M = \mathbb{Z} \times \mathbb{Z}$*. Then* M *is a* G*-graded* R*-module with* $M_0 = \mathbb{Z} \times \{0\}$ *and* $M_1 = \{0\} \times \mathbb{Z}$*. Consider the graded submodule* $S = (2\mathbb{Z} \times \{0\}) \oplus (\{0\} \times 2\mathbb{Z})$*. Clearly,* M *is not a graded comultiplication* R*-module. Suppose that* $\varphi : S^{gr}(R) \to S^{gr}(R) \cup \{\emptyset\}$ *and* $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ *be functions such that* $\varphi(I) = I$ *for each graded ideal* I *of* R *and* $\psi(S) = M$*. Then* $Ann_R(S) = 0$ *is a graded* φ*-prime ideal of* R *and* $\psi(S) = M = (0 :_M \varphi(Ann_R(S)))$*. But since* $4S \subseteq (8\mathbb{Z} \times \{0\}) \oplus (\{0\} \times 8\mathbb{Z})$*,* $S \nsubseteq (8\mathbb{Z} \times \{0\}) \oplus (\{0\} \times 8\mathbb{Z})$*, and* $4S \neq 0$*, we have* S *is not a graded* ψ*-second submodule of* M*.*

Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fractions $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R = \bigoplus_{g\in G}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in R, s \in S \text{ and } g = (\deg s)^{-1}(\deg r)\}$. We write $h(S^{-1}R) = \bigcup_{g\in G}(S^{-1}R)_g$ [8].

Proposition 1 *Let* M *be a graded* R*-module,* $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ *be a function and* N *be a graded* ψ*-second submodule of* M*. Then we have the following statements.*

(i) *If* K *is a graded submodule of* M *with* $K \subset N$ *and* $\psi_K : S^{gr}(M/K) \to S^{gr}(M/K) \cup \{\emptyset\}$ *be a function such that* $\psi_K(N/K) = \psi(N)/K$*, then* N/K *is a graded* ψ_K*-second submodule of* M/K*.*

(ii) *Let* N *be a graded finitely generated submodule of* M*,* S *be a multiplicatively closed subset of* R *with* $Ann_R(N) \cap S = \emptyset$*, and* $S^{-1}\psi : S^{gr}(S^{-1}M) \to S^{gr}(S^{-1}M) \cup \{\emptyset\}$ *be a function such that* $(S^{-1}\psi)(S^{-1}N) = S^{-1}\psi(N)$*. Then* $S^{-1}N$ *is a graded* $S^{-1}\psi$*-second submodule of* $S^{-1}M$*.*

Proof. (i) Since $K \subset N$, then $N/K \neq 0$. Let $r_g \in h(R)$, L/K be a graded submodule of M/K, $r_g(N/K) \subseteq L/K$ and $r_g\psi(N/K) \nsubseteq L/K$. We get $r_gN \subseteq L$ and $r_g\psi(N) \nsubseteq L$. Therefore, $r_gN = 0$ or $N \subseteq L$ since N is a graded ψ-second submodule of M. Hence $r_g(N/K) = 0$ or $N/K \subseteq L/K$, as needed.
(ii) Since N is graded finitely generated and $Ann_R(N) \cap S = \emptyset$, we get $S^{-1}(N) \neq 0$. Let $\frac{r}{s} \in h(S^{-1}R)$, $S^{-1}(K)$ be a graded submodule of $S^{-1}M$ and $\frac{r}{s}(S^{-1}\psi)(S^{-1}N) \nsubseteq S^{-1}K$. Thus we get $rN \subseteq K$ and $r\psi(N) \nsubseteq K$ ($(S^{-1}\psi)(S^{-1}N) = S^{-1}\psi(N)$). Hence $N \subseteq K$ or $rN = 0$ since N is a graded ψ-second submodule of M. Therefore, $S^{-1}N \subseteq S^{-1}K$ or $\frac{r}{s}\psi(S^{-1}N) = 0$, and so $S^{-1}N$ is a graded $S^{-1}\psi$-second submodule of $S^{-1}M$. □

Let $R = \bigoplus_{g\in G} R_g$ and $S = \bigoplus_{g\in G} S_g$ be two graded ring. The function $f : R \to S$ is called a graded homomorphism, if

(i) for any $a, b \in R$, $f(a+b) = f(a) + f(b)$,

(ii) for any $a, b \in R$, $f(ab) = f(a)f(b)$, and

(iii) $f(R_g) \subseteq S_g$ for any $g \in G$.

Proposition 2 *Let M and M' be graded R-modules and $f : M \to M'$ be a graded monomorphism. Let $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ and $\psi' : S^{gr}(M') \to S^{gr}(M') \cup \{\emptyset\}$ be functions such that $\psi(f^{-1}(N')) = f^{-1}(\psi'(N'))$, for each graded submodule N' of M'. If N' is a graded ψ-second submodule of M' such that $N' \subseteq Im(f)$, then $f^{-1}(N')$ is a graded ψ-second submodule of M.*

Proof. Since $N' \neq 0$ and $N' \subseteq Im(f)$, we have $f^{-1}(N') \neq 0$. Let $a_g \in h(R)$ and K be a graded submodule of M such that $a_g f^{-1}(N') \subseteq K$ and $a_g \psi(f^{-1}(N')) \nsubseteq K$. Then by assumptions, $a_g N' \subseteq f(K)$ and $a_g \psi'(N') \nsubseteq f(K)$. Thus $a_g N' = 0$ or $N' \subseteq f(K)$. Therefore, $a_g f^{-1}(N') = 0$ or $f^{-1}(N') \subseteq K$, as required. □

A proper graded submodule N of a graded R-module M is said to be graded completely irreducible if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of graded submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every graded submodule of M is an intersection of graded completely irreducible submodules of M.

Remark 1 *Let N, K be graded submodules of a graded R-module M. To prove $N \subseteq K$, it is enough to show that if L is a graded completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$.*

Proposition 3 *Let M be a graded R-module, $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be a function and let N be a graded ψ_1-second submodule of M . Then we have the following statements:*

(i) *If for $a_g \in h(R)$, $a_g N \neq N$, then $(N :_M Ann_R(N)) \subseteq (N :_M a_g)$.*

(ii) *If J is a graded ideal of R such that $Ann_R(N) \subseteq J$ and $JN \neq N$, then $(N :_M Ann_R(N)) = (N :_M J)$.*

Proof. (i) By Remark 1, there exists a graded completely irreducible submodule L of M such that $a_g N \subseteq L$ and $N \nsubseteq L$. If $a_g N = 0$, then we get $(N :_M Ann_R(N)) \subseteq (N :_M a_g)$. Hence let $a_g N \neq 0$. Since N is a graded ψ_1-second submodule of M, we have $a_g(N :_M Ann_R(N)) \subseteq L$. Now let H be a graded completely irreducible submodule of M such that $N \subseteq H$. Then $N \nsubseteq L \cap H$ and $a_g N \subseteq L \cap H$. Thus as N is a graded ψ_1-second submodule of M, we

have $a_g(N :_M Ann_R(N)) \subseteq L \cap H$. Hence $a_g(N :_M Ann_R(N)) \subseteq H$. Therefore, $a_g(N :_M Ann_R(N)) \subseteq N$ by Remark 1. Hence $(N :_M Ann_R(N)) \subseteq (N :_M a_g)$.
(ii) This follows from (i). □

Theorem 6 *Let M be a graded R-module, $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be a function and let $g \in G$. If $(0 :_{M_g} a_e)$ is a g-ψ_1-second submodule of R_e-module M_g such that $(0 :_{M_g} a_e) \subseteq a_e(0 :_{M_g} a_e Ann_R(0 :_{M_g} a_e))$, then $(0 :_{M_g} a_e)$ is a g-second submodule of M_g.*

Proof. Let $N = (0 :_{M_g} a_e)$ be a g-ψ_1-second submodule of M. Then $(0 :_{M_g} a_e) \neq 0$. Let $b_e \in R_e$ and K be a submodule of M_g such that $b_e(0 :_{M_g} a_e) \subseteq K$. If $b_e(N :_{M_g} Ann_{R_e}(N)) \nsubseteq K$, then $b_e(0 :_{M_g} a_e) = 0$ or $(0 :_{M_g} a_e) \subseteq K$ since $(0 :_{M_g} a_e)$ is a g-ψ_1-second submodule of M_g. So let $b_e(N :_{M_g} Ann_{R_e}(N)) \subseteq K$. Now we have $(a_e + b_e)(0 :_{M_g} a_e) \subseteq K$. If $(a_e + b_e)(N :_{M_g} Ann_{R_e}(N)) \nsubseteq K$, then as $(0 :_{M_g} a_e)$ is a g-ψ_1-second submodule of M_g, then $(a_e + b_e)(0 :_{M_g} a_e) = 0$ or $(0 :_{M_g} a_e) \subseteq K$ and we are done. Hence assume that $(a_e + b_e)(N :_{M_g} Ann_{R_e}(N)) \subseteq K$. Then $b_e(N :_{M_g} Ann_{R_e}(N)) \subseteq K$ gives that $a_e(N :_{M_g} Ann_{R_e}(N)) \subseteq K$. Therefore by assumption, $(0 :_{M_g} a_e) \subseteq K$ and the result follows from Theorem 1. □

Theorem 7 *Let M be a graded R-module, $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be a functions, and N be a non-zero graded submodule of M. Then the following are equivalent:*

(i) *N is a graded ψ-second submodule of M;*

(ii) *For graded completely irreducible submodule L of M with $N \nsubseteq L$, we have $(L :_R N) = Ann_R(N) \cup (L :_R \psi(N))$;*

(iii) *For graded completely irreducible submodule L of M with $N \nsubseteq L$, we have $(L :_R N) = Ann_R(N)$ or $(L :_R N) = (L :_R \psi(N))$;*

(iv) *For any graded ideal I of R and any graded submodule K of M, if $IN \subseteq K$ and $I\psi(N) \nsubseteq K$, then $IN = 0$ or $N \subseteq K$.*

(v) *For each $a_g \in h(R)$ with $a_g\psi(N) \nsubseteq a_gN$, we have $a_gN = N$ or $a_gN = 0$.*

Proof. (i) ⇒ (ii) Let for a graded completely irreducible submodule L of M with $N \nsubseteq L$, we have $a_g \in (L :_R N) \setminus (L :_R \psi(N))$. Then $a_g\psi(N) \nsubseteq L$. Since N is a graded ψ-second submodule of M, we have $a_g \in Ann_R(N)$. As we may assume that $\psi(N) \subseteq N$, the other inclusion always holds.

(ii) ⇒ (iii) This follows from the fact that if a graded ideal is a union of two graded ideals, it is equal to one of them.
(iii) ⇒ (iv) Let I be a graded ideal of R and K be a graded submodule of M such that $IN \subseteq K$ and $I\psi(N) \nsubseteq K$. Suppose $I \nsubseteq Ann_R(N)$ and $N \nsubseteq K$. We show that $I\psi(N) \subseteq K$. Let $a \in I$ and L is a graded completely irreducible submodule of M with $K \subseteq L$. First, let $a \notin Ann_R(N)$. Then since $aN \subseteq L$, we have $(L :_R N) \neq Ann_R(N)$. Hence by assumption $(L :_R N) = (L :_R \psi(N))$. So $a\psi(N) \subseteq L$. Now let $a \in I \cap Ann_R(N)$. Let $b \in I \setminus Ann_R(N)$. Then $a + b \in I \setminus Ann_R(N)$. Hence by the first case, for each graded completely irreducible submodule L of M with $K \subseteq L$ we have $b\psi(N) \subseteq L$ and $(b + a)\psi(N) \subseteq L$. This gives that $a\psi(N) \subseteq L$. Thus in any case $a\psi(N) \subseteq L$. Thus $I\psi(N) \subseteq L$. Therefore, $a\psi(N) \subseteq K$ by Remark 1.
(iv) ⇒ (i) The proof is straightforward.
(i) ⇒ (v) Let $a_g \in h(R)$ such that $a_g\psi(N) \nsubseteq a_gN$. Then $a_gN \subseteq a_gN$ implies that $N \subseteq a_gN$ or $a_gN = 0$ by part (i). Thus $N = a_gN$ or $a_gN = 0$, as required. (v) ⇒ (i) Let $a_g \in h(R)$ and K be a graded submodule of M such that $a_gN \subseteq K$ and $a_g\psi(N) \nsubseteq K$. If $a_g\psi(N) \subseteq a_gN$, then $a_gN \subseteq K$ implies that $a_g\psi(N) \subseteq K$, a contradiction. Thus by part (v), $a_gN = N$ or $a_gN = 0$. Therefore, $N \subseteq K$ or $a_gN = 0$, as needed. □

Example 2 *Let N be a non-zero graded submodule of a graded R-module M and let $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be a function. If $\psi(N) = N$, then N is a graded ψ-second submodule of M by Theorem 7 (v) ⇒ (i).*

Let R_1 and R_2 be two G-graded rings. Then $R = R_1 \times R_2$ becomes a G-graded ring with homogeneous elements $h(R) = \bigcup_{g\in G} R_g$, where $R_g = (R_1)_g \times (R_2)_g$ for all $g \in G$. Let M_1 be a graded R_1-module and M_2 be a graded R_2-module. Then $M = M_1 \times M_2$ is a graded $R = R_1 \times R_2$-module.

Theorem 8 *Let $R = R_1 \times R_2$ be a graded ring and $M = M_1 \times M_2$ be a graded R-module where M_1 is a graded R_1-module and M_2 is a graded R_2-module. Suppose that $\psi^i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function for $i = 1, 2$. Then $S_1 \times 0$ is a graded $\psi^1 \times \psi^2$-second submodule of M, where S_1 is a graded ψ^1-second submodule of M_1 and $\psi^2(0) = 0$.*

Proof. Let $(r_g, r'_g) \in h(R)$ and $K_1 \times K_2$ be a graded submodule of M such that $(r_g, r'_g)(S_1 \times 0) \subseteq K_1 \times K_2$ and

$$(r_g, r'_g)((\psi^1 \times \psi^2)(S_1 \times 0)) = r_g\psi^1(S_1) \times r'_g\psi^2(0) = r_g\psi^1(S_1) \times 0 \nsubseteq K_1 \times K_2$$

Then $r_g S_1 \subseteq K_1$ and $r_g \psi^1(S_1) \nsubseteq K_1$. Hence $r_g S_1 = 0$ or $S_1 \subseteq K_1$ since S_1 is a graded ψ^1-second submodule of M_1. Therefore, $(r_g, r'_g)(S_1 \times 0) = 0 \times 0$ or $S_1 \times 0 \subseteq K_1 \times K_2$, as needed. $\square$

3 Graded I-second submodules

Definition 6 *Let* R *be a graded ring,* M *be a graded* R*-module and* I *be a graded ideal of* R.
(a) *A proper graded ideal* P *of* R *is called graded* I*-prime, if* $a_g b_h \in P \setminus IP$, *then* $a_g \in P$ *or* $b_h \in P$.
(b) *A proper graded submodule* N *of* M *is called graded* I*-prime, if* $r_g m_h \in N \setminus IN$, *then* $m_h \in N$ *or* $r_g \in (N :_R M)$.

Theorem 9 *Let* I *be a graded ideal of a graded ring* R. *For a non-zero graded submodule* S *of a graded* R*-module* M *the following statements are equivalent:*

(i) *For each* $r_g \in h(R)$, *a submodule* K *of* M, $r_g \in (K :_R S) \setminus (K :_R (S :_M I))$ *implies that* $S \subseteq K$ *or* $r_g \in Ann_R(S)$;

(ii) *For each* $r_g \notin (r_g S :_R (S :_M I))$, *we have* $r_g S = S$ *or* $r_g S = 0$;

(iii) $(K :_R S) = Ann_R(S \cup (K :_R (S :_M I))$, *for any submodule* K *of* M *with* $S \nsubseteq K$;

(iv) $(K :_R S) = Ann_R(S)$ *or* $(K :_R S) = (K :_R (S :_M I))$, *for any submodule* K *of* M *with* $S \nsubseteq K$.

Proof. (i) $\Rightarrow$ (ii) Let $r_g \notin (r_g S :_R (S :_M I))$. Then as $r_g S \subseteq r_g S$, we have $S \subseteq r_g S$ or $r_g S = 0$ by part (i). Thus $r_g S = S$ or $r_g S = 0$.
(ii) $\Rightarrow$ (i) Let $r_g \in h(R)$ and K be a graded submodule of M such that $r_g \in (K :_R S) \setminus (K :_R (S :_M I))$. Then if $r_g \in (r_g S :_R (S :_M I))$, then $r_g \in (K :_R (S :_M I))$ which is a contradiction. Thus $r_g \notin (r_g S :_R (S :_M I))$. Now by part (ii), $r_g S = S$ or $r_g S = 0$. So $S \subseteq K$ or $r_g S = 0$, as needed.
(i) $\Rightarrow$ (iii) Let $r_g \in (K :_R S)$ and $S \nsubseteq K$. If $r_g \notin (K :_R (S :_M I))$, then $r_g \in Ann_R(S)$ by part (i). Hence, $(K :_R S) \subseteq Ann_R(S)$. If $r_g \in (K :_R (S :_M I))$, then $(K :_R S) \subseteq (K :_R (S :_M I))$. Therefore, $(K :_R S) \subseteq Ann_R(S) \cup (K :_R (S :_M I))$. The other inclusion always holds.
(iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i) are clear. $\square$

Definition 7 *Let* I *be a graded ideal of* R*. We say that a non-zero graded submodule* S *of a graded* R*-module* M *is a graded* I*-second submodule of* M*, if satisfies the equivalent conditions of Theorem 9.*

Let I be a graded ideal of R. Clearly, every graded second submodule is a graded I-second submodule. But the converse is not true in general.

Example 3 (a) *If* $I = 0$*, then every graded module is a graded* I*-second submodule of itself but every graded module is not a graded second module. for example, let* $G = \mathbb{Z}_2$*,* $R = \mathbb{Z}$ *be a* G*-graded ring with* $R_0 = \mathbb{Z}$ *and* $R_1 = \{0\}$*. Then it is clear that the graded* R*-module* $M = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ *with* $M_0 = \mathbb{Z}$ *and* $M_1 = i\mathbb{Z}$ *is not a graded second module.*
(b) *Let* $G = \mathbb{Z}_2$*,* $R = \mathbb{Z}$ *and* $M = \mathbb{Z}_{12}[i] = \{\bar{a} + \bar{b}i \mid \bar{a}, \bar{b} \in \mathbb{Z}_{12}\}$*. Then* R *is a* G*-graded ring with* $R_0 = \mathbb{Z}$*,* $R_1 = \{0\}$ *and* M *is a graded* R*-module with* $M_0 = \mathbb{Z}_{12}$*,* $M_1 = i\mathbb{Z}_{12}$*. Consider the grade ideal* $I = 4\mathbb{Z} \oplus \{0\}$ *and the graded submodule* $S = \bar{3}\mathbb{Z} \oplus \{0\}$*. Thus* S *is a graded* I*-second submodule of* M*, but it is not a graded second submodule of* M*.*

Let I be a graded ideal of R and M be a graded R-module . If $I = R$, then every graded submodule is a graded I-second submodule. So in the rest of this paper we can assume that $I \neq R$.

Theorem 10 *Let* M *be a graded* R*-module and* I, J *be graded ideals of* R *such that* $I \subseteq J$*. If* S *is a graded* I*-second submodule of* M*, then* S *is a graded* J*-second submodule of* M*.*

Proof. The result follows from the fact that $I \subseteq J$ implies that $(r_g S :_R S) \setminus (r_g S :_R (S :_M J)) \subseteq (r_g S :_R S) \setminus (r_g S :_R (S :_M I))$, for each $r_g \in R$. □

Theorem 11 *Let* M *be a graded* R*-module and* $g \in G$*. If* I *is an ideal of* R_e *and* S *a* g*-*I*-second submodule of* R_e*-module* M_g *which is not* g*-second, then* $Ann_{R_e}(S)(S :_{M_g} I) \subseteq S$*.*

Proof. Assume on the contrary that $Ann_{R_e}(S)(S :_{M_g} I) \nsubseteq S$. We show that S is g-second. Let $rS \subseteq K$ for some $r \in R_e$ and a submodule K of M_g. If $r \notin (K :_{R_e} (S :_{M_g} I))$, then S is a g-I-second submodule implies that $S \subseteq K$ or $r \in Ann_{R_e}(S)$ as needed. So assume that $r \in (K :_{R_e} (S :_{M_g} I))$. First, suppose that $r(S :_{M_g} I) \nsubseteq S$. Then there exists a graded submodule L of M such that $S \subseteq L$ but $r_g(S :_{M_g} I) \nsubseteq L$. Then $r \in (K \cap L :_{R_e} S) \setminus (K \cap L :_{R_e} (S :_{M_g} I))$. So $S \subseteq K \cap L$ or $r_g \in Ann_{R_e}(S)$ and hence $S \subseteq K$ or $r \in Ann_{R_e}(S)$. So we can assume that $r(S :_{M_g} I) \subseteq S$. On the other hand, if $Ann_{R_e}(S)(S :_{M_g} I) \nsubseteq K$, then there exists $t \in Ann_{R_e}(S)$ such that $t \notin (K :_{R_e} (S :_{M_g} I))$. Then $t + r \in (K :_{R_e} S) \setminus (K :_{R_e} (S :_{M_g} I))$. Thus $S \subseteq K$ or $t + r \in Ann_{R_e}(S)$ and hence $S \subseteq K$ or $r \in Ann_{R_e}(S)$. So we can assume that $Ann_{R_e}(S)(S :_{M_g} I) \subseteq K$. Since $Ann_{R_e}(S)(S :_{M_g} I) \nsubseteq S$, there exists $t \in Ann_{R_e}(S)$, a submodule L of M such that $S \subseteq L$ and $t(S :_{M_g} I) \nsubseteq L$. Now we have $r + t \in (K \cap L :_{R_e} S) \setminus (K \cap L :_{R_e} (S :_{M_g} I))$. So S is a g- I-second submodule gives $S \subseteq K \cap L$ or $r + t \in Ann_{R_e}(S)$. Hence $S \subseteq K$ or $r \in Ann_{R_e}(S)$, as requested. □

Theorem 12 *Let* I *be a graded ideal of* R*,* M *a graded* R*-module and* S *be a graded submodule of* M*. Then we have the following.*

(i) *If* S *is a graded* I*-second submodule of* M *such that* $Ann_R((S :_M I)) \subseteq IAnn_R(S)$*, then* $Ann_R(S)$ *is a graded* I*-prime ideal of R.*

(ii) *If* M *is a graded comultiplication* R*-module and* $Ann_R(S)$ *is a graded* I*-prime ideal of* R*, then* S *is a graded* I*-second submodule of* M*.*

Proof. (i) Let $a_g b_h \in Ann_R(S) \setminus IAnn_R(S)$ for some $a_g, b_h \in h(R)$. Then $a_g S \subseteq (0 :_M b_h)$. As $a_g b_h \notin IAnn_R(S)$ and $Ann_R((S :_M I)) \subseteq IAnn_R(S)$, we have $a_g b_h \notin Ann_R((S :_M I))$. This implies that $a_g \notin ((0 :_M b_h) :_R (S :_M I))$. Thus $a_g \in Ann_R(S)$ or $S \subseteq (0 :_M b_h)$. Hence $a_g \in Ann_R(S)$ or $b_h \in Ann_R(S)$, as needed.
(ii) Let $r_g \in (K :_R S) \setminus (K :_R (S :_M I))$ for some $r_g \in h(R)$ and graded submodule K of M. As M is a graded comultiplication R-module, there exists a graded ideal J of R such that $K = (0 :_M J)$. Thus $r_g J \subseteq Ann_R(S)$. Since $r_g \notin (K :_R (S :_M I))$, we have $J r_g (S :_M I) \neq 0$. This implies that $J r_g \nsubseteq Ann_R((S :_M I))$. Since always $IAnn_R(S) \subseteq Ann_R((S :_M I))$, we have $r_g J \nsubseteq IAnn_R(S)$. Thus by assumption, $r_g \in Ann_R(S)$ or $J \subseteq Ann_R(S)$ and so $S \subseteq (0 :_M J) = K$. □

Proposition 4 *Let* M *be a graded* R*-module and* I *a graded ideal of* R*. Let* N *be a graded* I*-second submodule of* M*. Then we have the following statements.*

(i) *If* K *is a graded submodule of* M *with* $K \subset N$*, then* N/K *is a graded* I*-second submodule of* M/K*.*

(ii) *Let* N *be a graded finitely generated submodule of* M*,* $S \subseteq h(R)$ *be a multiplicatively closed subset of* R *with* $Ann_R(N) \cap S = \emptyset$*. Then* $S^{-1}N$ *is a graded* $S^{-1}I$*-second submodule of* $S^{-1}M$*.*

Proof. (i) This follows from the fact that $r_g \notin (r_g(S/K) :_R (S/K :_{M/K} I))$ implies that $r_g \notin (r_gS :_R (S :_M I))$.
(ii) As $Ann_R(N) \cap S = \emptyset$ and N is graded finitely generated, $S^{-1}N \neq 0$. Now the claim follows from the fact that $r/s \notin ((r/s)S^{-1}N :_{S^{-1}} (S^{-1}N :_{S^{-1}M} S^{-1}I))$ implies that $r \notin (rN :_R (N :_M I))$. □

Proposition 5 *Let* I *be a graded ideal of* R*,* M *and* M' *be graded* R*-modules, and let* $f : M \to M'$ *be an R-monomorphism. If* N' *is a graded* I*-second submodule of* M' *such that* $N' \subseteq Im(f)$*, then* $f^{-1}(N')$ *is a graded* I*-second submodule of* M*.*

Proof. As $N' \neq 0$ and $N' \subseteq Im(f)$, we have $f^{-1}(N') \neq 0$. Let $r_g \notin (r_gf^{-1}(N') :_R (f^{-1}(N') :_M I))$; then one can see that $r_g \notin (r_gN' :_R (N' :_M I))$ using assumptions. Thus $r_gN' = 0$ or $r_gN' = N'$. This implies that $r_gf^{-1}(N') = 0$ or $r_gf^{-1}(N') = f^{-1}(N')$, as requested. □

Theorem 13 *Let* I *be a graded ideal of* R*,* M_1, M_2 *be graded* R*-modules, and let* N *be a graded submodule of* M_1*. Then* $N \oplus 0$ *is a graded* I*-second submodule of* $M_1 \oplus M_2$ *if and only if* N *is a graded* I*-second submodule of* M_1 *and for* $r_g \in (r_gN :_R (N :_{M_1} I))$*,* $r_gN \neq 0$*, and* $r_gN \neq N$*, we have* $r_g \in Ann_R((0 :_{M_2} I))$*.*

Proof. (⇒) Let $r_g \notin (r_gN :_R (N :_{M_1} I))$. Then $r_g \in (r_g(N \oplus 0) :_R (N \oplus 0 :_M I))$. Since $N \oplus 0$ is a graded I-second submodule, either $r_g(N \oplus 0) = N \oplus 0$ or $r_g(N \oplus 0) = 0 \oplus 0$. Thus either $r_gN = N$ or $r_gN = 0$, so N is graded I-second. Now, let $r_g \in (r_gN :_R (N :_{M_1} I))$, $r_gN \neq 0$, and $r_gN \neq N$. Assume on the contrary that $r_g \in Ann_R((0 :_{M_2} I))$. Then there exists $y_h \in M_2$ such that $Iy_h = 0$ and $r_gy_h \neq 0$. This implies that $r_g(0, y_h) \in r_g(N \oplus 0 :_M I) \setminus r_g(N \oplus 0)$. So since $N \oplus 0$ is a graded I-second submodule, either $r_g(N \oplus 0) = N \oplus 0$ or $r_g(N \oplus 0) = 0 \oplus 0$. Thus either $r_gN = N$ or $r_gN = 0$, which is a contradiction. Therefore, $r_g \in Ann_R((0 :_{M_2} I))$.
(⇐) Let $r_g \notin (r_g(N \oplus 0) :_R (N \oplus 0 :_M I))$. Then if $r_gN = N$ or $r_gN = 0$, the result is clear. So suppose that $r_gN \neq N$ and $r_gN \neq 0$. We show that $r_g(r_gN :_R (N :_{M_1} I))$ and this contradiction proves the result because N is a graded I-second submodule of M_1. Assume on the contrary that $r_g \in (r_gN :_R (N :_{M_1} I))$. Then by assumption, $r_g \in Ann_R((0 :_{M_2} I))$. This implies that if $(x_h, y_h) \in N \oplus (0 :_M I)$, then $r_g(x_h, y_h) \in r_g(N \oplus 0)$. Therefore, $r_g \in (r_g(N \oplus 0) :_R (N \oplus 0 :_M I))$, which is a desired contradiction. □

A non-zero graded R-module M is said to be graded secondary if for each $a_g \in h(R)$ the endomorphism of M given by multiplication by a_g is either surjective or nilpotent [4].

Corollary 3 *Let* I *and* P *be graded ideals of* R*,* M_1, M_2 *be graded* R*-modules, and let* N *be a graded submodule of* M_1*. Let* S_i *(*$1 \leq i \leq n$*) be graded* P*-secondary submodules of* M_1 *with* $\sum_{i=1}^{n} S_i = (N :_{M_1} I)$*. If* N *is a graded* I*-second submodule of* M_1 *and* $P \subseteq Ann_R((0 :_{M_2} I))$*, then* $N \oplus 0$ *is a graded* I*-second submodule of* $M_1 \oplus M_2$*.*

Proof. Let $r_g \in (r_g N :_R (N :_{M_1} I))$, $r_g N \neq 0$, and $r_g N \neq N$. Then we will prove that $r_g \in Ann_R((0 :_{M_2} I))$ and hence the result is obtained by Theorem 13. Assume on the contrary that $r_g \notin Ann_R((0 :_{M_2} I))$. Hence $r \notin P$. On the other hand, $r_g(\sum_{i=1}^{n} S_i) = r_g(N :_{M_1} I) \subseteq r_g N$. But $\sum_{i=1}^{n} S_i$ is a graded P-secondary submodule by [4], so either $r_g(\sum_{i=1}^{n} S_i) = \sum_{i=1}^{n} S_i$ or $r_g \in P$. This implies that $r_g N = N$ or $r_g \in P$, which is a contradiction. Thus $r_g \in Ann_R((0 :_{M_2} I))$. □

Theorem 14 *Let* I *be a graded ideal of* R *and* M *be a graded* R*-module. Then we have the following.*

(i) *If* $\bigcap_{n=1}^{\infty} I^n M = 0$ *and every proper graded submodule of* M *is graded* I*-prime, then every non-zero graded submodule of* M *is graded* I*-second.*

(ii) *If* $\sum_{n=1}^{\infty}(0 :_M I^n) = M$ *and every non-zero graded submodule of* M *is graded* I*-second, then every proper graded submodule of* M *is graded* I*-prime.*

Proof. (i) Let S be a non-zero graded submodule of M, $r_g \in (K :_R S) \setminus (K :_R (S :_M I))$ for some $r_g \in h(R)$ and a graded submodule K of M and $r_g S \neq 0$. If $r_g S \nsubseteq IK$, then as K is graded I-prime, we have $r_g M \subseteq K$ or $S \subseteq K$. If $r_g M \subseteq K$, then $r_g(S :_M I) \subseteq K$ which is a contradiction. So $S \subseteq K$ and we are done. Now suppose that $r_g S \subseteq IK$. As $r_g S \neq 0$ and $\bigcap_{n=1}^{\infty} I^n K = 0$, there exists a positive integer t such that $r_g S \nsubseteq I^t K$. Therefore, there is a positive integer h such that $r_g S \subseteq I^{h-1} K$ but $r_g S \nsubseteq I^h K$, where $2 \leq h \leq t$. Thus since $I^{h-1} K$ is graded I-prime, $S \subseteq I^{h-1} K$ or $r_g M \subseteq I^{h-1} K$. If $r_g M \subseteq I^{h-1} K$, then $r_g(S :_M I) \subseteq K$ which is a contradiction. So $S \subseteq I^{h-1} K$ as needed.
(ii) Let P be a proper graded submodule of M, $r_g K \subseteq P \setminus IP$ for some $r_g \in h(R)$ and a graded submodule K of M and $r_g M \nsubseteq P$. If $r_g(K :_M I) \nsubseteq P$, then as K is graded I-second, we have $r_g K = 0$ or $K \subseteq P$. If $r_g K = 0$, then $r_g K \subseteq IP$ which is a contradiction. So $K \subseteq P$ and we are done. Now suppose that $r_g(K :_M I) \subseteq P$.

As $r_g M \nsubseteq P$ and $\sum_{n=1}^{\infty}(K :_M I^n) = M$, there exists a positive integer t such that $r_g(K :_M I^t) \nsubseteq P$. Therefore, there is a positive integer h such that $r_g(K :_M I^{h-1}) \subseteq P$ but $r_g(K :_M I^h) \nsubseteq P$, where $2 \leq h \leq t$. Thus since $(K :_M I^{h-1})$ is graded I-second, $(K :_M I^{h-1}) \subseteq P$ or $r_g(K :_M I^{h-1}) = 0$. If $r_g(K :_M I^{h-1}) = 0$, then $0 = r_g K \subseteq IP$ which is a contradiction. So $K \subseteq (K :_M I^{h-1}) \subseteq P$, as needed. □

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